

JACOBIAN PROBLEMS IN DIFFERENTIAL EQUATIONS AND ALGEBRAIC GEOMETRY

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Dedicated to Professor Lloyd K. Jackson
on the occasion of his sixtieth birthday.

CONTENTS

1. Introduction.
2. Jacobian problems in differential equations.
3. Jacobian problems in algebraic geometry.
4. Some transformations of the plane.
5. The groups \mathcal{P} and \mathcal{G} .
6. Connections with differential equations.
7. Appendix: Aizerman's original stability conjecture.
Bibliography.

1. Introduction. This paper compares two open problems concerning the relationship (under different conditions) between the global invertibility of a transformation

$$(T) \quad \begin{aligned} u &= P(x, y) \\ v &= Q(x, y) \end{aligned}$$

of the real plane \mathbb{R}^2 into itself and properties of its Jacobian matrix

$$J = J(T) = \begin{bmatrix} \partial P/\partial x & \partial P/\partial y \\ \partial Q/\partial x & \partial Q/\partial y \end{bmatrix}.$$

One of these problems, described in §3, concerns polynomial transformations (i.e., transformations T where P and Q are polynomials in x and y) and, hence, is of interest to algebraic geometers, although they would prefer to consider transformations of \mathbb{C}^2 . The author is indebted to Sylvia and Roger Wiegand for first bringing to his attention this Jacobian problem of algebraic geometry; and to Irwin Fischer, Steven Minsker, and Jean Dieudonné for several references on this problem.

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The other problem (where P and Q are required to have continuous first-order partials) is related to a question about the global asymptotic stability of a critical point of the 2-dimensional autonomous (i.e., time-independent) system

$$(D) \quad \begin{aligned} \dot{x} &= P(x, y) \\ \dot{y} &= Q(x, y) \end{aligned} \quad (\dot{x} \equiv dx/dt)$$

of ordinary nonlinear differential equations. A “critical point”, also called an “equilibrium state”, is merely a point at which P and Q both vanish. We will always assume that our systems (D) have at least one critical point which we take to be the origin so that $P(0, 0) = Q(0, 0) = 0$. The author is indebted to Philip Hartman and Czesław Olech for information regarding this Jacobian problem in the stability theory of differential equations which is the subject of §2.

Both problems have n -dimensional counterparts and various generalizations and variations, but we have restricted ourselves to the 2-dimensional case over the field \mathbf{R} of real numbers (even though algebraic geometers usually prefer an algebraically closed field) because this seems to be the simplest nontrivial case in which the two problems can be easily compared. Although these two problems are not identical, they nevertheless have some overlap. In §6 we combine some of the known partial results on these two problems (as described in §2 through §5) to obtain a large family \mathcal{D} of explicit examples of systems (D) , many of which have the origin as a globally asymptotically stable critical point, and all of which can be explicitly analyzed to determine the extent of asymptotic stability of the origin. Included in this family \mathcal{D} are the famous equations of Liénard and van der Pol. The set of transformations T , associated with the systems (D) in \mathcal{D} , forms a large group \mathcal{G} of one-to-one area-preserving (actually, area-preserving modulo a constant factor) transformations of the plane \mathbf{R}^2 onto itself. This group \mathcal{G} has as a subgroup \mathcal{P} the set of all one-to-one polynomial transformations T whose inverses T^{-1} are also polynomial transformations. Both of these groups (described in §5) are “infinite continuous” groups in the sense of Sophus Lie. In fact, for their parametrizations, \mathcal{P} requires an infinite sequence of real numbers and \mathcal{G} requires, in addition, an infinite sequence of functions f (of one variable) where each parameter f ranges over a suitable function space. The group \mathcal{G} (and even its subgroup \mathcal{P}) is so large that we wonder if it is not dense (in some useful sense) in the larger group \mathcal{A} of all one-to-one area-preserving transformations of the plane (at least under suitable smoothness conditions on the elements of \mathcal{A}).

We have tried to write this paper in an elementary and self-contained manner in order to make it accessible to as wide an audience as possible and also because many readers who may be familiar with stability theory

are not likely to be familiar with algebraic geometry and vice versa. It is interesting to note that there was, until recently, at least one person among us who was quite familiar with both subjects. I refer of course to Solomon Lefschetz [1884–1972] who, as is well known, worked in both areas. We hope that this paper will serve as a readable and useful introduction to the various open questions associated with these Jacobian problems.

2. Jacobian problems in differential equations. The following problem was stated explicitly by Markus and Yamabe [33] where they gave affirmative answers for several special cases. It was evidently motivated for them by a slightly different problem formulated by Aizerman [3]. See the appendix of the present paper for a statement of Aizerman’s original problem and see §13 of the book by Hahn [16] for a detailed discussion of it. The two problems are not equivalent and neither has been completely solved. However, the Aizerman problem has evidently been solved in the 2-dimensional case (see Hahn [16]) whereas the Markus-Yamabe problem has not.

PROBLEM 1. (Markus and Yamabe [33].) Assume that the Jacobian matrix J of the transformation (T) has, at each point of \mathbf{R}^2 , characteristic roots with negative real parts; that is, assume that

$$(I) \quad \operatorname{tr} J \equiv \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) < 0 \quad \text{on } \mathbf{R}^2,$$

and

$$(II) \quad \det J \equiv \left(\frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x} \right) > 0 \quad \text{on } \mathbf{R}^2.$$

Does it then follow that the solution $(x, y) = (0, 0)$ of the system (D) is globally asymptotically stable? That is, does every solution curve of (D) approach $(0, 0)$ as $t \rightarrow +\infty$?

An autonomous vector system $\dot{X} = T(X)$, for X in \mathbf{R}^n , is said to have an *equilibrium state* (or *critical point*) at $X = 0$ if $T(0) = 0$. Such an equilibrium state is said to be *stable* if to each $\varepsilon > 0$ there is a $\delta > 0$ such that $\|X^\circ\| < \delta$ implies $\|X(t, X^\circ)\| < \varepsilon$ for all $t \geq 0$. Here $X(t, X^\circ)$ denotes the solution with the initial condition $X(0, X^\circ) = X^\circ$. The equilibrium state $X = 0$ is said to be *asymptotically stable* if it is stable and if there is a positive δ_0 such that $\|X^\circ\| < \delta_0$ implies $X(t, X^\circ) \rightarrow 0$ as $t \rightarrow \infty$. The set of all X° for which $X(t, X^\circ) \rightarrow 0$ as $t \rightarrow \infty$ is called the *region of asymptotic stability*. When this region is all of \mathbf{R}^n the equilibrium state $X = 0$ is said to be *globally asymptotically stable*. All of the systems (D) discussed in this paper easily meet well-known conditions which ensure the local existence and uniqueness of solutions. See Coddington and Levinson

[12], Hartman [18], or Dieudonné [14]. Since asymptotic stability requires solutions to extend for all $t > 0$, and since, as is well-known, this can fail to happen for solutions of even very regular systems (e.g., consider the 2-dimensional system $\dot{x} = -x^2$, $\dot{y} = -y$), "continuation of solutions" becomes a concern. However, there are also well-known methods for dealing with this. See, e.g., §3 of Chapter II of Hartman [18] and also the treatment by Bebernes, Fulks, and Meisters [4]. Asymptotic stability is treated in extensive detail by Bellman [5], Cesari [11], Hahn [16], LaSalle and Lefschetz [30], and LaSalle [26–29].

Although Problem 1 is still open, some notable advances were made on it around 1961, primarily by Hartman and Olech who had been spurred by results of Krasovskii [23] and the interesting paper by Markus and Yamabe [33]. We will mention here only a few representative results which are neither the only ones known nor necessarily the sharpest ones known. In particular, the following results of Olech are of prime importance for our considerations.

THEOREM 1. (OLECH). *Problem 1 has an affirmative answer under any one of the following additional conditions.*

(III) *There are two positive constants ρ and r such that $P^2 + Q^2 \geq \rho^2$ whenever $x^2 + y^2 \leq r^2$.*

(IV) *The mapping (T) associated with (D) is (globally) one-to-one.*

(V) *At least one of the products $(\partial P/\partial x) \cdot (\partial Q/\partial y)$ or $(\partial P/\partial y) \cdot (\partial Q/\partial x)$ never vanishes on \mathbf{R}^2 .*

PROOF. See the paper by Olech [40] for the proof of (III), and for the reduction of (V) to (IV). For the convenience of the reader we will give Olech's reduction of (IV) to (III) as well as his proof, starting from (III), that Problem 1 is equivalent to the following problem, concerning the global invertibility of the associated transformation T .

PROBLEM 2. (OLECH [40]). Consider the transformation (T) where P and Q are real-valued class C^1 functions on \mathbf{R}^2 , and where we may as well assume (by translation of the (u, v) -coordinates) that $P(0, 0) = Q(0, 0) = 0$. Does it then follow from the two inequalities

$$(I) \quad \operatorname{tr} J < 0 \quad \text{on } \mathbf{R}^2,$$

and

$$(II) \quad \det J > 0 \quad \text{on } \mathbf{R}^2,$$

that the transformation (T) is globally one-to-one?

