ON THE INITIAL BOUNDARY-VALUE PROBLEM FOR NON-HOMOGENEOUS INCOMPRESSIBLE HEAT CONDUCTING FLUIDS

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ABSTRACT. The existence of a weak global solution of the following system of initial boundary-value problem:

$$\rho(u' + u \cdot \nabla u) - \nabla(\varepsilon(\rho)\nabla u) + \text{grad } p = \rho f,$$
$$\text{div}(u) = 0 \quad \text{on } (0, T) \times G,$$
$$u(x, t) = 0 \quad \text{on } (0, T) \times \partial G, u(x, O) = u^0 \quad \text{on } G$$

and of

$$\rho' + u \cdot \text{grad } \rho = 0,$$
$$\rho(x, t) > 0 \quad \text{on } (0, T) \times G, \rho(x, 0) = \rho^0 \quad \text{on } G$$

with

$$\rho(\theta' + u \cdot \text{grad } \theta) - \nabla(\chi(\rho)\nabla\theta) = \rho g + h \quad \text{on } (0, T) \times G,$$
$$\theta(x, t) = 0 \quad \text{on } (0, T) \times \partial G, \theta(x, 0) = \theta^0 \quad \text{on } G$$

is established by the method of successive approximations.

The purpose of this paper is to show the existence of a weak global solution of the first initial boundary-value problem for non-homogeneous viscous, incompressible heat conducting fluids. Let u, ρ, θ be the velocity, the density and the temperature of the fluid respectively. The motion of the fluid is described by the initial boundary-value problem

(0.1)
$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) - \nabla(\varepsilon(\rho)\nabla u) + \text{grad } p = \rho f,$$
$$\text{div}(u) = 0 \quad \text{on } (0, T) \times G,$$
$$u(x, t) = 0 \quad \text{on } (0, T) \times \partial G \text{ and } u(x, 0) = u^0(x) \quad \text{on } G,$$

where G is a bounded open subset of $\mathbf{R}^3$.

The conservation of mass is expressed by

(0.2)
$$\frac{\partial \rho}{\partial t} + u \cdot \text{grad } \rho = 0, \ \rho(x, t) > 0 \quad \text{on } (0, T) \times G,$$
$$\rho(x, 0) = \rho^0(x) \quad \text{on } G.$$

The conservation of internal energy is described by the initial boundary value problem

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Received by the editors on June 2, 1979 and in revised form on September 11, 1979.

$$\rho\left(\frac{\partial \theta}{\partial t} + u \cdot \operatorname{grad} \theta\right) - \nabla(\chi(\rho)\nabla\theta) = h + \rho g \quad \text{on } (0, T) \times G,$$

(0.3)

$$\theta(x, t) = 0 \quad \text{on } (0, T) \times \partial G \text{ and } \theta(x, 0) = \theta^0(x) \quad \text{on } G.$$

In the system (0.3) we have used a standard argument to reduce the case of a non-homogeneous boundary condition to that of a homogeneous one. The viscosity of the fluid is $\varepsilon(\rho)$ and the coefficient of heat conduction is $\chi(\rho)$.

The system (0.1)–(0.3) does not belong to any of the three traditional types of classification of partial differential equations.

When $\varepsilon(\rho)$ is a positive constant, the existence of a weak solution of (0.1) — (0.2) has been established by Kajikov [2] and reported in [5], using the semi-Galerkin approximation method and a new type of estimates for fractional time-derivatives.

In this paper we shall use the method of successive approximations. The notations, the assumptions on $\varepsilon(\rho)$ and on $\chi(\rho)$ and the main result of the paper are given in §1. A detailed outline of the proof of the theorem is at the end of that section.

Acknowledgment. After the original version of the paper was submitted, Prof. P. Fife called the attention of the writer to Kajikov's result. The writer is indebted to Prof. Fife for pointing out to him the references [2] and [5] as well as the physical relevance of the dependence of ε, χ on ρ.

1. Let G be a bounded open subset of $\mathbf{R}^3$ with a smooth boundary ∂G. For each triple $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of non-negative integers we write

$$D^\alpha = \prod_{j=1}^{3} D_j^{\alpha_j} \text{ with } |\alpha| = \sum_{j=1}^{3} \alpha_j \text{ and } D_j = \partial/\partial x_j.$$

The inner product and the norm in $H = L^2(G)$ are denoted by $(\cdot, \cdot)$ and by $\|\cdot\|$ respectively.

The Sobolev space $H^k = \{\psi: \psi \text{ in } H, D^\alpha\psi \text{ in } H \text{ for } |\alpha| \leq k\}$ is a Hilbert with the norm

$$\|\psi\|_{k, 2} = \left\{\sum_{|\alpha|\leq k} \|D^\alpha\psi\|^2\right\}^{1/2}$$

and the usual inner product.

H_0^k is the closure of the set of testing functions in the H^k-norm, We denote by H^{-k} the dual of H_0^k and by abuse of notations, we shall write $(\cdot, \cdot)$ for the pairing between H_0^k and its dual.

V^k is the closure of the set $\{w: w = (w_1, w_2, w_3), w \text{ in } C_0^\infty(G), \operatorname{div}(w) = 0\}$ in the H^k-norm. We shall write V for V^0 and V^{-k} for $(V^k)^*$.

Throughout the paper we shall make repeated uses of the following results of the Sobolev imbedding theorem:

$$H^2 \subset C(\text{cl } G) \quad \text{and } H^1 \subset L^6(G).$$

The above natural injection mappings are all continuous and H^2 is an algebra with respect to pointwise multiplication.

