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STEFFENSEN TYPE INEQUALITIES

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

1. Introduction. Stephensen's inequality has a long and varied history, see Mitrinović [3, pg. 107–119], for example. The simplest version is the following theorem.

THEOREM A. *Let F be non-decreasing and* $0 \leqq g \leqq 1$, *both functions continuous. Then*

$$\int_0^a f\,dx \leqq \int_0^1 fg\,dx \leqq \int_{1-a}^1 f\,dx \tag{1}$$

where $a = \int_0^1 g\,dx$.

Recently Milovanović and Pečarič [2] have shown that the same conclusions hold if $0 \leqq g \leqq 1$ is replaced by

(i) $$\int_x^1 g\,dt \geqq 0 \text{ and } \int_0^x g\,dt \leqq x,\ x \in [0, 1];$$

for the left hand inequality of (1) and for the right hand inequality

(ii) $$\int_x^1 g\,dt < 1-x,\ \ \int_0^x g\,dt \geqq 0,\ x \in [0, 1].$$

They further prove versions of (1) with f satisfying a higher monotonicity.

In this paper we show that Theorem A as well as the versions of Theorem A proved in [2] are simple corollaries of Theorem D and its extensions proved in this paper.

THEOREM B. *Let* M_0 *be the class of non-negative non-decreasing integrable functions, and* μ *a (signed) regular Borel measure. Then*

$$\int_0^1 f\,d\mu \geqq 0 \tag{2}$$

holds for all $f \in M_0$ *if and only if*

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(3) $$\int_x^1 d\mu \geqq 0 \text{ for } x \in [0, 1].$$

If M_0^* is the class of non-decreasing functions, then the necessary and sufficient condition for (2) *is* (3) *and*

(4) $$\int_0^1 d\mu = 0.$$

Using this result we can prove

THEOREM C. *Let λ be a regular Borel measure such that $\int_0^1 |d\lambda| < \infty$ and let* dx *denote Lebesuge measure, then*

(5) $$\int_0^1 f\, d\lambda \geqq \int_0^a f\, dx$$

holds for all $f \in M_0$ if and only if

(6) $$\int_x^1 d\lambda \geqq 0, \; x \in [0, 1]$$

and

(7) $$a \leqq \min_{0 \leqq t \leqq 1} \left\{ t + \int_t^1 d\lambda \right\}$$

Therefore $a = \min_{0 \leqq t \leqq 1} \{t + \int_t^1 d\lambda\}$ is the best possible choice.

We prefer to replace $g\, dx$ by $d\lambda$ in order to include the discrete versions and in order to make our results necessary and sufficient. We will generalize Theorem C to functions with higher monotonicity as well as getting an upper bound for $\int_0^1 f\, d\lambda$ which does not appear in [**2**]. Furthermore, our methods also give multi-dimensional versions of Steffensen's inequalities which appear to be completely new.

2. Preliminaries. Let f be a non-negative non-decreasing function. Then $f(x) = \int_0^x dv(t)$ for some non-negative Borel measure. If $f(0) > 0$, then this includes an atom at 0. In order to facilitate the arithmetic we introduce the notation $x_+ = \max(x, 0)$. Also x_+^n means $(x_+)^n$ except that 0^0 will be interpreted as 1. Thus the characteristic function of $[t, \infty)$ is $(x - t)_+^0$. Now the above formula for $f \in M_0$ may be written

(8) $$f(x) = \int_0^1 (x - t)_+^0 \, dv(t).$$

The class of functions which we consider generalize this formula. Let M_k denote the class of functions f with the representation

(9) $$f(x) = \int_0^1 (x - t)_+^k \, dv(t), \; x \in [0, 1],$$

for ν some non-negative regular Borel measure.

Note that k need not be an integer, although the integral case is the most important. M_1 is the class of increasing convex functions with a zero at 0. More generally, if $f \in C^{(n+1)}(0, 1)$ with $f^{(i)}(0) = 0$, $i = 0, \ldots, n - 1$, and $f^{(n)} \geqq 0, f^{(n+1)} \geqq 0$ on $[0, 1]$, then $f \in M_n$.