REDUCTION OF (IV) TO (III). Assume that the transformation (T) satisfies Conditions (II) and (IV). Condition (II) implies that T is a local homeomorphism (see any good advanced calculus book or Meisters and Olech

[35]). In particular, T maps some open (x, y) -neighborhood of the origin, $x^2 + y^2 < \varepsilon^2$, one-to-one and onto some open (u, v) -neighborhood of the origin which therefore contains an open disk, $u^2 + v^2 < \delta^2$, for some $\delta > 0$. But then, by (IV), no point outside $x^2 + y^2 < \varepsilon^2$ can be mapped inside $u^2 + v^2 < \delta^2$. That is, $P^2 + Q^2 \geq \delta^2$ whenever $x^2 + y^2 \geq \varepsilon^2$, which is Condition (III).

PROOF THAT PROBLEMS 1 AND 2 ARE EQUIVALENT. First suppose that Conditions (I) and (II) imply that T is one-to-one. That is, suppose that the answer to Problem 2 is “yes”. Then, as just seen above, T also satisfies Condition (III), so that the origin is globally asymptotically stable for the system (D) by virtue of Olech’s Theorem 1 (III). That is, the answer to Problem 1 is also “yes”.

Conversely, assume that the answer to Problem 1 is “yes”. That is, assume that the origin is globally asymptotically stable for any system (D) which satisfies Conditions (I) and (II). Suppose further that T satisfies Conditions (I) and (II), but that it is not one-to-one. Then there exist points X_1 and X_2 in \mathbf{R}^2 such that $T(X_1) = T(X_2) = U_0$. This means that the system $\dot{X} = H(X)$, where $H(X) \equiv T(X + X_1) - U_0$, has two distinct critical points, one at the origin and one at $X_3 = X_2 - X_1 \neq 0$. But then, contrary to hypothesis, the origin can not be globally asymptotically stable for the system $\dot{X} = H(X)$ even though H satisfies Conditions (I) and (II) along with T .

Olech’s Condition (V) of Theorem 1 contains the following two interesting special cases which had been established earlier by Markus and Yamabe and by Hartman.

(Va) Markus and Yamabe [33]. One of the four functions P_x, P_y, Q_x, Q_y is identically zero on \mathbf{R}^2 . This is easily seen to imply (V) in the presence of (II).

(Vb) Hartman [17]. The symmetric part of J , i.e., $(J + J^*)/2$, is negative definite on \mathbf{R}^2 .

Now $J + J^*$ is negative definite if and only if $\text{tr}(J + J^*) < 0$ and $\det(J + J^*) > 0$. But $\text{tr}(J + J^*) = 2 \text{tr} J$, and $\det(J + J^*) = 4P_xQ_y - (Q_x + P_y)^2 = 4 \det J - (Q_x - P_y)^2$, so that (Vb) implies (I), (II) and (V).

Hartman [17] and Hartman and Olech [20] dealt primarily with the n -dimensional version of Problem 1 and with certain of its variations including a simplification and extension of an earlier result of Borg [6]. All of these and related results known by 1964 are described in Part III of Chapter XIV of Hartman [18]. Olech [41] returned to the 2-dimensional case but dealt with deeper generalizations of his earlier results on a variation of Problem 1 and he obtained nothing new for Problem 1 itself. Hartman [19] also contains no new results for Problem 1. The same can be said of

the otherwise interesting papers by Datko [13] and McCann [34]. The author's 1963 paper with Olech [35] was motivated in part by Problem 2, but it has produced no new results for Problems 1 and 2. Minty's Theorem, as described for example by Browder [8], is an interesting condition for global invertibility of nonlinear maps (somewhat analogous to Hartman's Condition (Vb) above), but it too seems useless for our two Problems 1 and 2.

3. Jacobian problems in algebraic geometry. The following conjecture is a famous unsolved problem of algebraic geometry.

THE JACOBIAN CONJECTURE OF ALGEBRAIC GEOMETRY. Assume that k is a field of characteristic zero and that n is a positive integer. If f_1, \dots, f_n are polynomials over k (i.e., members of the ring $k[x_1, \dots, x_n]$) such that the Jacobian matrix $(\partial f_i / \partial x_j)$ has a nonzero constant determinant, then it is conjectured that $k[f_1, \dots, f_n] = k[x_1, \dots, x_n]$. That is, it is conjectured that there are polynomials g_1, \dots, g_n in $k[y_1, \dots, y_n]$ such that $x_i = g_i(f_1, \dots, f_n)$ for $i = 1, \dots, n$; or, in other words, that the polynomial map $f = (f_1, \dots, f_n)$ from k_x^n to k_y^n has a polynomial inverse.

Contributions to this conjecture can be found in Keller [22], Engel [15], Vitushkin [46], Campbell [9], Abhyankar [1], Lane [25], Nagata [38], Razar [42], Wang [47], Wright [49], and Moh [36]. The question was evidently first raised by Keller [22] for $k = \mathbf{C}$. Engel [15] announced an affirmative solution when $n = 2$ and $k = \mathbf{C}$, but Vitushkin [46] pointed out two essential mistakes in Engel's argument which completely invalidate his proof. Campbell [9], by using the theory of complex manifolds, proved that the answer is affirmative when $k = \mathbf{C}$ if (and obviously only if) the field extension $\mathbf{C}(f_1, \dots, f_n) \subset \mathbf{C}(x_1, \dots, x_n)$ is a normal extension. Wright [49] gave a purely algebraic proof of Campbell's theorem, but now valid for any field k of characteristic zero. An appendix to Wright's paper contains a reduction (but by increasing dimension) to the case of polynomials of degree ≤ 3 , and also another reduction (due to E.H. Connell) to the case where the Jacobian matrix is of the form $I + N$ with N nilpotent. Wang [47] showed that the conjecture holds for $f = (f_1, \dots, f_n)$ provided that the degree of each f_i is ≤ 2 . Abhyankar [1] discusses the 2-dimensional case ($n = 2$) including the example $f_1 = x_1 + (x_1)^p, f_2 = x_2$, which shows that the corresponding conjecture for positive (prime) characteristic p is trivially false. Further partial results for $n = 2$ (and mostly for $k = \mathbf{C}$) are contained in Vitushkin [45], Abhyankar and Moh [2], Wright [48], Wang [47], and Moh [36].

In this paper we are concerned only with the special case $n = 2$ and $k = \mathbf{R}$ which we restate as follows.

PROBLEM 3. Consider the transformation

$$(T) \quad \begin{aligned} u &= P(x, y) \\ v &= Q(x, y) \end{aligned}$$

of the real plane \mathbf{R}^2 into itself where P and Q are polynomials in x and y with real coefficients. Under what conditions will T be one-to-one and onto with an inverse T^{-1} whose components are polynomials in u and v ? The condition

$$(VI) \quad \det J = \text{nonzero constant}$$

is obviously necessary because it is the only way that the product of the two polynomials $\det J$ and $\det J^{-1}$ can be identically equal to 1, as it must be since

$$(\det J) \cdot (\det J^{-1}) = \det(J \cdot J^{-1}) = \det I.$$

It has been conjectured (by Keller and others) that the condition (VI) is also sufficient.

Recall how 2-dimensional Lebesgue measure transforms under mappings T . If Ω is a region in the (x, y) -plane, then the measure of its image $T(\Omega)$ in the (u, v) -plane is given by the formula

$$\iint_{T(\Omega)} du dv = \iint_{\Omega} \left(\frac{\partial P}{\partial x} \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial y} \frac{\partial Q}{\partial x} \right) dx dy = \iint_{\Omega} (\det J) dx dy.$$

Therefore, transformations T satisfying condition (VI) are, up to a constant factor, "area-preserving".

4. Some transformations of the plane. We now give some explicit examples of transformations T of \mathbf{R}^2 into itself.

EXAMPLE 1. It may at first seem impossible that there can be nonlinear polynomial maps T with polynomial inverses. Presently (in §5) we shall describe all such maps in complete detail and it will then be seen that there is a large group \mathcal{P} of such maps including many complicated polynomials of every degree. In the meantime, here is a simple example for each degree $d \geq 2$.

$$(T_1) \quad \begin{aligned} u &= (x - y)^d + y - 2x \\ v &= (x - y)^d - x \\ x &= (v - u)^d - v, \\ (T_1^{-1}) \quad y &= (v - u)^d + u - 2v. \end{aligned}$$

$$J = \begin{bmatrix} d(x - y)^{d-1} - 2 & -d(x - y)^{d-1} + 1 \\ d(x - y)^{d-1} - 1 & -d(x - y)^{d-1} \end{bmatrix}$$

$\det J = +1$ and $\text{tr } J = -2$. This transformation T_1 is a one-to-one area-

preserving nonlinear polynomial map of \mathbf{R}^2 onto itself with a polynomial inverse.

As we have seen in §3, every polynomial map with a polynomial inverse must be area-preserving (up to a constant factor). The open question is: Must every area-preserving polynomial map have a polynomial inverse? In particular, must every area-preserving polynomial map be one-to-one?