$L^2(0, T; V^k)$ is the set of equivalence classes of functions $w(\cdot, t)$ from $(0, T)$ to V^k which are L^2-integrable over $(0, T)$. It is a Hilbert space with the norm

$$\|w\|_{L_2(0,T;V^k)} = \left\{ \int_0^T \|w(\cdot, t)\|_{k,2}^2 dt \right\}^{1/2}$$

and the usual inner product.

$L^\infty(0, T; V^k)$ is silmilary defined with the obvious modification. The derivative of w with respect to t is denoted by $\partial w / \partial t$ or simply by w' when there is no confusion possible.

The following assumption on the viscosity and on the coefficient of heat conduction shall be made throughout the paper.

ASSUMPTION (I).

1) $0 < \varepsilon_0 \leqq \varepsilon(\rho) \leqq c, \ 0 < \chi_0 \leqq \chi(\rho) \leqq c$ for $0 < a \leqq \rho \leqq b$.

2) $|\partial \varepsilon / \partial \rho|, |\partial \chi / \partial \rho| \leqq M$ for $0 < a \leqq \rho \leqq b$.

3) If $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$ and if $\rho_n' \to \rho'$ weakly in $L^2(0, T; H^{-1})$, then $\varepsilon(\rho_n) \to \varepsilon(\rho)$, $\chi(\rho_n) \to \chi(\rho)$ both in the weak*-topology of $L^\infty(0, T; L^\infty(G))$.

DEFINITION 1. Let u be in $L^\infty(0, T; V) \cap L^2(0, T; V^1)$ and $\rho^0 > 0$ be a scalar function in $L^\infty(G)$. Then a scalar function ρ in $L^\infty(0, T; L^\infty(G))$ with $\rho > 0$ is said to be a weak solution of (0.2) if

$$-\int_0^T (\rho, \phi')dt - \int_0^T (u \cdot \text{grad } \phi, \rho)dt = (\rho^0, \phi(\cdot, 0))$$

for all scalar functions ϕ in $L^2(0, T; H^1)$ with ϕ' in $L^2(0, T; H)$ and $\phi(\cdot, T) = 0$.

DEFINITION 2. Let f be in $L^2(0, T; H)$, u^0 be in V and $\rho^0 > 0$, be in $L^\infty(G)$. Then $\{u, \rho\}$ in $\{L^\infty(0, T; V) \cap L^2(0, T; V^1)\} \times L^\infty(0, T; L^\infty(G))$ is said to be a weak solution of (0.1)–(0.2) if

(i) ρ is a weak solution of (0.2) in the sense of Definition 1, and

$$\text{(ii)} \quad -\int_0^T (\rho u, w')dt + \int_0^T (\varepsilon(\rho)\nabla u, \nabla w)dt - \int_0^T (\rho u \cdot \nabla w, u)dt$$

$$= \int_0^T (\rho f, w)dt + (\rho^0 u^0, w(\cdot, 0))$$

for all vector functions w in $L^2(0, T; V^3)$ with w' in $L^2(0, T; V)$ and $w(\cdot, T) = 0$.

Finally for the system (0.3) we have the following definition.

DEFINITION 3. *Let g, h be in $L^2(0, T; H)$, θ^0 in H. Let $\{u, \rho\}$ be a weak solution of (0.1)–(0.2) in the sense of Definition 2. Then a scalar function θ in $L^\infty(0, T; H) \cap L^2(0, T; H_0^1)$ is said to be a weak solution of (0.3) if*

$$-\int_0^T (\rho\theta, \psi')dt + \int_0^T (\chi(\rho)\nabla\theta, \ \nabla\psi)dt - \int_0^T (\rho u \cdot \text{grad}\psi, \theta) \ dt$$

$$= \int_0^T (\rho g + h, \psi)dt + (\rho^0\theta^0, \psi(\cdot, 0))$$

for all scalar functions ψ in $L^2(0, T; H_0^3)$ with ψ' in $L^2(0, T; H)$ and $\psi(\cdot, T) = 0$.

The main result of the paper is the following theorem.

THEOREM 1.1. *Let u^0 be in V, let ρ^0 and θ^0 be two scalar functions with θ^0 in H and $0 < a \leq \rho^0(x) \leq b$ on G. Suppose that Assumption (I) is satisfied. Then for any $\{f, g, h\}$ in $L^2(0, T; H)$ there exists a weak solution $\{u, \rho, \theta\}$ of (0.1)-(0.3) in the sense of Definitions 2 and 3. Moreover $0 < a \leq \rho(x, t) \leq b$ on $(0, T) \times G$.*
Furthermore u is in $L^\infty(0, T; V) \cap L^2(0, T; V^1)$ and θ is in $L^\infty(0, T; H) \cap L^2(0, T; H_0^1)$.

REMARKS. 1) The problem of the unicity of the solution of (0.1)–(0.3) is open even in the case of constant density. (Cf. [3] for the Navier-Stokes equations). 2) As noted earlier the system (0.1)–(0.3) is not of standard type. An application of the Galerkin method or of its variants seems to give rise to difficulties. A priori estimates are fairly easy to establish and unlike the case of constant density, the difficulty lies in the proof of the existence of a local solution of the Galerkin approximating solutions. We shall circumvent the difficulty by using the method of successive approximations.

Before going into the details we shall give an outline of the proof of Theorem 1.1.