It is for the class M_k that we prove a theorem which has Theorem B as the special case $k = 0$.

THEOREM D. *Let μ be a (signed) regular Borel measure such that $\int_0^1 |d\mu| < \infty$. Then*

$$\int_0^1 f\, d\mu \geqq 0 \textit{ for all } f \in M_k \tag{10}$$

if and only if

$$\int_0^1 (x - t)_+^k\, d\mu(x) \geqq 0 \textit{ for } t \in [0, 1]. \tag{11}$$

PROOF. Using the representation (9) in (10) and Fubini's Theorem, (10) is equivalent to

$$\int_0^1 d\nu(t) \int_0^1 (x - t)_+^k\, d\mu(x) \geqq 0$$

for all non-negative Borel measures ν. This holds if and only if (11).

COROLLARY 1. *Let M_k^* be the function $f \in C^{(k+1)}(0, 1)$ with $f^{(k+1)}(x) \geqq 0$ on $[0, 1]$. Then $\int_0^1 f\, d\mu \geqq 0$ for all $f \in M_k^*$ if and only if (11) holds and*

$$\int_0^1 x^j\, d\mu = 0, \quad j = 0, \ldots, k. \tag{12}$$

PROOF. Since $\pm x^j \in M_k^*, j = 0, \ldots, k$, (12) is necessary, and thus (11) and (12) are necessary. For the sufficiency, we apply Theorem D to

$$f(x) - \sum_{j=0}^{k} f^{(j)}(0)\, x^j/j! \in M_k.$$

3. One-dimensional Steffensen inequalities. We are now in a position to prove the inequalities for the classes M_k and M_k^*.

THEOREM E. *Let λ be a (signed) regular Borel measure such that $\int_0^1 |d\lambda| < \infty$. Then*

$$\int_0^1 f\, d\lambda \geqq \int_0^a f\, dx \tag{13}$$

for all $f \in M_k$ if and only if

$$\int_0^1 (x - t)_+^k\, d\lambda(x) \geqq 0, \quad t \in [0, 1]; \tag{14}$$

and

$$a \leqq \min_{0 \leqq t \leqq 1} \left\{ t + \left((k+1) \int_0^1 (x-t)_+^k \, d\lambda(x) \right)^{1/k} \right\}. \tag{15}$$

Therefore the best possible choice is for equality in (15).

PROOF. We apply Theorem D to the measure $d\mu = d\lambda - (a-x)_+^0 dx$. Then (13) is equivalent to $\int_0^1 f \, d\mu \geqq 0$ for all $f \in M_k$. Thus the condition is

$$\int_0^1 (x-t)_+^k \, d\lambda(x) \geqq \int_0^1 (x-t)_+^k (a-x)_+^0 \, dx. \tag{16}$$

Since the right hand side is non-negative, (14) is necessary. Now taking $0 \leqq t \leqq 1$, (16) is

$$\int_0^1 (x-t)_+^k \, d\lambda(x) \geqq (a-t)^{k+1}/(k+1) \tag{17}$$

and in turn

$$a \leqq t + \left((k+1) \int_0^1 (x-t)_+^k \, d\lambda(x) \right)^{1/(k+1)}, \; 0 \leqq t \leqq a. \tag{18}$$

But since (14) holds, the inequality (18) is true if $t \geqq a$. Thus (15) is necessary and sufficient since we may reverse all of the above steps.

COROLLARY 2. *Inequality* (13) *holds for all* $f \in M_k^*$ *if* (14) *and* (15) *hold as well as*

$$\int_0^1 x^j d\lambda = a^{j+1}/(j+1), \; j = 0, \ldots, k. \tag{19}$$

It is worthwhile to set down conditions which give an exact formula for a. This will give the result of Milovanović and Pečarič [2]. Our further results do not have corresponding results in [2].