EXAMPLE 2. In the complex case (i.e., with \mathbf{R} replaced by \mathbf{C} in Problem 3 so that T maps \mathbf{C}^2 into \mathbf{C}^2 and the polynomials P and Q can have complex coefficients) it suffices to require that $\det J$ never vanishes. For then, since \mathbf{C} is an algebraically closed field and $\det J$ is a polynomial, it automatically follows that $\det J$ is a nonzero constant. But in the real case the nonvanishing of $\det J$ is not enough to ensure that T should have a polynomial inverse, even under the additional hypothesis that T be a homeomorphism of \mathbf{R}^2 onto itself. Consider for example the transformation

$$(T_2) \quad \begin{aligned} u &= x^3 + x, \\ v &= y. \end{aligned}$$

This transformation T_2 is a one-to-one polynomial map of \mathbf{R}^2 onto itself; but, while $\det J$ is nonvanishing (in fact ≥ 1), T_2 cannot have a polynomial inverse since $\det J \neq \text{constant}$. Indeed,

$$(T_2^{-1}) \quad \begin{aligned} x &= \sqrt[3]{(u/2) + \sqrt{(u/2)^2 + (1/3)^3}} + \sqrt[3]{(u/2) - \sqrt{(u/2)^2 + (1/3)^3}}, \\ y &= v. \end{aligned}$$

EXAMPLE 3. On the other hand, the mere nonvanishing of $\det J$, even for all complex values of the variables, does not ensure that every analytic transformation T is one-to-one. The transformation

$$(T_3) \quad \begin{aligned} u &= e^x \cos y \\ v &= e^x \sin y \end{aligned}$$

has $\det J = e^{2x}$ which is positive for all real x (and never zero even for complex x), but, nevertheless is obviously not globally one-to-one since it is periodic in y . The nonvanishing of $\det J$ does always ensure, of course, that the transformation T will be locally one-to-one: that is, the restriction of T to some neighborhood, of any point (x_0, y_0) at which $\det J \neq 0$, will be one-to-one. See Meisters and Olech [35] for a discussion of locally one-to-one maps and some theorems which tell when such maps must be globally one-to-one. Does there exist a polynomial map T with nonvanishing (but not necessarily constant) Jacobian determinant which nevertheless fails to be one-to-one (or onto)?

EXAMPLE 4. A continuous and area-preserving rational map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ need not be one-to-one even if $\det J > 0$ at all points of \mathbb{R}^2 .

$$(T_4) \quad \begin{aligned} u &= (10x^3y^2 - x^5 - 5xy^4)/\sqrt{5}(x^2 + y^2)^2, \\ v &= (10x^2y^3 - y^5 - 5x^4y)/\sqrt{5}(x^2 + y^2)^2. \end{aligned}$$

If we assume that T_4 maps $(0, 0)$ to $(0, 0)$, then T_4 is a continuous map of \mathbb{R}^2 onto itself and has first-order partials everywhere on \mathbb{R}^2 . However, its first-order partials are not continuous at the origin. Moreover, $\det J = +1$ everywhere except at the origin where $\det J = 1/5$. Since T_4 maps both $(1, 0)$ and $(\cos 2\pi/5, \sin 2\pi/5)$ to $(-1/\sqrt{5}, 0)$, this map is not one-to-one. For $(x, y) = (0, 0)$,

$$J = \begin{bmatrix} -1/\sqrt{5} & 0 \\ 0 & -1/\sqrt{5} \end{bmatrix},$$

while for $(x, y) = (r \cos \theta, r \sin \theta) \neq (0, 0)$, $J = -(1/\sqrt{5}) A$ where A is the 2×2 matrix

$$\begin{bmatrix} \cos \theta \cos 5\theta + 5 \sin \theta \sin 5\theta & \sin \theta \cos 5\theta - 5 \cos \theta \sin 5\theta \\ \cos \theta \sin 5\theta - 5 \sin \theta \cos 5\theta & \sin \theta \sin 5\theta + 5 \cos \theta \cos 5\theta \end{bmatrix}.$$

Note that for $(x, y) = (0, 0)$, $\text{tr } J = -2/\sqrt{5}$, while for $(x, y) \neq (0, 0)$, $\text{tr } J = -6(\cos 4\theta)/\sqrt{5}$. Does there exist a class C^1 , area-preserving, rational map $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which fails to be one-to-one? Even with $\text{tr } J < 0$ at all points of \mathbb{R}^2 ?

EXAMPLE 5. Let $C_{\mathbb{R}}^1[0, \infty)$ denote the set of all real-valued functions ϕ defined and continuous on $[0, \infty]$ with continuous first derivatives on $(0, \infty)$ such that $\lim_{r \rightarrow 0^+} r\phi'(r) = 0$. This function space is obviously a real vector space and closed under products as well. For each ϕ in $C_{\mathbb{R}}^1[0, \infty)$ let R_ϕ denote the transformation of \mathbb{R}^2 into itself defined by the equations

$$(R_\phi) \quad \begin{aligned} u &= x \cos \phi(r) - y \sin \phi(r) \\ v &= x \sin \phi(r) + y \cos \phi(r) \end{aligned}$$

where $r = (x^2 + y^2)^{1/2}$. Note that $R_0 = I$, the identity map, and that $R_\phi^{-1} = R_{-\phi}$. Furthermore, since $u^2 + v^2 = x^2 + y^2 = r^2$, it follows that $R_\phi \circ R_\psi = R_{\phi+\psi}$ so that $R_\phi \circ R_\psi = R_\psi \circ R_\phi$. Thus the set $\mathcal{R} = \{R_\phi: \phi \in C_{\mathbb{R}}^1[0, \infty)\}$ forms a commutative group under composition of maps. This group \mathcal{R} contains the group of all "rigid rotations" of the plane (i.e., R_ϕ with $\phi \equiv \text{constant}$). In fact, each R_ϕ in \mathcal{R} rigidly rotates each circle centered at the origin, but it may rotate circles of different radii r through different angles

$\phi(r)$. We will call this group \mathcal{R} the group of "nonlinear rotations" of \mathbf{R}^2 . The Jacobian of R_ϕ for $r \neq 0$ is $J_\phi =$

$$\begin{bmatrix} \cos \phi(r) - x\phi'(r) \sin(\theta + \phi(r)) & -\sin \phi(r) - y\phi'(r) \sin(\theta + \phi(r)) \\ \sin \phi(r) + x\phi'(r) \cos(\theta + \phi(r)) & \cos \phi(r) + y\phi'(r) \cos(\theta + \phi(r)) \end{bmatrix}$$

where, as usual, $x = r \cos \theta$ and $y = r \sin \theta$. Thus for $r \neq 0$,

$$\det J_\phi = +1 \quad \text{and} \quad \text{tr } J_\phi = 2 \cos \phi(r) - r\phi'(r) \sin \phi(r).$$

At $(x, y) = (0, 0)$ the Jacobian of R_ϕ is

$$J_\phi = \begin{bmatrix} \cos \phi(0) & -\sin \phi(0) \\ \sin \phi(0) & \cos \phi(0) \end{bmatrix}$$

and the condition $\lim_{r \rightarrow 0^+} r\phi'(r) = 0$ is seen to ensure that all four first-order partials are continuous at the origin. Thus each transformation R_ϕ in \mathcal{R} is class C^1 , area-preserving, and maps \mathbf{R}^2 one-to-one onto itself. None of these maps can be a nonlinear polynomial map. For if R_ϕ were a polynomial map, then there would be two real polynomials $P(x, y)$ and $Q(x, y)$ such that $\cos \phi(r) = (xP + yQ)/r^2$ and $\sin \phi(r) = (xQ - yP)/r^2$. But then $\sin^2 \phi + \cos^2 \phi = 1$ would imply that $P^2 + Q^2 = x^2 + y^2$ which by consideration of degree, would force both P and Q to be linear (i.e., $\phi(r) \equiv \text{constant}$).

However, because of the elementary identities

$$\sin \phi = \frac{2 \tan(\phi/2)}{1 + \tan^2(\phi/2)} \quad \text{and} \quad \cos \phi = \frac{1 - \tan^2(\phi/2)}{1 + \tan^2(\phi/2)}$$

it is possible to choose $\phi(r)$ in many ways to make R_ϕ a rational map. For example, if $f(s)$ is any rational function of s which is defined for all real $s \geq 0$, then one may choose

$$\phi(r) = 2 \arctan f(r^2)$$

to obtain the rational functions

$$\sin \phi(r) = \frac{2f(r^2)}{1 + f^2(r^2)} \quad \text{and} \quad \cos \phi(r) = \frac{1 - f^2(r^2)}{1 + f^2(r^2)}$$

so that R_ϕ will be a rational map. Thus there are many class C^1 , area-preserving, one-to-one, non-polynomial, rational maps of \mathbf{R}^2 onto itself. Does every class C^1 , area-preserving, one-to-one, rational map have a rational inverse?