STEP 1. (to be carried out in §2). Let v be in $L^\infty(0, T; V) \cap L^2(0, T; V^1)$. By a standard method we show the existence of a weak solution of

(1.1) $\rho' + v \cdot \text{grad} \ \rho = 0,$
          $0 < a \leq \rho(x, t) \leq b \quad \text{on } (0, T) \times G, \ \rho(x, 0) = \rho^0(x) \quad \text{on } G.$

Moreover

$$\|\rho'\|_{L^2(0, T; H^{-1})} + \|(\varepsilon(\rho))'\|_{L^2(0, T; H^{-1})}$$
$$+ \|(\chi(\rho))'\|_{L^2(0, T; H^{-1})} \leq C(1 + \|v\|_{L^2(0, T; V^1)}).$$

C is independent of v.

STEP 2. (to be studied in §3). Let v be as before and let ρ be a weak

solution of (1.1) given by Step 1. The Galerkin approximation method gives the existence of a weak global solution of

$$(1.2) \quad \begin{aligned} &\rho(u' + v \cdot \nabla u) - \nabla(\varepsilon(\rho)\nabla u) + \operatorname{grad} \rho = \rho f, \\ &\operatorname{div}(u) = 0 \quad \text{on } (0, T) \times G, \\ &u(x, t) = 0 \quad \text{on } (0, T) \times \partial G, \ u(x, 0) = u^0(x) \quad \text{on } G. \end{aligned}$$

moreover

$$\|u\|_{L^\infty(0(T;V)} + \varepsilon_0^{1/2}\|u\|_{L^2(0,T;V^1)} \leqq C;$$
$$\|(\rho u)'\|_{L^2(0,T;V^{-3})} \leqq C(1 + \|v\|_{L^\infty(0,T;V)}).$$

C is independent of v and ρ.

Similarly there exists a weak global solution of

$$(1.3) \quad \begin{aligned} &\rho(\theta' + v \cdot \operatorname{grad} \theta) - \nabla(\chi(\rho)\nabla\theta) = h + \rho g \quad \text{on } (O, T) \times G, \\ &\theta(x, t) = 0 \quad \text{on } (0, T) \times \partial G, \ \theta(x, 0) = \theta^0(x) \quad \text{on } G. \end{aligned}$$

Furthermore

$$\|\theta\|_{L^\infty(0,T;H)} + \chi_0^{1/2}\|\theta\|_{L^2(0,T;H_0^1)} \leqq C \text{ and}$$
$$\|\rho\theta'\|_{L^2(0,T;H^{-3})} < C(1 + \|v\|_{L^\infty(0,T;V)}).$$

C is independent of v and ρ.

STEP 3. We now construct a sequence of successive approximations of (0.1)–(0.3). Consider the system

$$(1.4) \quad \begin{aligned} &\rho_n' + u_{n-1} \cdot \operatorname{grad} \rho_n = 0, \quad \rho_n(x, t) > 0 \quad \text{on } (0, T) \times G, \\ &\rho_n(x, 0) = \rho^0(x) \quad \text{on } G \text{ with } u_0 = 0, \ n = 1, 2 \dots \end{aligned}$$

and

$$(1.5) \quad \begin{aligned} &\rho_n(u_n' + u_{n-1} \cdot \nabla u_n) - \nabla(\varepsilon(\rho_n)\nabla u_n) + \operatorname{grad} p_n = \rho_n f, \\ &\operatorname{div}(u_n) = 0 \quad \text{on } (0, T) \times G, \\ &u_n(x, t) = 0 \quad \text{on } (0, T) \times \partial G, \ u_n(x, 0) = u^0(x) \quad \text{on } G \end{aligned}$$

and

$$(1.6) \quad \begin{aligned} &\rho_n(\theta_n' + u_{n-1} \cdot \operatorname{grad} \theta_n) - \nabla(\chi(\rho_n)\nabla\theta_n) = h + \rho_n g \quad \text{on } (0, T) \times G, \\ &\theta_n(x, t) = 0 \quad \text{on } (0\ T) \times \partial G, \ \theta_n(x, 0) = \theta^0(x) \quad \text{on } G. \end{aligned}$$

It follows from Steps 1 and 2 that

$$\begin{aligned} \|u_n\|_{L^\infty(0,T;V)} + \|u_n\|_{L^2(0,T;V^1)} &+ \|\theta_n\|_{L^\infty(0,T;H)} + \|\theta_n\|_{L^2(0,T;H_0^1)} \\ &+ \|\rho_n'\|_{L^2(0,T;H^{-1})} + \|(\rho_n u_n)'\|_{L^2(0,T;V^{-3})} + \|(\rho_n\theta_n)'\|_{L^2(0,T;H^{-3})} \\ &+ \|(\varepsilon(\rho_n))'\|_{L^2(0,T;H^{-1})} + \|(\chi(\rho_n))'\|_{L^2(0,T;H^{-1})} \leqq C. \end{aligned}$$

C is a constant independent of n. Let $n \to +\infty$ and by standard compact-

ness theorems as well as by the compensated compactness arguments of Murat (Cf. [5]) we get the theorem.

2. We shall now carry out the proof of Step 1.

THEOREM 2.1. *Let v be in $L^\infty(0, T; H) \cap L^2(0, T; V^1)$ and let ρ^0 be a scalar function in $L^\infty(G)$ with $0 < a \leqq \rho^0(x) \leqq b$ on G. Then there exists a weak solution ρ in $L^\infty(0, T; L^\infty(G))$ of (1.1) in the sense of Definition 1. Moreover $0 < a \leqq \rho(x, t) \leqq b$ on $(0, T) \times G$ and*

$$\|\rho'\|_{L^2(0,T;H^{-1})} + \|(\varepsilon(\rho))'\|_{L^2(0,T;H^{-1})}$$
$$+ \|(\chi(\rho))'\|_{L^2(0,T;H^{-1})} \leqq C(1 + \|v\|_{L^2(0,T;V^1)}).$$

C is a constant independent of v.