COROLLARY 3. If $\int_0^x t^k d\lambda(t) \leqq x^{k+1}/(k+1)$ and $\int_x^1 t^k d\lambda(t) \geqq 0$ *and* $a = [(k+1)\int_0^1 s^k d\lambda(s)]^{1/(k+1)}$, *then* (13) *holds for all* $f \in M_k$.

PROOF. We compute the left hand side of (17) as follows. Let $0 \leqq t \leqq a$. Then

$$\begin{aligned} \int_0^1 (x-t)_+^k d\lambda(x) &= \int_t^1 (1 - t/x)^k x^k d\lambda(x) \\ &= \int_t^1 (tk/x^2)(1-t/x)^{k-1} \int_x^1 s^k d\lambda(s) \, dx \\ &\geqq \int_t^a tk(1-t/x)^{k-1}/x^2 \int_x^1 s^k d\lambda(s) \, dx \\ &= \int_t^a tk(1-t/x)^{k-1}/x^2 \left[a^{k+1}/(k+1) - \int_0^x s^k d\lambda(s) \right] dx \end{aligned}$$

$$\geqq \int_t^a tk(1 - t/x)^{k-1}/x^2[a^{k+1}/(k + 1) - x^{k+1}/(k + 1)]\, dx$$
$$= (a - t)^{k+1}/(k + 1).$$

Note that according to Corollary 2, (13), holds from all $f \in M_k$ only if

$$\int_0^1 x^j d\lambda(x) = \frac{\left((k + 1)\int_0^1 x^k d\lambda(x)\right)^{(j+1)/(k+1)}}{j + 1}, \quad j = 0, \ldots, k.$$

In particular, this is true for $k = 0$.

COROLLARY 4. *Under the conditions of Corollary* 3, (13) *holds for* $f \in M_0$.

We turn to deriving upper bounds for $\int_0^1 f\, d\lambda$.

THEOREM F. *If* $\int_0^1 |d\lambda| < \infty$, *then the inequality*

(20) $$\int_0^1 f d\lambda(x) \leqq \int_a^1 f\, dx$$

holds for all $f \in M_k$ *if and only if*

(21) $$\int_0^1 (x - t)_+^k d\lambda(x) \leqq (1 - t)^{k+1}/(k + 1), \quad t \in [0, 1];$$

and

(22) $$a \leqq \min_{0 \leqq t \leqq 1} \left\{ t + [(1 - t)^{k+1} - (k + 1)\int_0^1 (x - t)_+^k d\lambda(x)]^{1/(k+1)} \right\}.$$

In particular, the best choice for a is equality in (22).

PROOF. We apply Theorem D to the measure $d\mu = (x - a)_+^0 dx - d\lambda$. The details are the same as in the proof of Theorem E.

In several important instances, as in Corollary 3, the formula for a is given by a specific choice of t, in that case the minimum is attained at $t = 0$. These instances can be checked directly by using our ideas. It is shown in Fink and Jodeit [1], that if $f \in M_k$, then $f(x)x^{-k} \in M_0$. In general, the converse is not true. We offer versions of the Steffensen inequality for this class.

THEOREM G. *Let* $f(x)x^{-1} \in M_0$, *then*

(i) $$\int_0^{a_1} f\, dx \leqq \int_0^1 f\, d\lambda$$

holds when

(23) $$\int_t^1 x^k d\lambda \geqq 0, \quad t \in [0, 1],$$

and

$$a_1 = \min_{0 \leqq t \leqq 1} [t^{k+1} + (k+1)\int_t^1 x^k d\lambda(x)]^{1/(k+1)}; \tag{24}$$

(ii)
$$\int_0^1 f\, d\lambda \leqq \int_{1-a_2}^1 f\, dx$$

holds when

$$\int_t^1 x^k d\lambda \leqq (1 - t^{k+1})/(k+1),\ t \in [0, 1], \tag{25}$$

and

$$1 - a_2 = \min_{0 \leqq t \leqq 1} [1 - (k+1)\int_t^1 x^k d\lambda]^{1/(k+1)}. \tag{26}$$