5. The groups \mathcal{P} and \mathcal{S} . In this section we first solve the Jacobian Conjecture as formulated in Problem 3 (in §3) for the special case of quadratic

polynomials. This reveals the explicit form that polynomial maps with polynomial inverses must have and leads to a complete and detailed description (Theorem of van der Kulk—Jung) of the group \mathcal{P} of all such (1-1) polynomial maps of \mathbf{R}^2 onto itself. It is then easily seen that \mathcal{P} is a subgroup of a much larger group \mathcal{S} of area-preserving diffeomorphisms of \mathbf{R}^2 . These groups \mathcal{P} and \mathcal{S} can be combined with the group \mathcal{Q} of Example 5 to form even larger groups of area-preserving diffeomorphisms of the plane. We wonder if the resulting groups are not nearly all of (or at least dense in) the group \mathcal{A} of all area-preserving diffeomorphisms of \mathbf{R}^2 .

THEOREM 2. *Consider the transformation*

$$(T) \quad \begin{aligned} u &= P(x, y) \\ v &= Q(x, y) \end{aligned}$$

where P and Q are polynomials of degree at most 2 (but not both linear) in x and y over the field \mathbf{R} or \mathbf{C} (or, for that matter, over any real closed field k or any algebraically closed field k of characteristic zero). Then in order that

$$(VI) \quad \det J = \text{nonzero constant } \delta$$

it is both necessary and sufficient that

$$(i) \quad \begin{aligned} P(x, y) &= \mu\lambda(\alpha y - \beta x)^2 + ax + by \\ Q(x, y) &= \mu\lambda'(\alpha y - \beta x)^2 + a'x + b'y \end{aligned}$$

where $\alpha, \beta,$ and μ are any constants (in k) satisfying $(\alpha, \beta) \neq (0, 0)$ and $\mu \neq 0$; $a, b, a',$ and b' are any constants satisfying

$$(ii) \quad ab' - a'b = \delta \neq 0;$$

and

$$(iii) \quad \lambda = \alpha a + \beta b \text{ and } \lambda' = \alpha a' + \beta b'.$$

Furthermore, if (VI) holds, then the transformation T maps \mathbf{R}^2 (or \mathbf{C}^2 or k^2) one-to-one onto itself and the inverse transformation is also a (quadratic) polynomial map.

PROOF. (Sufficiency) If the conditions (i), (ii), and (iii) hold, then $\det J = ab' - ba' \neq 0$, as can be seen by a direct calculation of the determinant of the Jacobian matrix

$$J = \begin{bmatrix} (-2\beta\mu\lambda(\alpha y - \beta x) + a) & (2\alpha\mu\lambda(\alpha y - \beta x) + b) \\ (-2\beta\mu\lambda'(\alpha y - \beta x) + a') & (2\alpha\mu\lambda'(\alpha y - \beta x) + b') \end{bmatrix}$$

(Necessity) On the other hand, if $P(x, y) = Ax^2 + Bxy + Cy^2 +$

$ax + by$, and $Q(x, y) = A'x^2 + B'xy + C'y^2 + a'x + b'y$, then (VI) leads to the following six equations:

$$\begin{aligned} (1) \quad AB' - A'B &= 0, & (4) \quad 2(Ab' - A'b) &= a'B - aB', \\ (2) \quad AC' - A'C &= 0, & (5) \quad 2(aC' - a'C) &= (B'b - Bb'), \\ (3) \quad BC' - B'C &= 0, & (6) \quad (ab' - a'b) &= \delta \neq 0. \end{aligned}$$

The system of equations (1), (2), (3) is equivalent to the pair of vector equations

$$(7) \quad \begin{aligned} (A, B, C) &= \lambda(X, Y, Z) \\ (A', B', C') &= \lambda'(X, Y, Z), \end{aligned}$$

for some constants X, Y, Z, λ and λ' , in terms of which equations (4) and (5) become

$$\begin{aligned} (4') \quad 2X(\lambda b' - \lambda'b) + Y(a\lambda' - \lambda a') &= 0, \\ (5') \quad Y(\lambda b' - \lambda'b) + 2Z(a\lambda' - \lambda a') &= 0. \end{aligned}$$

Therefore, either

$$(8) \quad \begin{vmatrix} 2X & Y \\ Y & 2Z \end{vmatrix} = 4XZ - Y^2 = 0,$$

or

$$(9) \quad \lambda b' - \lambda'b = 0 \quad \text{and} \quad \lambda a' - \lambda'a = 0.$$

But (9) and (6) imply that $\lambda = \lambda' = 0$ which means, by (7), that $A = B = C = 0$ so that P and Q are both linear polynomials contrary to hypothesis. Therefore (9) is false and (8) is true, so that it is possible to choose numbers α and β , not both zero, so that

$$(10) \quad (X, Y, Z) = \pm (\beta^2, -2\alpha\beta, \alpha^2).$$

But then the condition (8) is automatically satisfied, and one obtains

$$(11) \quad \begin{aligned} P(x, y) &= \lambda(\alpha y - \beta x)^2 + ax + by \\ Q(x, y) &= \lambda'(\alpha y - \beta x)^2 + a'x + b'y. \end{aligned}$$

Furthermore, equations (4') and (5') become

$$(4'') \quad \beta^2(\lambda b' - \lambda'b) = -\alpha\beta(\lambda a' - \lambda'a),$$

and

$$(5'') \quad -\alpha\beta(\lambda b' - \lambda'b) = \alpha^2(\lambda a' - \lambda'a),$$

which are equivalent to the linear system

$$(12) \quad \begin{aligned} \lambda b' - \lambda' b &= -\mu\alpha \\ \lambda a' - \lambda' a &= +\mu\beta \end{aligned}$$

for some number μ . But then (6) applies again to imply that (12) has unique solutions for λ and λ' in terms of the parameters $\mu, \alpha, \beta, a, b, a', b'$ as follows:

$$(13) \quad \lambda = -\mu(\alpha a + \beta b)/\delta \quad \text{and} \quad \lambda' = -\mu(\alpha a' + \beta b')/\delta.$$

By substituting these values for λ and λ' into (11) and by renaming $\lambda, \lambda', \mu,$ and μ' , one obtains (i), (ii), and (iii) as desired. This completes the proof of Theorem 2.

Wang [47] has also proved this quadratic theorem, but more generally for every dimension $n \geq 2$.

We are now in a position to describe the group \mathcal{P} of all (1-1) polynomial maps of \mathbf{R}^2 onto itself which have polynomial inverses, and the associated larger group \mathcal{G} . Let $C^1(\mathbf{R})$ denote the real vector space of all real-valued functions f which are defined and continuous with continuous first derivatives f' on the entire real line \mathbf{R} . For each matrix L in the general linear group $GL_2(\mathbf{R})$ let L^* denote the transpose of L . Vectors V in \mathbf{R}^2 will be column vectors so that LV denotes the matrix product of L and V . For each vector $V = (v_1, v_2)^*$ in \mathbf{R}^2 let V^\perp (read “ V perp”) denote the vector $(-v_2, v_1)^*$ which is perpendicular to V with respect to the usual Euclidean inner-product of \mathbf{R}^2 , $V \cdot U = V^*U$. It will be useful to note that L^{-1} can be expressed in terms of transpose and “perp” as follows.

$$(14) \quad L^{-1}V = -(\det L)^{-1}(L^*(V^\perp))^\perp.$$

In terms of this vector and matrix notation the polynomial transformation T defined by (i) of Theorem 1 above can be written as follows.

$$(15) \quad U = T(X) = LX + f(X \cdot V^\perp)LV,$$

where $f(s) \equiv \mu s^2$, $U = (u, v)^*$, $X = (x, y)^*$, $V = (\alpha, \beta)^*$, and L is the nonsingular matrix.

$$L = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \quad \text{with } \det L = \delta \neq 0.$$

Thus T defined by (i) of Theorem 1 is the composition, $T = L \circ S(f, V)$, of the non-singular linear map L with a special (nonlinear) map $S(f, v)$ defined by

$$(16) \quad U = S(f, V)X = X + f(X \cdot V^\perp)V, \quad f \in C^1(\mathbf{R}).$$

This $S(f, V)$ maps \mathbf{R}^2 one-to-one onto itself with inverse $S(f, V)^{-1}$ given by

$$(17) \quad X = S(f, V)^{-1} U = U - f(U \cdot V^\perp)V.$$

Formula (17) is easily checked by noting that $U \cdot V^\perp = X \cdot V^\perp$ whenever U and X are related by equations (16) or (17). Thus the inverse of $S(f, V)$ has the same form as $S(f, V)$ itself. That is,

$$(18) \quad S(f, V)^{-1} = S(-f, V).$$

Also note that for any pair f, g in $C^1(\mathbf{R})$,

$$(19) \quad S(f, V) \circ S(g, V) = S(f + g, V).$$

However, if $V_1 \neq V_2$, then $S(f, V_2) \circ S(g, V_1)$ is not of the form $S(h, V_3)$ for any h in $C^1(\mathbf{R})$ and V_3 in \mathbf{R}^2 , even if $f = g$. Rather, there is the more complicated formula

$$(20) \quad \begin{aligned} S(f, V_2) \circ S(g, V_1) X &= X + g(X \cdot V_1^\perp) V_1 \\ &\quad + f(X \cdot V_1^\perp + g(X \cdot V_1^\perp) V_1 \cdot V_2^\perp) V_2. \end{aligned}$$

The Jacobian matrix of $S(f, V)$, denoted $J(f, V)$, or $S'(f, V)$ since it is the (Fréchet) derivative of $S(f, V)$, can also be expressed in vector-matrix notation:

$$(21) \quad S'(f, V) = I + f'(X \cdot V^\perp) V (V^\perp)^*,$$

where $V(V^\perp)^*$ is the dyadic matrix product

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot [-\beta, \alpha] = \begin{bmatrix} -\alpha\beta & \alpha^2 \\ -\beta^2 & \alpha\beta \end{bmatrix}.$$

Consequently, for all f in $C^1(\mathbf{R})$ and for all V in \mathbf{R}^2 ,

$$(22) \quad \det S'(f, V) = 1 \quad \text{and} \quad \text{tr } S'(f, V) = 2.$$

In particular, each special map $S(f, V)$ is area-preserving.