PROOF. 1) Since v is in $L^2(0, T; V^1)$ there exists $\{v_n\}$ in $C(0, T; C_0^\infty(G))$ with $\operatorname{div}(v_n) = 0$ and such that $v_n \to v$ in $L^2(0, T; V^1)$ as $n \to +\infty$.

Consider the initial-value problem

$$\rho_n' + v_n \cdot \operatorname{grad} \rho_n = 0 \quad \text{on } (0, T) \times G, \; \rho_n(x, 0) = \rho^0(x) \quad \text{on } G.$$

It is standard to show that there exists ρ_n, solution of the above problem. Since $D/Dt(\rho_n) = \rho_n' + v_n \cdot \operatorname{grad} \rho_n = 0$, the Lagrangian derivative of ρ_n is zero. Therefore ρ_n is constant along a particle path. On the other hand we know that $0 < a \leqq \rho_n(x, 0) = \rho^0(x) \leqq b$ on G. Hence $0 < a \leqq \rho_n(x, t) \leqq b$ on $(0, T) \times G$.

Let ψ be a scalar function in H_0^1. Then

$$(\rho_n', \psi) = -(v_n \cdot \operatorname{grad} \rho_n, \psi) = (v_n \cdot \operatorname{grad} \psi, \rho_n).$$

Therefore

$$|(\rho_n', \psi)| \leqq \|\rho_n(\cdot, t)\|_{L^\infty(G)} \|v_n(\cdot, t)\| \|\psi\|_{1,2}.$$

Hence

$$\|\rho_n'\|_{L^2(0,T;H^{-1})} \leqq b\|v_n\|_{L^2(0,T;V^1)} \leqq C(1 + \|v\|_{L^2(0,T;V^1)}).$$

C is independent of n and v.

2) Similarly, we have

$$(\rho_n' \cdot \partial\varepsilon(\rho_n)/\partial\rho_n, \psi) + ((v_n \cdot \operatorname{grad} \rho_n)\partial\varepsilon(\rho_n)/\partial\rho_n, \psi) = 0.$$

Thus,

$$((\varepsilon(\rho_n))', \psi) + (v_n \cdot \operatorname{grad} \varepsilon(\rho_n), \psi) = 0.$$

Hence

$$((\varepsilon(\rho_n))', \psi) = (v_n \cdot \operatorname{grad} \psi, \varepsilon(\rho_n)).$$

Therefore

$$\|(\varepsilon(\rho_n))'\|_{L^2(0,T;H^{-1})} \leqq \|v_n\|_{L^2(0,T;H_0^1)} \|\varepsilon(\rho_n)\|_{L^\infty(0,T;L^\infty(G))}.$$

It follows from the first part and from Assumption (I) that

$$\|(\varepsilon(\rho_n))'\|_{L^2(0,T;H^{-1})} \leqq C(1 + \|v\|_{L^2(0,T;V^1)}).$$

In exactly the same fashion, we obtain

$$\|(\chi(\rho_n))'\|_{L^2(0,T;H^{-1})} \leqq C(1 + \|v\|_{L^2(0,T;V^1)}).$$

The different constants C are all independent of n and v.

3) From the above estimates we get by taking subsequences $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$, $\rho_n' \to \rho'$ weakly in $L^2(0, T; H^{-1})$. It is clear that $0 < a \leqq \rho(x, t) \leqq b$ on $(0, T) \times G$ and

$$\|\rho'\|_{L^2(0,T;H^{-1})} \leqq C(1 + \|v\|_{L^2(0,T;H^1)}).$$

It follows from Assumption (I) and from the estimates of part (2) that

$$\|(\varepsilon(\rho))'\|_{L^2(0,T;H^{-1})} + \|(\chi(\rho))'\|_{L^2(0,T;H^{-1})} \leqq C(1 + \|v\|_{L^2(0,T;V^1)}).$$

C is independent of v.

Let ψ be a scalar function in $L^2(0, T; H^1)$ with ψ' in $L^2(0, T; H)$ and $\psi(\cdot, T) = 0$. Then

$$- \int_0^T (\rho_n, \psi')dt - \int_0^T (v_n \cdot \text{grad } \psi, \rho_n)dt = (\rho^0, \psi(\cdot, 0)).$$

Let $n \to +\infty$ and we get the theorem.

3. We now proceed to Step 2.

Let $\{w_j\}$ be a basis of V^3 and since u^0 is in V and V is dense in V^3, there exist real numbers α_{jn} such that

$$\sum_{j=1}^n \alpha_{jn} w_j \to u^0 \text{ in } V.$$

Set

$$u_n = \sum_{j=1}^n c_{jn}(t)w_j.$$

Let ρ be as in Theorem 2.1 and consider the following system of linear ordinary differential equations in $c_{jn}(t)$:

(3.1)
$$(\rho u_n', w_j) + (\varepsilon(\rho)\nabla u_n, \nabla w_j) + (\rho v \cdot \nabla u_n, w_j) = (\rho f, w_j),$$
$$c_{jn}(0) = \alpha_{jn}; 1 \leqq j \leqq n.$$

LEMMA 3.1. *Let b and ρ be as in Theorem 2.1. Then for any f in $L^2(0, T; H)$, there exists a local solution u_n in $C(0, T_n; V^3)$ of the system* (3.1).

PROOF. We have

$$(\rho u_n', w_j) = \sum_{k=1}^{n} c_{kn}'(t)(\rho w_k, w_j).$$

Since $0 < a \leqq \rho(x, t) \leqq b$ on $(0, T) \times G$, the norm induced by the inner product $((u, v)) = (\rho(\cdot, t)u, v)$ is equivalent to the H-norm. By hypotheses $\{w_j\}$ is linearly independent in H, thus $\det(\rho w_k, w_j) \neq 0$. The lemma from the Caratheodory theorem.