In particular, if

$$\int_0^t x^k d\lambda \leqq t^{k+1}/(k+1), \tag{27}$$

then

$$a_1 = [(k+1)\int_0^1 x^k d\lambda]^{1/(k+1)}.$$

If (23) *holds as well as* (25), *then*

$$1 - a_2 = [1 - (k+1)\int_0^1 x^k d\lambda]^{1/(k+1)}.$$

Proof. We apply Theorem D with $k = 0$ and f replaced by $f(x)x^{-k}$, and $d\mu = x^k d\lambda - x^k(a_1 - x)_+^0\, dx$ to prove (i). The equivalent statement is

$$\int_t^1 x^k d\lambda \geqq \int_t^1 x^k(a_1 - x)_+^0\, dx \quad \text{for} \quad t \in [0, 1]$$

which is (23) for $t \geqq 0$ and $a_1^{k+1} \leqq t^{k+1} + (k+1)\int_t^1 x^k d\lambda$ for $0 \leqq t \leqq a_1$. Thus (24) is the best possible choice.

To prove (ii) we use the measure $d\mu = x^k(x - 1 + a_2)_+^0\, dx - x^k d\lambda$ in Theorem D to get

$$\int_t^1 x^k d\lambda \leqq \int_t^1 x^k(x - 1 + a_2)_+^0\, dx.$$

This is

$$(1 - a_2)^{k+1} - t^{k+1} \leqq 1 - (k+1)\int_t^1 x^k d\lambda - t^{k+1},\ t \leqq 1 - a_2;$$

and

$$0 \leqq 1 - (k+1)\int_t^1 x^k d\lambda - t^{k+1},\ t \geqq 1 - a_2.$$

Since $(1 - a_2)^{k+1} - t^{k+1} \leqq 0$ for $t \geqq 1 - a_2$, the first of these holds for all t. Thus (25) and (26) imply the validity of the inequalities.

If both (23) and (25) hold, then by (23)

$$[1 - (k+1)\int_t^1 x^k d\lambda]^{1/(k+1)} \geqq [1 - (k+1)\int_0^1 x^k d\lambda]^{1/(k+1)}$$

so

$$1 - a_2 = [1 - (k+1)\int_0^1 x^k d\lambda]^{1/(k+1)}.$$

Furthermore, using (27),

$$t^{k+1} + (k+1)\int_t^1 x^k d\lambda = t^{k+1} + (k+1)\int_0^1 x^k d\lambda - (k+1)\int_0^t x^k d\lambda$$
$$\geqq (k+1)\int_0^1 x^k d\lambda.$$

Note that $a_1 = a_2$ if $k = 0$. Furthermore, if $d\lambda = g(x)dx$ for $0 \leqq g(x) \leqq 1$, then the hypotheses of Theorem G are satisfied.

4. Multi-dimensional inequalities. To derive a multi-dimensional version of Steffensen's inequality we need a counterpart to Theorem D. If $x \in \mathbf{R}^n$ with non-negative components then $\int_0^x dv(t)$ means the multiple integral

$$\iint_{0 \leqq t_i \leqq x_i} \cdots \int dv(t_1, \ldots, t_n).$$

Let $\overline{M_0}$ be the functions which have the representation

$$f(x) = \int_0^x dv(t) \tag{28}$$

for some non-negative regular Borel measure v. A function $f \in \overline{M_0}$ if it "increases away from 0". For example in $\mathbf{R}^2$, if $f_1 \geqq 0, f_2 \geqq 0$ and $f_{12} \geqq 0$, then

$$f(x, y) = f(0, 0) + \int_0^x f_1(s, 0)ds + \int_0^y f_2(0, t)dt + \int_0^x\int_0^y f_{12}(s, t)\, ds\, dt$$

so $f \in M_0$ if $f(0, 0) \geqq 0$ and

$$dv = f(0, 0)\delta_{00}(x, y) + \delta_0(y)f_1(x, 0)dx + \delta_0(x)f_2(0, y)dy + f_{12}(x, y)dx\, dy.$$

Let $\mathbf{1} = (1, \ldots, 1)$.