Note that in equation (16), which defined $S(f, V)$, we may assume that V is a unit vector by replacing f by $g(s) = \|V\|f(\|V\|s)$. When this is done, then f and V are unique in (16).

Finally, we note that each of these special (usually nonlinear) maps $S(f, V)$ has a simple geometric interpretation: if $f \equiv 0$, then $S(f, V)$ is the identity map. If $f \equiv c$ (a nonzero constant), then $S(f, V)$ is translation by cV . If $f(s) \equiv cs$, then $S(f, V)$ is the linear transformation, $I + cV(V^\perp)^*$, identity-plus-dyad. But in any case (i.e., for each f in $C^1(\mathbf{R})$ and for any unit vector V in \mathbf{R}^2), the map $S(f, V)$ translates any two points (e.g., the vectors X and X' in Figure 1) which lie on the same line parallel to V by the same amount, namely $f(X \cdot V^\perp)\|V\|$, in the direction of the vector V . That is, each straight line ℓ parallel to V is rigidly slid or shifted along itself (in the direction of V) by the amount $f(d)$, where d is the perpendicular distance of this line ℓ from the origin.

Therefore we will call $S(f, V)$, defined by formula (16) with f in $C^1(\mathbf{R})$

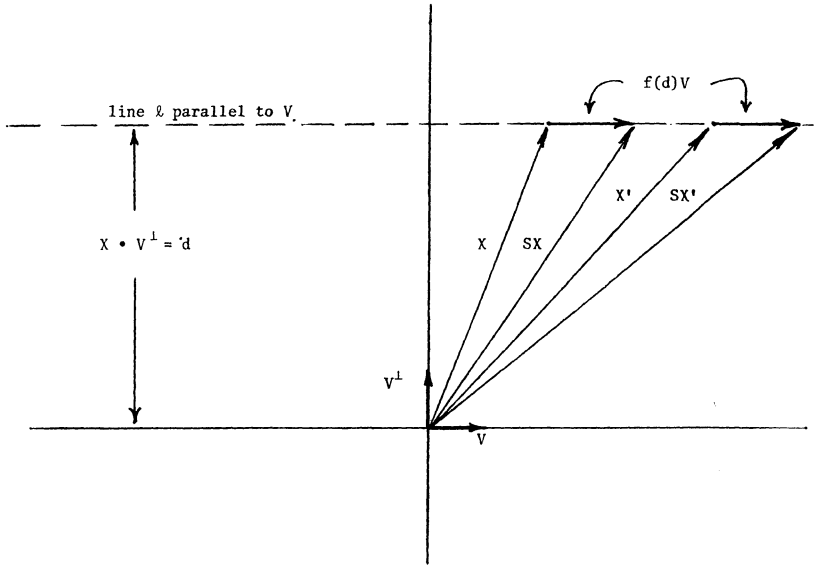


Fig. 1 Geometric Interpretation of $S(f, V)$

and unit vector V in \mathbf{R}^2 , a *shift*. Note that unless $f(s) \equiv \text{constant}$, different lines l parallel to V get shifted by different amounts; and that unless $f(s) \equiv cs$, $S(f, V)$ is a nonlinear transformation of the plane.

LEMMA 1. *The conjugates $\tilde{S}(f, V) \equiv LS(f, V)L^{-1}$ of a shift $S(f, V)$ by a non-singular matrix L is another shift. More precisely,*

$$(23) \quad \tilde{S}(f, V) \equiv LS(f, V)L^{-1} = S(\tilde{f}, \tilde{V}),$$

where $\tilde{V} = LV/\|LV\|$, and $\tilde{f}(s) = \|LV\| \cdot f(s/\|LV\| \cdot \det L^{-1})$.

PROOF. Apply the identity (14) to $(L^{-1})^*V^\perp$ in $\tilde{S}(f, V)$ and compare the result with formula (16) which defines $S(f, V)$. Remember that we are now requiring that $\|V\| = \|\tilde{V}\| = 1$.

DEFINITION OF THE GROUPS \mathcal{S} , \mathcal{G} AND \mathcal{P} . Let \mathcal{G} denote the set of all (mostly nonlinear) transformations T of \mathbf{R}^2 into itself which can be written in the form $T = L \circ S_1 \circ S_2 \circ \dots \circ S_m$, where m ranges over the positive integers, L ranges over the group $GL_2(\mathbf{R})$, and each S_i denotes a shift $S(f_i, V_i)$ defined by formula (16) with f_i in $C^1(\mathbf{R})$ and unit vector V_i in \mathbf{R}^2 . Let \mathcal{S} denote the set of all products (i.e., compositions) $S_1 \circ S_2 \circ \dots \circ S_m$ of shifts only. Let \mathcal{P} denote those elements of \mathcal{G} in which each f_i (in each factor S_i) is a polynomial in one variable. Finally let \mathcal{SP} denote the set theoretic intersection of \mathcal{S} and \mathcal{P} . Then it follows easily from Lemma 1

that \mathcal{G} , \mathcal{S} , \mathcal{P} , and \mathcal{SP} are groups under composition of maps and that \mathcal{S} is a normal subgroup of \mathcal{G} while \mathcal{SP} is a normal subgroup of \mathcal{P} .

THEOREM 3. (VAN DER KULK-JUNG). *The group \mathcal{P} defined above is identical to the group of all (1-1) polynomial maps of \mathbf{R}^2 onto itself which have polynomial inverses. Equivalently, \mathcal{P} is the automorphism group of the ring of polynomials $\mathbf{R}[x, y]$.*

This theorem was evidently first proved by Jung [21]. See also van der Kulk [24], Nagata [37, 38], and Abhyankar [1].

The Jacobian Problem 3 stated in §3 (i.e., the 2-dimensional Jacobian conjecture of algebraic geometry) can now be restated as follows (for $k = \mathbf{R}$): Is it true that every area-preserving polynomial map T of the plane \mathbf{R}^2 into itself must be a member of the group \mathcal{P} ? We have shown (Theorem 1) that the answer is "yes" at least for quadratic maps. As mentioned before, Wright [49] has shown that the general case (i.e., for dimensions $n \geq 2$ and arbitrary degree) can be reduced to the case of degree ≤ 3 (but for increased dimension). Recently T.-T. Moh [36] has shown with the help of a computer that the answer is "yes" in the 2-dimensional case provided that the degrees of P and Q do not exceed 100. So, if the Jacobian conjecture is false, there is a third-degree counter-example in some large dimension and there is also a 2-dimensional counter-example involving polynomials of some degree greater than 100.

Even a simple composition $L \circ S_2 \circ S_1$ of three factors can produce a fairly complicated polynomial transformation. Since the computation of such a harmless looking composition can be rather lengthy, we exhibit one here as an illustration for the convenience of the reader.

The polynomial transformation $U = TX$ given by the equations

$$\begin{aligned} u &= x - y - 3x^2 - 12y^2 - 12xy + 54x^3 - 2y^3 + 18xy^2 - 54x^2y \\ &\quad + 1134x^4 - 84y^4 + 714xy^3 - 1890x^2y^2 + 1134x^3y - 107,163x^6 \\ &\quad - 147y^6 + 2,646xy^5 - 19,845x^2y^2 + 79,380x^3y^3 - 178,605x^4y^2 \\ &\quad + 214,326x^5y, \\ v &= x + y - x^2 - 4y^2 - 4xy - 180x^3 - 36xy^2 + 108x^2y + 4y^3 \\ &\quad - 28y^4 + 378x^4 + 238xy^3 - 630x^2y^2 + 378x^3y - 35,721x^6 \\ &\quad - 49y^6 + 902xy^5 - 6,615x^2y^4 + 26,460x^3y^3 - 59,535x^4y^2 \\ &\quad + 71,442x^5y, \end{aligned}$$

has the form $T = L \circ S_2 \circ S_1$ where

$$L = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$

$f_2(s) \equiv s^2$, and $f_1(s) \equiv s^3$. This transformation appears somewhat simpler in factored form. (We give its polynomial inverse only in factored form.)

$$\begin{aligned}
 (T) \quad & u = x - y - 2(y - 3x)^3 - 3(x + 2y + 7(x - 3x)^3)^2, \\
 & v = x + y + 4(y - 3x)^3 - (x + 2y + 7(y - 3x)^3)^2. \\
 & x = \frac{1}{2}u + \frac{1}{2}v + 2\left(\frac{u - 3v}{2}\right)^2 + \left(2u + v + 7\left(\frac{u - 3v}{2}\right)^2\right)^3. \\
 (T^{-1}) \quad & y = -\frac{1}{2}u + \frac{1}{2}v - \left(\frac{u - 3v}{2}\right)^2 + 3\left(2u + v + 7\left(\frac{u - 3v}{2}\right)^2\right)^3.
 \end{aligned}$$

6. Connections with differential equations. We can now combine the results of §2, §4, and §5 to construct a large family \mathcal{D} of examples of 2-dimensional autonomous systems

$$\begin{aligned}
 (D) \quad & \dot{x} = P(x, y) & (\dot{x} = dx/dt) \\
 & \dot{y} \equiv Q(x, y)
 \end{aligned}$$

of nonlinear differential equations with critical points at the origin (i.e., with $P(0, 0) = Q(0, 0) = 0$) whose stability and extent of asymptotic stability can be explicitly analyzed by means of the methods of Liapunov, LsSalle, Olech, and others. We shall restrict ourselves to just a few of the possibilities to illustrate our point.