LEMMA 3.2. *Let v, ρ and u_n be as in Lemma 3.1. Then*

$$\|u_n\|_{L^\infty(0,T;V)} + \varepsilon_0^{1/2}\|u_n\|_{L^2(0,T;V^1)} \leqq C.$$

C is constant independent of n, v and ρ. It depends only on f and on the bounds of ρ^0 on G as well as on u^0.

PROOF. We shall show that u_n is a global solution and establish some apriori estimates using the conservation equation for the energy density.

Multiplying (3.1) by $c_{jn}(t)$ and taking the summation with respect to j from 1 to n, we obtain

(3.2) $(\rho u_n', u_n) + \varepsilon_0\|\nabla u_n\|^2 + (\rho v \cdot \nabla u_n, u_n) \leqq (\rho f, u_n).$

since u_n is in $C(0, T_n; V^3)$, $|u_n|^2$ is in $C(0, T_n; H_0^1)$. On the other hand ρ is a weak solution of (1.1) and thus,

$$(\rho', \psi) - (\rho, v \cdot \text{grad } \psi) = 0$$

for all ψ in H_0^1. Hence with $\psi = |u_n|^2$, we get

(3.3) $\dfrac{1}{2}(\rho', |u_n|^2) - \dfrac{1}{2}(\rho, v \cdot \text{grad } |u_n|^2) = 0.$

Noting that

$$(\rho v \cdot \nabla u_n, u_n) = \frac{1}{2}(\rho, v \cdot \text{grad } |u_n|^2),$$

we have by adding (3.2)–(3.3)

(3.4) $\dfrac{1}{2}\dfrac{d}{dt}(\rho u_n, u_n) + \varepsilon_0\|\nabla u_n\|^2 \leqq (\rho f, u_n).$

Therefore $\frac{1}{2}\rho|u_n|^2$, which is the kinetic energy, satisfies the following inequality:

$$\frac{1}{2}(\rho(\cdot, t)u_n(\cdot, t), u_n(\cdot, t))$$

$$\leqq \frac{1}{2}(\rho^0 u^0, u^0) + \int_0^t (\rho f, u_n) \, dt$$

$$\leqq \frac{1}{2}(\rho^0 u^0, u^0) + \frac{1}{2}\int_0^t (\rho f, f) \, dt + \frac{1}{2}\int_0^t (\rho u_n, u_n) \, dt$$

$$\leqq C + \frac{1}{2}\int_0^t (\rho u_n, u_n) \, dt.$$

C is a constant independent of n, v and ρ. It depends on f, u^0 and on the upper bound b of ρ^0 on G. In the above estimate we have applied the Holder inequality. The Gronwall lemma gives

$$(\rho(\cdot, t)u_n(\cdot, t), u_n(\cdot, t)) \leq C.$$

C is independent of n, v, t and ρ. It depends only of f, u^0 and b. Since $0 < a \leq \rho(x, t) \leq b$ on $(0, T) \times G$, it follows that $\|u_n\|_{L^\infty(0,T;V)} \leq C$. Returning to (3.4), we get

$$\varepsilon_0^{1/2}\|u_n\|_{L^2(0,T;V^1)} \leq C.$$

The different constants C are as in the lemma.

THEOREM 3.1. *Let* v, ρ *and* ρ^0 *be as in Theorem 2.1. Suppose all the hypotheses of Theorem 1.1 are satisfied. Then there exists* u *in* $L^\infty(0, T; V) \cap L^2(0, T; V^1)$, *a weak solution of (1.2) in the sense of Definition 2, i.e.,*

$$-\int_0^T (\rho u, w')dt + \int_0^T (\varepsilon(\rho)\nabla u, \nabla w)dt - \int_0^T (\rho v \cdot \nabla w, u)\ dt$$
$$= \int_0^T (\rho f, w)dt + (\rho^0 u^0, w(\cdot, 0))$$

for all w *in* $L^2(0, T; V^3)$ *with* w' *in* $L^2(0, T; V)$ *and* $w(\cdot, T) = 0$. *Moreover*

$$\|u\|_{L^\infty(0,T;V)} + \varepsilon_0^{1/2}\|u\|_{L^2(0,T;V^1)} \leq C \text{ and}$$
$$\|(\rho u)'\|_{L^2(0,T;V^{-3})} \leq C(1 + \|v\|_{L^\infty(0,T;V)}).$$

C is a constant independent of v, ρ *but depends on* f, u^0 *and on the bounds of* ρ^0 *on* G.

PROOF. 1) Let u_n be as in Lemmas 3.1–3.2. From the estimates of Lemma 3.2 we get by taking subsequences: $u_n \to u$ weakly in $L^2(0, T; V^1)$ and in the weak*-topology of $L^\infty(0, T; V)$. Moreover

$$\|u\|_{L^\infty(0,T;V)} + \varepsilon_0^{1/2}\|u\|_{L^2(0,T;V^1)} \leq C.$$

C is as stated in the theorem.