THEOREM H. (See [1].) *Let μ be a (signed) regular Borel measure with $\int_0^{\mathbf{1}} |d\mu| < \infty$. Then*

(29) $$\int_0^1 f\,d\mu \geqq 0 \text{ for all } f \in \overline{M_0}$$

if and only if

(30) $$\int_x^1 d\mu \geqq 0 \text{ for all } x \in [0, 1]^n.$$

PROOF. If one writes (28) as

$$f(x) = \int_0^1 \prod_1^n (x_i - t_i)_+^0 dv(t)$$

and inserts this into (29), changes order of integration by Fubini's Theorem, then

$$\int_0^1 f\,d\mu = \int_0^1 dv(t) \int_0^1 \prod_1^n (x_i - t_i)_+^0 d\mu(x) = \int_t^1 dv(t) \int_t^1 d\mu(x).$$

The result easily follows since dv is an arbitrary non-negative measure.

THEOREM I. *Let $f \in \overline{M_0}$ and λ be a regular Borel measure such that $\int_0^1 |d\lambda| < \infty$ and for every union of cubes E $\int_E d\lambda \leqq$ volume (E). If $\int_t^1 d\lambda \geqq 0$ for all $t \in [0, 1]^n$, then*

(31) $$\int_0^a f\,dx \leqq \int_0^1 f\,d\lambda$$

where a is the vector $(c, c, \ldots, c)$ for $c = 1 - (1 - \int_0^1 d\lambda)^{1/n}$.

PROOF. We apply Theorem H to the measure $d\mu = d\lambda - \prod_1^n (c - x_i)_+^0 dx_1 \ldots dx_n$. Then (31) is equivalent to

(32) $$\int_x^1 d\lambda \geqq \prod_1^n (c - x_i)_+.$$

If some $x_i > c$, then this is true so we may assume $x \leqq a$. Now $\int_x^1 d\lambda = \int_0^1 d\lambda - \int_E d\lambda$ where E is a union of cubes whose volume is $1 - \prod_1^n (1 - x_i)$. Thus

$$\int_x^1 d\lambda \geqq \int_0^1 d\lambda - 1 + \prod_1^n (1 - x_i).$$

The inequality (32) is valid if $\int_0^1 d\lambda - 1 + \prod_1^n (1 - x_i) \geqq \prod_1^n (c - x_i)_+$. Since $1 - \int_0^1 d\lambda = (1 - a)^n$ we may write this as $\prod_1^n (1 - x_i) \geqq \prod_1^n (a - x_i) + \prod_1^n (1 - a)$. Since $1 - x_i = 1 - a + (a - x_i)$, the product on the left is the sum of the two terms on the right plus many more non-negative terms. Hence (31) follows.

To get an upper bound seems to be more difficult.

THEOREM J. *Assume $\int_t^1 d\lambda \leqq \prod_1^n (1 - t_i)$ and $f \in \overline{M_0}$. Then*

$$\int_0^1 f\,d\lambda \leqq \int_a^1 f\,dx \tag{33}$$

if

$$\sup_{0 \leqq t_i \leqq 1} \prod_1^n (1 - t_i)^{-1} \int_t^1 d\lambda \leqq (1 - c)^n, \; a = (c, c, \ldots, c).$$

PROOF. We apply Theorem H to get the equivalent condition,

$$\int_t^1 d\lambda \leqq \prod_1^n \int_{t_i}^1 (x - c)_+^0 dx = \prod_1^n \min(1 - t_i, 1 - c). \tag{34}$$

Let $S = \{i | 1 - t_i < 1 - c\}$, $S^c = \{i | 1 - t_i \geqq 1 - c\}$, with S having k elements. Then

$$\int_t^1 d\lambda \leqq (1 - c)^n \prod_1^n (1 - t_i) = [\prod_S (1 - t_i) \prod_{S^C} (1 - c)][(1 - c)^k \prod_{S^c} (1 - t_i)]$$

$$\leqq \prod_S (1 - t_i) \prod_{S^C} (1 - c) = \prod_1^n \min(1 - t_i, 1 - c).$$

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