THEOREM 4. *For each positive integer m and each matrix L in $GL_2(\mathbf{R})$ with $\det L > 0$ choose*

- (i) *functions f_1, \dots, f_m in $C^1(\mathbf{R})$, each zero at $s = 0$, and*
- (ii) *unit vectors V_1, \dots, V_m in \mathbf{R}^2 .*

Then the 2-dimensional autonomous system

$$(24) \quad \dot{X} = L \circ S(f_1, V_1) \circ \dots \circ S(f_m, V_m) X$$

has the origin as a globally asymptotically stable critical point provided

$$\text{trace } J(L \circ S_1 \circ \dots \circ S_m) < 0$$

throughout \mathbf{R}^2 .

PROOF. As we have seen in §5, the transformation of \mathbf{R}^2 into itself defined by $T = L \circ S_1 \circ \dots \circ S_m$ is globally one-to-one with $\det J = \det L > 0$, so that the result follows from Olech's Condition (IV) of Theorem 1, stated in §2.

Notice that one can greatly increase the number of examples given in Theorem 4 by including as factors in T an arbitrary number of the "non-linear rotations" R_ϕ described in Example 5 of §4. But instead we will limit ourselves to the following two special cases.

SPECIAL CASE 1. The simplest case of Theorem 4 is the system

$$(25) \quad \dot{X} = -X + f(X \cdot V^\perp)V, f(0) = 0,$$

for which $\det J = +1$ and $\text{tr } J = -2$. Actually in this simple (but non-linear) case one can explicitly solve the differential equation as follows. By treating the differential equation like a linear first-order equation one obtains the equivalent (nonlinear) vector integral equation

$$X(t) = e^{-t}X(0) + \left(e^{-t} \int_0^t e^{sf}(X(s) \cdot V^\perp) ds \right) V.$$

But then $X(t) \cdot V^\perp = e^{-t}X(0) \cdot V^\perp$, so that the solution $X(t)$ can be written explicitly in terms of its initial condition $X(0)$ as follows.

$$X(t) = e^{-t}X(0) + \left(e^{-t} \int_0^t e^{sf}(e^{-s}X(0) \cdot V^\perp) ds \right) V.$$

In case $X(0) \cdot V^\perp = 0$, then (since $f(0) = 0$) one has $X(t) = e^{-t}X(0)$ for all t ; so in this case it is easily seen that $X(t) \rightarrow 0$ as $t \rightarrow +\infty$. If, on the other hand, $X(0) \cdot V^\perp = c \neq 0$, then one can integrate by parts to obtain

$$e^{-t} \int_0^t e^{sf}(ce^{-s}) ds = f(ce^{-t}) - f(c)e^{-t} + ce^{-t} \int_0^t f'(ce^{-s}) ds.$$

But continuity of f' implies that $f'(ce^{-s})$ is bounded on the interval $0 \leq s < +\infty$. Thus (again since $f(0) = 0$) it follows that $X(t) \rightarrow 0$ as $t \rightarrow +\infty$ in this case too.

SPECIAL CASE 2. The next simplest cases of Theorem 4 are the differential equations

$$(26) \quad \dot{X} = LX + f(X \cdot V^\perp)LV = L \circ S(f, V)X,$$

where $L \in \text{GL}_2(\mathbf{R})$ with $\det L > 0$. Then

$$\text{tr } J(L \circ S) \equiv (\text{tr } L) + f'(X \cdot V^\perp) \cdot (LV) \cdot V^\perp.$$

It is evidently not possible to explicitly solve all equations of this form since (as we show below) they include Liénard's second-order scalar equation

$$(27) \quad \ddot{x} + F(x)\dot{x} + x = 0,$$

which in turn includes van der Pol's equation

$$(28) \quad \ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0,$$

important in the analysis of vacuum tube circuits. See van der Pol [43, 44] and Liénard [32].

Incidentally, transistors may not have rendered vacuum tubes obsolete after all, in light of the recent awakening to the electromagnetic pulse

effect (EMP) produced by nuclear bombs detonated above the Earth's atmosphere. See Broad [7]. Evidently the EMP from a single nuclear bomb detonated 200 miles above Nebraska would knock out "unprotected" communications equipment all across the United States. Transistor circuits connected to any sizeable antenna or metal cable, such as long leads to stereo speakers, would be especially vulnerable to extensive damage. Well-built vacuum tube circuits are more capable of withstanding the shock of this 25,000 – 50,000 volt per meter microsecond pulse. One wonders what effect such a pulse would have on humans and other life forms.

As is well known, Liénard's equation (27) can be rewritten as an equivalent 2-dimensional first-order system

$$(29) \quad \begin{aligned} \dot{x} &= y - f(x) \\ \dot{y} &= -x \end{aligned}$$

by the introduction of a second dependent variable y defined by the equation $y = \dot{x} + f(x)$, where $f(x) = \int_0^x F(x)dx$. Equation (29) is then seen to be expressible in our form (26) with

$$L = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$

and f belongs to $C^1(\mathbf{R})$ provided that F is continuous. Also note that $f(0) = 0$ so that (29) has a critical point at the origin. Furthermore,

$$J = \begin{bmatrix} -F(x) & 1 \\ -1 & 0 \end{bmatrix},$$

so that $\det J = +1$ and $\text{tr } J = -F(x)$. Hence by Theorem 4, the origin is a globally asymptotically stable critical point for Liénard's equation if $F(x)$ is positive for all x . To obtain the corresponding equations for van der Pol's equation (28) one chooses $F(x) = \mu(x^2 - 1)$; and therefore $f(x) = \mu(-x + x^3/3)$. Hence (29) becomes

$$(30) \quad \begin{aligned} \dot{x} &= \mu x + y - \mu x^3/3 \\ \dot{y} &= -x, \end{aligned}$$

which can be put into the form (26) in at least two obvious ways. But, regardless, one has for (30), $\det J = +1$ and $\text{tr } J = \mu(1 - x^2)$. Hence for van der Pol's equation the trace fails to be negative at all points of \mathbf{R}^2 . In fact, while the origin is (locally) asymptotically stable when $\mu < 0$, it can not be globally asymptotically stable since van der Pol's equation is known to have (surrounding the origin) a unique periodic solution, or "limit-cycle", to which every other solution tends (as $t \rightarrow -\infty$ when μ is negative and as $t \rightarrow +\infty$ when μ is positive). For a discussion of van

der Pol's equation as well as various other forms of Liénard's equation see Cesari [11], LaSalle [27], LaSalle and Lefschetz [30], Lefschetz [31], or Nemytskii and Stepanov [39]. When, as in the case of van der Pol's equation, the trace of the Jacobian is not negative throughout the plane, then one can apply the so-called Direct or Second Method of Liapunov, especially as refined by LaSalle and others, to determine the extent of the region of asymptotic stability. See LaSalle [26-29].

What kinds of differential equations can be written in the form of equation (26)? We will now answer this question by showing that aside from those of the form of equations (25) and (27), just considered, there are essentially no others.

THEOREM 5. *Every autonomous system of differential equations of the form (26) $\dot{X} = L \circ S(f, V)X$ is either explicitly solvable (essentially in the same way as (25) was solved above), or else it is equivalent (by an appropriate change of variables) to a single second-order scalar equation of the form*

$$(31) \quad \ddot{u} + (\nu \cdot f'(-u) - \text{tr } L)\dot{u} + (\det L)u = 0,$$

which is an equation of Liénard's type (27).

PROOF. First make a change of variables in (26) by means of the equations

$$U = RX, \quad U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix},$$

where

$$R = \begin{bmatrix} \beta & -\alpha \\ \alpha & \beta \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Then equation (26) becomes

$$(32) \quad \dot{U} = MU + f(-u)M \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

where

$$M = RLR^{-1} = \begin{bmatrix} \mu & \nu \\ \mu' & \nu' \end{bmatrix}.$$

The vector equation (32) is the system

$$(33) \quad \begin{aligned} \dot{u} &= \mu u + \nu u + \nu f(-u) \\ \dot{v} &= \mu' u + \nu' v + \nu' f(-u). \end{aligned}$$

CASE 1. If $\nu = 0$, then (33) can be solved explicitly (first for u and then for ν).