2) By definition, we have

(3.5) $(\rho u_n', w_j) + (\varepsilon(\rho)\nabla u_n, \nabla w_j) + (\rho v \cdot \nabla u_n, w_j) = (\rho f, w_j); j \leq n.$

Since ρ is a weak solution of (1.1) we get

$$(\rho', \psi) - (\rho, v\ \text{grad}\ \psi) = 0$$

for all ψ in H_0^1. But $u_n \cdot w_j$ is in $H_0^1 \cap H^2$ and hence by taking $\psi = u_n \cdot w_j$ we obtain

(3.6) $(\rho', u_n \cdot w_j) - (\rho, v \cdot \text{grad}\ u_n \cdot w_j) = 0.$

Adding (3.6) to (3.5) we have

$$((\rho u_n)', w_j) + (\varepsilon(\rho)\nabla u_n, \nabla w_j) - (\rho v \cdot \nabla w_j, u_n) = (\rho f, w_j); \quad j \leqq n.$$

Let ϕ be a function in $C^1(0, T)$ with $\phi(T) = 0$. Then

$$-\int_0^T (\rho u_n, \phi' w_j)dt + \int_0^T (\varepsilon(\rho)\nabla u_n, \phi \nabla w_j)dt - \int_0^T (\rho v \cdot \phi \nabla w_j, u_n)dt$$

$$= \int_0^T (\rho f, \phi w_j)dt + (\rho^0 u_n(\cdot, 0), w_j\phi(0)).$$

Thus, from the first part we obtain by letting $n \to +\infty$,

$$-\int_0^T (\rho u, \phi' w_j)dt + \int_0^T (\varepsilon(\rho)\nabla u, \phi \nabla w_j)dt - \int_0^T (\phi v \cdot \nabla w_j, \rho u)dt$$

$$= \int_0^T (\rho f, \phi w_j)dt + (\rho^0 u^0, w_j\phi(0)).$$

Now by standard arguments we have

$$(3.7) \qquad -\int_0^T (\rho u, w')dt + \int_0^T (\varepsilon(\rho)\nabla u, \nabla w)dt - \int_0^T (\rho v \cdot \nabla w, u)dt$$

$$= \int_0^T (\rho f, w)dt + (\rho^0 u^0, w(\cdot, 0))$$

for all w in $L^2(0, T; V^3)$ with w' in $L^2(0, T; V)$ and $w(\cdot, T) = 0$.

3) From (3.7) we have

$$\left| -\int_0^T (\rho u, w')dt \right| \leqq \|w\|_{L^2(0, T;V^3)}(\|u\|_{L^2(0, T;V^1)} + b\|f\|_{L^2(0, T;H)}$$

$$+ b\|v\|_{L^\infty(0,T;V)}\|u\|_{L^2(0, T;V^1)})$$

for all w in $C_1^0(0, T, V^3)$.

Taking into account the estimates for u we obtain

$$\left| -\int_0^T (\rho u, w')dt \right| \leqq \|w\|_{L^2(0, T;V^3)} C(1 + \|v\|_{L^\infty(0, T;V)})$$

for all w in $C_1^0(0, T; V^3)$. Hence

$$\|(\rho u)'\|_{L^1(0, T;V^{-3})} \leqq C(1 + \|v\|_{L^\infty(0, T;V)}).$$

C is a constant independent of v, ρ and depends only on f, u^0 and the bounds of ρ^0 on G.

THEOREM 3.2 *Let v, ρ be as in Theorem 2.1 and suppose all the hypotheses of Theorem 1.1 are satisfied. Then there exists θ in $L^\infty(0, T; H) \cap L^2(0, T; H_0^1)$, a weak solution of (1.3) in the sense of Definition 3. Moreover*

$$\|\theta\|_{L^\infty(0,T;H)} + \chi_0^{1/2}\|\theta\|_{L^2(0, T;H_0^1)} \leqq C \text{ and}$$

$$\|(\rho\theta)'\|_{L^2(0, T;H^{-3})} \leqq C(1 + \|v\|_{L^\infty(0, T;V)})$$

C is a constant independent of v, ρ and depends only on f, g, h, u^0, θ^0 and the bound of ρ^0 on G.

The proof is the same as that of Theorem 3.1. We shall not reproduce it.

4. We now proceed to the proof of Step 3.

LEMMA 4.1. *Suppose all the hypotheses of Theorem 1.1 are satisfied. Then for each n, there exists $\{u_n, \rho_n, \theta_n\}$, a solution of the system (1.4)–(1.6) in the sense of Definitions 1–3. Moreover*

(i) $0 < a \leqq \rho_n(x, t) \leqq b$ *on* $(0, T) \times G$,

$\|\rho_n'\|_{L^2(0, T; H^{-1})} + \|(\varepsilon(\rho_n))'\|_{L^2(0, T; H^{-1})} + \|(\chi(\rho_n))'\|_{L^2(0, T; H^{-1})} \leqq C$.

(ii) $\|u_n\|_{L^\infty(0,T;V)} + \|u_n\|_{L^2(0, T;V^1)} + \|(\rho_n u_n)'\|_{L^2(0, T;V^{-3})} \leqq C$.

(iii) $\|\theta_n\|_{L^\infty(0, T;H)} + \|\theta_n\|_{L^2(0, T;H_0^1)} + \|(\rho_n\theta_n)'\|_{L^2(0, T;H^{-3})} \leqq C$.

C is a positive constant independent of n.

PROOF. Consider the system (1.4)–(1.6). From Theorems 2.1, 3.1 and 3.2 we know that there exists $\{u_n, \rho_n, \theta_n\}$, a solution of the system (1.4)–(1.6) in the sense of Definitions 1–3. From the estimates of Theorem 2.1 we have

$$0 < a \leqq \rho_n(x, t) \leqq b \quad \text{on} \ (0, T) \times G \text{ and}$$

$$\|\rho_n'\|_{L^2(0, T; H^{-1})} + \|(\varepsilon(\rho_n))'\|_{L^2(0, T; H^{-1})}$$
$$+ \|(\chi(\rho_n))'\|_{L^2(0 (T; H^{-1})} \leqq C(1 + \|u_{n-1}\|_{L^2(0, T; V^1)}).$$

C is independent of n.