CASE 2. If $\nu \neq 0$, then the first equation of (33) can be solved for ν in terms of u and \dot{u} to obtain

$$(34) \quad \nu = \dot{u}/\nu - \mu u/\nu - f(-u).$$

This equation (34) can be used to eliminate ν and $\dot{\nu}$ from the second equation in (33) to obtain equation (31) as desired, because $\text{tr } M = \text{tr } L$ and $\det M = \det L$. The parameter ν appearing in (31) can be expressed explicitly in terms of

$$L = \begin{bmatrix} a & b \\ a' & b' \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

by the equation $\nu = \alpha\beta(a - b') - \alpha^2a' + \beta^2b$. This completes the proof of Theorem 5.

Finally we mention another connection of the Jacobian problem of algebraic geometry with differential equations. It so happens that many area-preserving diffeomorphisms of the plane can be realized as “flows” (i.e., solutions)

$$(F_t) \quad \begin{aligned} p &= p(t; p_0, q_0) \\ q &= q(t; p_0, q_0) \end{aligned}$$

of 2-dimensional autonomous systems

$$(D) \quad \begin{aligned} \dot{p} &= P(p, q) \\ \dot{q} &= Q(p, q) \end{aligned}$$

of differential equations. Here p_0 and q_0 denote the initial conditions so that

$$(F_0) \quad \begin{aligned} p_0 &= p(0; p_0, q_0) \\ q_0 &= q(0; p_0, q_0) \end{aligned}$$

represents the identity map, and F_t is the transformation of the “initial” point (p_0, q_0) to the point (p, q) at a later time $t > 0$. Furthermore, under suitable hypotheses on the functions P and Q , the inverse transformation F_t^{-1} is given by F_{-t} . That is,

$$(F_{-t}) \quad \begin{aligned} p_0 &= p(-t; p, q) \\ q_0 &= q(-t; p, q). \end{aligned}$$

In particular, the celebrated Liouville Theorem of statistical mechanics states that the flow F_t defined by Hamilton’s Equations

$$\begin{aligned}
 (\mathcal{H}) \quad \dot{p} &= -\partial H/\partial q \\
 \dot{q} &= +\partial H/\partial p
 \end{aligned}$$

is necessarily (i.e., automatically) area-preserving. (The function $H(p, q)$, which usually represents the total energy of a classical mechanical system, is called the "Hamiltonian" function for the system.) There is also a kind of converse result which states that every system (D) which possesses an invariant integral

$$\frac{d}{dt} \iint_{\Omega_t} \rho(p, q) dp dq = 0,$$

is equivalent to some Hamiltonian system (\mathcal{H}). Invariant integrals of dynamical systems were introduced by Poincaré and are discussed by Cartan [10] and by Nemytskii and Stepanov [39].

Thus, in addition to the transformation $T = (P, Q)$ which we have been associating with the system (D), there is another transformation F_t that can be associated with (D). But, in order that a flow F_t effect a transformation of the plane satisfying Condition (VI) of §3, it is not necessary that F_t be area-preserving (a Hamiltonian flow); rather, it is only necessary that the Jacobian determinant of F_t be independent of p_0 and q_0 . That is, Condition (VI) for flows F_t becomes

$$(\text{VI}_t) \quad \det J(F_t) = \delta(t) \neq 0,$$

so that under the transformation F_t the area of a region Ω gets multiplied by the nonzero factor $\delta(t)$, possibly different for different values of t . However, if a flow F_t is polynomial in p_0 and q_0 , then evidently so is its inverse F_{-t} , and Condition (VI_t) would be automatically satisfied. Therefore, by the theorem of van der Kulk-Jung (Theorem 3 in §5), a flow F_t which is polynomial in its initial condition p_0 and q_0 must belong, for each t , to the group \mathcal{P} .

These considerations give rise to the following questions. Which systems (D), other than Hamiltonian systems (\mathcal{H}), produce flows F_t which satisfy Condition (VI_t) ? Which of these flows F_t are polynomial in the initial conditions p_0 and q_0 ? Which polynomial transformations (area-preserving or not) can be realized as an F_{t_0} for some time t_0 and some flow F_t ?

The Hamiltonian $H(p, q) = p^2/2 + q^2/2$, for one-dimensional simple harmonic motion (along the q -axis), gives rise to the flow $p = p_0 \cos t - q_0 \sin t$ and $q = p_0 \sin t + q_0 \cos t$ which is linear in the initial conditions. Generally speaking, the manner in which the solution of a system (D) depends on initial conditions is a reflection of the manner in which the functions P and Q depend on their variables p and q . Evidently polynomial dependence of solutions on initial conditions is produced only by poly-

nomial functions P and Q . But polynomials P and Q do not always produce polynomial dependence of solutions on initial conditions. For example, the system $\dot{x} = -x - x^3$ and $\dot{y} = -y$ has the solution

$$x = x_0 e^{-t} [1 + x_0(1 - e^{-2t})]^{-1/2} \quad \text{and} \quad y = y_0 e^{-t}$$

and the system $\dot{x} = y + x(1 - x^2 - y^2)$ and $\dot{y} = -x + y(1 - x^2 - y^2)$ has the solution

$$x = \frac{x_0 \cos t + y_0 \sin t}{(r_0^2 + (1 - r_0^2)e^{-2t})^{1/2}}$$

and

$$y = \frac{-x_0 \sin t + y_0 \cos t}{(r_0^2 + (1 - r_0^2)e^{-2t})^{1/2}},$$

where $r_0^2 = x_0^2 + y_0^2$.

Since neither of these examples are Hamiltonian systems, consider Hamilton's equations $\dot{p} = 6q^2$ and $\dot{q} = p$ for the Hamiltonian function $H(p, q) = p^2/2 - 2q^3$. The energy integral yields $(\dot{q})^2 = 4q^3 + 2H_0$, where $H_0 = p_0^2/2 - 2q_0^3$. If $H_0 = 0$ (i.e., if $q_0 \geq 0$ and $p_0 = \pm 2q_0^{3/2}$), then $p = (p_0/2)(1 - \sqrt{q_0}t)^{-2}$ and $q = q_0(1 - \sqrt{q_0}t)^{-2}$. On the other hand, if $H_0 \neq 0$, then the solution can be expressed in terms of the Weierstrass elliptic function $\mathcal{P}(z)$ which satisfies the differential equation $[\mathcal{P}'(z)]^2 = 4[\mathcal{P}(z)]^3 + 1$. One obtains

$$p = \sqrt[3]{2H_0} \mathcal{P}'(\lambda t + z_0) \quad \text{and} \quad q = \sqrt[3]{2H_0} \mathcal{P}(\lambda t + z_0),$$

where $\lambda = \sqrt[3]{2H_0}$ and

$$p_0 = \sqrt[3]{2H_0} \mathcal{P}'(z_0) \quad \text{and} \quad q_0 = \sqrt[3]{2H_0} \mathcal{P}(z_0).$$

Then the addition formula

$$\begin{vmatrix} \mathcal{P}(\lambda t) & \mathcal{P}'(\lambda t) & 1 \\ \mathcal{P}(z_0) & \mathcal{P}'(z_0) & 1 \\ \mathcal{P}(\lambda t + z_0) & -\mathcal{P}'(\lambda t + z_0) & 1 \end{vmatrix} = 0,$$

or

$$\mathcal{P}(\lambda t + z_0) = (1/4) \left[\frac{\mathcal{P}'(\lambda t) - \mathcal{P}'(z_0)}{\mathcal{P}(\lambda t) - \mathcal{P}(z_0)} \right]^2 - \mathcal{P}(\lambda t) - \mathcal{P}(z_0),$$

can be used along with the differential equation for $\mathcal{P}(z)$ to show that the above solutions for p and q can be expressed as algebraic functions of the initial conditions, except for the factor λ which appears as a factor of t and apparently can not be extricated from the elliptic function without making a different change of time-scale along each orbit.

However, nonlinear polynomial dependence on initial conditions is possible. For example, the non-autonomous Hamiltonian system

$$\begin{aligned}\dot{p} &= 2\alpha\dot{\theta}t^2[\alpha q - \beta p]^3 + (\dot{\alpha}t + \alpha)[\alpha q - \beta p]^2 + 2t\alpha[\alpha q - \beta p][\dot{\alpha}q - \dot{\beta}p], \\ \dot{q} &= 2\beta\dot{\theta}t^2[\alpha q - \beta p]^3 + (\dot{\beta}t + \beta)[\alpha q - \beta p]^2 + 2t\beta[\alpha q - \beta p][\dot{\alpha}q - \dot{\beta}p],\end{aligned}$$

has the solution

$$p = p_0 + \alpha t[\alpha q_0 - \beta p_0]^2 \quad \text{and} \quad q = q_0 + \beta t[\alpha q_0 - \beta p_0]^2.$$

Here $\alpha = \cos \theta$, $\beta = \sin \theta$, and $\theta = \theta(t)$ is any class C^1 function of t . This map is (for each fixed value of t) of the van der Kulk-Jung type since

$$\begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} + f(\alpha q_0 - \beta p_0) \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

with $f(s) \equiv t \cdot s^2$. The Hamiltonian function is

$$\begin{aligned}H(p, q, t) &= -\frac{1}{2}\dot{\theta}t^2[\alpha q - \beta p]^4 - \frac{1}{3}q^3[\alpha^3 - 3\alpha^2\beta\dot{\theta}t] \\ &\quad + \frac{1}{3}p^3[\beta^3 + 3\alpha\beta^2\dot{\theta}t] + p^2q[3\beta^3\dot{\theta}t - 2\beta\dot{\theta}t - \alpha\beta^2] \\ &\quad + pq^2[\alpha^2\beta + \alpha^3\dot{\theta}t - 2\alpha\beta^2\dot{\theta}t].\end{aligned}$$

7. Appendix: Aizerman's original stability conjecture. Evidently, the various Jacobian problems in differential equations (as discussed in this paper) stem from the following conjecture contained in Aizerman [3]. See Hahn [16] and Markus and Yamabe [33]. Aizerman's problem concerns the equations of motion of an automatic control system containing, as frequently occurs in practice, a single nonlinear component.

AIZERMAN'S CONJECTURE. For each integer k , $1 \leq k \leq n$, the real nonlinear system

$$(N) \quad \begin{aligned}\dot{x}_1 &= \sum_{j=1}^n a_{1j}x_j + f(x_k) \\ \dot{x}_i &= \sum_{j=1}^n a_{ij}x_j\end{aligned} \quad (i = 2, 3, \dots, n)$$

has the origin as a globally asymptotically stable equilibrium point provided that $f(x)$ is continuous, $f(0) = 0$, and, for each $x \neq 0$, $\alpha < f(x)/x < \beta$ for every pair of real numbers α, β for which all the characteristic roots of the companion linear system

$$(L) \quad \begin{aligned}\dot{x}_1 &= \sum_{j=1}^n a_{1j}x_j + \alpha x_k \\ \dot{x}_i &= \sum_{j=1}^n a_{ij}x_j\end{aligned} \quad (i = 2, 3, \dots, n)$$

have negative real parts whenever $\alpha < a < \beta$.

BIBLIOGRAPHY

1. S.S. Abhyankar, *Lectures on Expansion Techniques in Algebraic Geometry. Part Two: The Jacobian Problem*, Tata Institute of Fundamental Research, Bombay, 1977.
2. ——— and T.-T. Moh, *Embeddings of the line in the plane*, Journal für die reine und angewandte Mathematik **276** (1975), 148–166.
3. M.A. Aizerman, *On a problem concerning the stability in the large of dynamical systems*, Uspehi Mat. Nauk. N.S. **4** (1949), No. 4, 187–188. (in Russian.)
4. J. Bebernes, W. Fulks and G.H. Meisters, *Differentiable paths and the continuation of solutions of differential equations*, Journal of Differential Equations **2** (1966), 102–106.
5. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.
6. G. Borg, *A condition for the existence of orbitally stable solutions of dynamical systems*, Kungl. Tekn. Högsk. Handl. **153** (1960).
7. W.J. Broad, *Nuclear pulse* (I), (II), (III), Science Vol. **212**: No 4498, 29 May 1981, 1009–1012; No. 4499, 5 June 1981, 1116–1120; No. 4500, 12 June 1981, 1248–1251.
8. F.E. Browder, *The solvability of nonlinear functional equations*, Duke Math. Journal **30** (1963), 557–566.
9. L.A. Campbell, *A condition for a polynomial map to be invertible*, Math. Ann **205** (1973), 243–248.
10. E. Cartan, *Leçons sur les invariants intégraux*, Hermann, Paris, 1958.
11. L. Cesari, *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*, Springer-Verlag, New York, 1959.
12. E.A. Coddington, and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.
13. R. Datko, *Global stability of second order autonomous differential equations*, Journal of Differential Equations **2** (1966), 412–419.
14. J. Dieudonné, *Foundations of Modern Analysis*, Vol. I of Treatise on Analysis, Academic Press, New York, 1969.
15. W. Engel, *Ein Satz über ganze Cremona Transformationen der Ebene*, Mathematische Annalen **130** (1955), 11–19.
16. W. Hahn, *Theory and Application of Liapunov's Direct Method*, English transl. of first German edition (1959), Prentice-Hall, New Jersey, 1963.
17. P. Hartman, *On stability in the large for systems of ordinary differential equations*, Canadian Journal of Mathematics **13** (1961), 480–492.
18. ———, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964.
19. ———, *The existence and stability of stationary points*, Duke Math Journal **33** (1966), 281–290.
20. ——— and C. Olech, *On global asymptotic stability of solutions of differential equations*, Transactions of the American Mathematical Society **104** (1962), 154–178.
21. H.W.E. Jung, *Über ganze birationale Transformationen der Ebene*, Journal für die reine und angewandte Mathematik **184** (1942), 161–174.
22. O.-H. Keller, *Ganze Cremona-Transformationen*, Monatshefte für Mathematik und Physik **47** (1939), 299–306.
23. N.N. Krasovskii, *Some problems in the stability theory of motion*, Gosudartv, Izdat. Fiz-Mat. Lit. Moscow, 1959, (in Russian).

24. W. van der Kulk, *On polynomial rings in two variables*, Nieuw Archief voor Wetkunde (3) I (1953), 33–41.
25. D.R. Lane, *Fixed points of affine Cremona transformations of the plane over an algebraically closed field*, American Journal of Mathematics **97** (1975), 707–732.
26. J.P. LaSalle, *The extent of asymptotic stability*, Proceedings of the National Academy of Sciences **46** (1960), 363–365.
27. ———, *Some extensions of Liapunov's second method*, IRE Transactions of the professional group on circuit theory, CT-7 (1960), 520–527.
28. ———, *Asymptotic stability criteria*, Proceedings of Symposia in Applied Mathematics, Volume 13: Hydrodynamic Instability, Amer. Math. Soc., Providence, 1962.
29. ———, *Recent Advances in Liapunov stability theory*, SIAM Review **6** (1964), 1–11.
30. ——— and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, 1961.
31. S. Lefschetz, *Differential Equations: Geometric Theory*, 2nd edition, Interscience, John-Wiley, New York, 1960.
32. A. Liénard, *Étude des oscillations entretenues*, Revue générale de l'Électricité **23** (1928), 901–912, 946–954.
33. L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J. **12** (1960), 305–317.
34. R.C. McCann, *Asymptotically stable dynamical systems are linear*, Pacific Journal of Mathematics **81** (1979), 475–479.
35. G.H. Meisters and C. Olech, *Locally one-to-one mappings and a classical theorem on schlicht functions*, Duke Math. Journal. **30** (1963), 63–80.
36. T.-T. Moh, *On the global Jacobian conjecture for polynomials of degree less than 100*, To appear.
37. M. Nagata, *On Automorphism Group of $k[x, y]$* , Lectures in Mathematics 5, Department of Mathematics, Kyoto University, Published by Kinokuniya Book-Store Co., Ltd. Tokyo, Japan and Kinokuniya Book Stores of America Co., Ltd., San Francisco, California, 1972.
38. ———, *Polynomial Rings and Affine Spaces*, Conference Board of the Mathematical Sciences, Regional Conference Series in Mathematics, Supported by the National Science Foundation, Number 37, Published for CBMS by the American Mathematical Society, 1977.
39. V.V. Nemytskii and V.V. Stepanov, *Qualitative Theory of Differential Equations*, Princeton University Press, Princeton, N.J., 1960.
40. C. Olech, *On the global stability of an autonomous system on the plane*, RIAS Technical Report 1961, Contributions to Differential Equations **1** (1963), 389–400.
41. ———, *Global phase-portrait of a plane autonomous system*, Ann. Inst. Fourier Grenoble **14** (1964), 87–98.
42. M. Razer, *Polynomial maps with constant Jacobian*, Israel J. of Math. **32** (1979), 97–106.
43. B. van der Pol, *On oscillation hysteresis in a triode generator with two degrees of freedom*, Phil. Mag. (6) **43** (1922), 700–719.
44. ———, *On relaxation-oscillations*, Phil. Mag. (7) **2** (1926), 978–992.
45. A.G. Vitushkin, *Certain examples in connection with the problem of polynomial transformations of C^n* , Izv. Akad. Nauk SSSR ser. Mat. **35** (1971), 269–279, (in Russian): English translation in Math USSR Izv. **5** (1971), 278–288.
46. ———, *On polynomial transformations of C^n* , Proceedings of the 1973 Tokyo International Conference on Manifolds, Tokyo University Press, Tokyo, 1975, 415–417.

47. S.S. Wang, *A Jacobian criterion for separability*, Journal of Algebra **65** (1980), 453–494.

48. D. Wright, *The amalgamated free product structure of $GL_2(k[x_1, \dots, x_n])$ and the weak Jacobian theorem for two variables*, Journal of Pure and Applied Algebra **12** (1978), 235–251.

49. ———, *On the Jacobian conjecture*, Illinois Journal **25** (1981), 423–440.

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ADDED IN PROOF. The two questions on page 700 lines 8–11 from the bottom have been answered in [51] for the autonomous case.

50. H. Bass, E.H. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse*, Bull. AMS, to appear.

51. H. Bass and G.H. Meisters, *Polynomial flows in the plane*, to appear.