From Theorem 3.1 we obtain

$$\|u_n\|_{L^\infty(0, T;V)} + \|u_n\|_{L^2(0, T;V^1)} \leqq C \text{ and}$$

$$\|(\rho_n u_n)'\|_{L^2(0, T;V^{-3})} \leqq C(1 + \|u_{n-1}\|_{L^\infty(0, T;V)}).$$

C is again independent of n.

We get from Theorem 3.2

$$\|\theta_n\|_{L^\infty(0, T;H)} + \|\theta_n\|_{L^2(0, T;H_0^1)} \leqq C \text{ and}$$

$$\|(\rho_n\theta_n)'\|_{L^2(0, T;H^{-3})} \leqq C(1 + \|u_{n-1}\|_{L^\infty(0, T;V)}).$$

C is independent of n.

The results stated in the lemma are an immediate consequence of the above estimates.

We need a technical lemma before going into the proof of Theorem 1.1.

LEMMA 4.2. *Let $\{u_n, \rho_n\}$ be as in Lemma 4.1. Then there exists a subsequence denoted again by $\{u_n, \rho_n\}$ such that*

(i) $\rho_n \to \rho$ *in the weak*-topology of $L^\infty(0, T; L^\infty(G))$ and in $L^2(0, T; H^{-1})$,*

(ii) $u_n \to u$ *in $L^2(0, T; V)$ and weakly in $L^2(0, T; V^1)$,*

(iii) $\rho_n u_n \to \rho u$ *weakly in $L^2(0, T; H)$ and in $L^2(0, T; V^{-1})$,*

(iv) $(\varepsilon(\rho_n))' \to (\varepsilon(\rho))'$ *weakly in $L^2(0, T; H^{-1})$,*

(v) $\varepsilon(\rho_n)\nabla u_n \to \varepsilon(\rho)\nabla u$ *weakly in $L^2(0, T; H)$.*

PROOF. 1) We have by taking subsequences $\rho_n \to \rho$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$, $\rho_n' \to \rho'$ weakly in $L^2(0, T; H^{-1})$.

The natural injection mapping of H_0^1 into H is compact and hence by Schauder's theorem that of H into $H^{-1} = (H_0^1)^*$ is also compact. It follows from the estimates of Lemma 4.1 and from Aubin's theorem [1] that $\rho_n \to \rho$ in $L^2(0, T; H^{-1})$.

2) We have again by taking subsequences $u_n \to u$ weakly in $L^2(0, T; V^1)$ and in the weak*-topology of $L^\infty(0, T; V)$. It is clear that $\rho_n u_n \to \xi$ in the weak*-topology of $L^\infty(0, T; H)$, and $(\rho_n u_n)' \to \xi'$ weakly in $L^2(0, T; V^{-3})$. Thus, as in the first part we have $\rho_n u_n \to \xi$ in $L^2(0, T; V^{-1})$.

We now show that $\xi = \rho u$. Let ψ be in $C_0^\infty(0, T; C_0^\infty(G))$ with $\mathrm{div}(\psi) = 0$. Then

$$\int_0^T (\rho_n \psi, u_n)dt \to \int_0^T (\rho \psi, u)dt = \int_0^T (\xi, \psi)dt.$$

Hence $\xi = \rho\, u$.

3) We now prove one of the key assertions of the lemma, namely that $u_n \to u$ in $L^2(0, T; V)$. Indeed

$$\|u_n - u\|_{L^2(0, T;V)}^2 \leqq \int_0^T (\rho_n(u_n - u), u_n - u)dt$$

$$= \int_0^T (\rho_n u_n - \rho u, u_n - u)dt - \int_0^T (\rho_n u_n, u)dt$$

$$+ \int_0^T (\rho_n u, u)dt + \int_0^T (\rho u, u_n - u)dt.$$

Since $\rho_n u_n - \rho u \to 0$ in $L^2(0, T; V^{-1})$ and $u_n - u \to 0$ weakly in $L^2(0, T; V^1)$, we have $u_n - u \to 0$ in $L^2(0, T; V)$.

4) It follows from Assumption (I) that $(\varepsilon(\rho_n))' \to (\varepsilon(\rho))'$ weakly in $L^2(0, T; H^{-1})$. It remains to prove the assertion (v) of the lemma. First let us note that

$$\|D_j(\varepsilon(\rho_n))\|_{L^\infty(0, T;H^{-1})} \leqq C \text{ and}$$

$$\|D_j(\varepsilon(\rho_n))'\|_{L^2(0, T;H^{-2})} \leqq C.$$

C is a constant independent of n.

Thus, $D_j(\varepsilon(\rho_n)) \to D_j(\varepsilon(\rho))$ weakly in $L^2(0, T; H^{-1})$ and $D_j(\varepsilon(\rho_n))' \to D_j(\varepsilon(\rho))'$ weakly in $L^2(0, T; H^{-2})$ as $n \to +\infty$. Since $u_n \to u$ weakly in $L^2(0, T; V^1)$, it now follows from the compensated compactness arguments of Murat as applied by Lions in [5] (p.72, relation 1.64) that $u_n D_j(\varepsilon(\rho_n)) \to u D_j(\varepsilon(\rho))$ in the distribution sense on $(0, T) \times G$.

It is clear that by taking subsequences we have $\varepsilon(\rho_n)D_j u_n \to \xi_j$ weakly in $L^2(0, T; H)$ as $n \to +\infty$. Using the above results we shall show that $\varepsilon(\rho)D_j u = \xi_j$.

Let ψ be in $C_0^\infty(0, T; C_0^\infty(G))$. Then

$$\int_0^T (\varepsilon(\rho_n) D_j u_n, \; \psi) dt = - \int_0^T (u_n, \; D_j\{\varepsilon(\rho_n)\psi\}) dt$$

$$= - \int_0^T (u_n, \; \varepsilon(\rho_n) D_j\psi) dt - \int_0^T (u_n D_j\varepsilon(\rho_n), \; \psi) dt.$$

The integrals make sense since u_n is in $L^\infty(0, T; V) \cap L^2(0, T; V^1)$. Let $n \to +\infty$ and it follows from the above arguments that

$$\int_0^T (\varepsilon(\rho_n) D_j u_n, \; \psi) \; dt$$

$$\longrightarrow - \int_0^T (u, \; \varepsilon(\rho) D_j\psi) dt - \int_0^T (u D_j\varepsilon(\rho), \; \psi) \; dt.$$

$$\longrightarrow - \int_0^T (u, \; D_j\{\psi\varepsilon(\rho)\}) dt = \int_0^T (\varepsilon(\rho) D_j u, \; \psi) \; dt.$$

On the other hand

$$\int_0^T (\varepsilon(\rho_n) D_j u_n, \; \psi) \; dt \to \int_0^T (\xi_j, \; \psi) \; dt.$$

Hence

$$\int_0^T (\xi_j, \; \psi) \; dt = \int_0^T (\varepsilon(\rho) D_j u, \; \psi) \; dt$$

for all ψ in $C_0^\infty(0, T; C_0^\infty(G))$ and therefore $\xi_j = \varepsilon(\rho) D_j u$.
The lemma is proved.

We have a similar result for ρ_n, θ_n.

LEMMA 4.3. *Let $\{\rho_n, \theta_n\}$ be as in Lemma 4.1. Then there exists a subsequence denoted again by $\{\rho_n, \theta_n\}$ such that*
  (i) *$\theta_n \to \theta$ in $L^2(0, T; H)$ and weakly in $L^2(0, T; H_0^1)$,*
  (ii) *$\rho_n\theta_n \to \rho\theta$ weakly in $L^2(0, T; H)$ and in $L^2(0, T; H^{-1})$,*
  (iii) *$\chi(\rho_n)\nabla\theta_n \to \chi(\rho)\nabla\theta$ weakly in $L^2(0, T; H)$.*

The proof is identical to that of Lemma 4.2. It suffice to replace u_n by θ_n.

PROOF OF THEOREM 1.1. 1) Let u_n, ρ_n and θ_n be as in Lemma 4.1. By definition we have

$$- \int_0^T (\rho_n, \; \psi') dt - \int_0^T (u_{n-1} \cdot \text{grad } \psi, \; \rho_n) dt = (\rho^0, \; \psi(\cdot, 0))$$

for all scalar functions ψ in $C^1(0, T; H^3)$ with $\psi(\cdot, T) = 0$. Since $\rho_n \to \rho$ in $L^2(0, T; H^{-1})$ and $u_{n-1} \cdot \text{grad } \psi \to u \cdot \text{grad } \psi$ weakly in $L^2(0, T; H_0^1)$, it is clear that

$$- \int_0^T (\rho, \; \psi') dt - \int_0^T (u \cdot \text{grad } \psi, \; \rho) dt = (\rho^0, \; \psi(\cdot, 0))$$

for all ψ in $C^1(0, T; H^3)$ with $\psi(\cdot, T) = 0$. By a standard argument we get

$$- \int_0^T (\rho, \, \phi')dt \; - \; \int_0^T (u \cdot \text{grad } \phi, \, \rho)dt \; = \; (\rho^0, \, \phi(\cdot, \, 0))$$

for all ϕ in $L^2(0, T; H_0^1)$ with ϕ' in $L^2(0, T; H)$ and $\phi(\cdot, T) = 0$.

2) For the system (1.5) we have

$$- \int_0^T (\rho_n u_n, \, w')dt \; + \; \int_0^T (\varepsilon(\rho_n)\nabla u_n, \, \nabla w)dt \; - \; \int_0^T (\rho_n u_{n-1} \cdot \nabla w, \, u_n)dt$$

$$= \int_0^T (\rho_n f, \, w)dt \; + \; (\rho^0 u^0, \, w(\cdot, \, 0))$$

for all w in $C^1(0, T; V^3)$ with $w(\cdot, T) = 0$.

Since $u_{n-1} \to u$ in $L^2(0, T; V)$ and $\rho_n u_n \to \rho u$ weakly in $L^2(0, T; H)$, it is easy to see that

$$- \sum_{i,j=1}^3 \int_0^T \int_G \rho_n u_{n-1}^j u_n^k D_j w^k \, dx \, dt \; \to \; - \sum_{j,k=1}^3 \int_0^T \int_G \rho u^j u^k D_j w^k \, dx \, dt.$$

Applying Lemma 4.2 we obtain

$$- \int_0^T (\rho u, \, w')dt \; + \; \int_0^T (\varepsilon(\rho)\nabla u, \, \nabla w)dt \; - \; \int_0^T (\rho u \cdot \nabla w, \, u)dt$$

$$= \int_0^T (\rho f, \, w)dt \; + \; (\rho^0 u^0, \, w(\cdot, \, 0))$$

for all w in $C^1(0, T; V^3)$ and $w(\cdot, T) = 0$. By a standard argument we have the above equation for all w in $L^2(0, T; V^3)$ with w' in $L^2(0, T; V)$ and $w(\cdot, T) = 0$.

3) An argument as in part (2) using Lemma 4.3 shows that θ is a weak solution of (0.3).

The theorem is proved.

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