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A DIFFERENTIAL CHARACTERIZATION OF MULTIPLICITY SEQUENCES OVER ARBITRARY FIELDS

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Introduction. Let $(\mathcal{O}, m, k)$ denote an excellent, local, domain of Krull dimension one with maximal ideal m and residue class field k. We assume that $\mathcal{O}$ is equicharacteristic and geometrically unibranched. Let $\bar{\mathcal{O}}$ denote the integral closure of $\mathcal{O}$ in its quotient field $K(\mathcal{O})$, and let $\mathscr{S}: \mathcal{O} = \mathcal{O}_0 \to \mathcal{O}_1 \to \mathcal{O}_2 \to \cdots \to \mathcal{O}_n \to \bar{\mathcal{O}}$ be the blow up sequence of $\mathcal{O}$ in $\bar{\mathcal{O}}$. Here the notation has been chosen to mean that $\mathcal{O}_n$ is the last nonregular local ring in the blow up sequence of $\mathcal{O}$ (If $\mathcal{O}$ is regular, we write $\mathscr{S}$ as $\mathscr{S}: \mathcal{O} = \bar{\mathcal{O}}$). Let $\bar{\mathcal{O}}/\bar{m}$ and $\mathcal{O}_i/m_i$, $i = 0, \ldots, n$, denote the residue class fields of $\bar{\mathcal{O}}$ and $\mathcal{O}_i$ respectively. Set $f_i = [\bar{\mathcal{O}}/\bar{m}: \mathcal{O}_i/m_i]$. Finally, let $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ denote the $\bar{\mathcal{O}}$-module of q-th order $\mathcal{O}$-differentials on $\bar{\mathcal{O}}$.

In [1] and [2], K. Fischer showed that if $\mathcal{O}$ is complete, and k is algebraically closed, then for all $q \gg 1$, $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ uniquely determines the multiplicity sequence $\{\mu(\mathcal{O}_i)\}$ of $\mathscr{S}$. In this paper, we shall prove a similar result when $\mathcal{O}$ is not necessarily complete, and k is not necessarily algebraically closed. Specifically, we shall show that if $(\mathcal{O}, m, k)$ is an excellent, local, domain of Krull dimension one, of equal characteristic and geometrically unibranched, then, for all q sufficiently large ($q \gg 1$), $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ and the residue class sequence $\{f_0, \ldots, f_n\}$ uniquely determine the multiplicity sequence $\{\mu(\mathcal{O}_i)\}$ of $\mathscr{S}$. We shall also give an example which shows that $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ by itself does not determine the multiplicity sequence of $\mathscr{S}$.

We shall assume that the reader is familiar with the contents of [1] and [4]. We shall use much of the notation from those two papers. In particular, $\mu(\mathcal{O})$ will denote the multiplicity of a local ring $\mathcal{O}$, and $\lambda(M)$ will denote the length of an $\mathcal{O}$-module M, and $K(A)$ will denote the total quotient ring of any ring A.

Now let $(\mathcal{O}, m, k)$ be as above. We shall explain why we must assume $\mathcal{O}$ is geometrically unibranched instead of just unibranched. In the theory that we shall present here (as well as that in [1]) the module $I(\bar{\mathcal{O}}/\mathcal{O})$ is the object which plays the principal role in determining $\{\mu(\mathcal{O}_i)\}$. Because of the good functorial properties the module $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ enjoys, we would like to continue to deal with a class of rings in which $I(\bar{\mathcal{O}}/\mathcal{O}) = D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ for all $q \gg 1$. If k is not algebraically closed, then a unibranched domain $(\mathcal{O},$

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m, k) need not have $D^q_{\mathcal{O}}(\bar{\mathcal{O}}) = I(\bar{\mathcal{O}}/\mathcal{O})$ for any q. The following example illustrates this point.

EXAMPLE 1. Let Q denote the field of rational numbers. Let X and Y be indeterminates over Q. Set $f(X, Y) = Y^2 - X^2(X + 2)$. Clearly f is an irreducible polynomial in $Q[X, Y]$ and, thus, $R = Q[X, Y]/(f)$ is a finitely generated integral domain of Krull dimension one. Let x and y denote the images of X and Y respectively in R. Then $R = Q[x, y]$.

If we set $z = y/x$, then the reader can easily check that $Q(z]$ is the integral closure of R in $K(R)$. The only maximal ideal in $Q[z]$ which lies over (x, y) in R is the principal ideal $(z^2 - 2) = (x)$. Thus, one easily checks that the local ring $\mathcal{O} = R_{(x,y)}$ is a unibranched, local domain with integral closure $\bar{\mathcal{O}} = Q[z]_{(z^2-2)}$. Let m and $\bar{m}$ denote the maximal ideals of $\mathcal{O}$ and $\bar{\mathcal{O}}$ respectively. We pass to the completion $\hat{\mathcal{O}} = Q[[x, y]]$ of $\mathcal{O}$, and note that the triple $(\hat{\mathcal{O}}, m\hat{\mathcal{O}}, Q)$ is an excellent, equicharacteristic, complete, local domain of Krull dimension one which is unibranched. Since $\bar{\mathcal{O}}/\bar{m} \cong Q(\sqrt{2})$, we see that B, the integral closure of $\hat{\mathcal{O}}$, has the form $B = Q(\sqrt{2})[[t]]$. Here t can be taken to be x.

Now we claim that $I(B/\hat{\mathcal{O}})$ is not nilpotent. To see this, we consider the following commutative diagram with exact rows.

$$(1)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & I(B/\hat{\mathcal{O}}) & \longrightarrow & B \otimes_{\hat{\mathcal{O}}} B & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\pi} & & \\ 0 & \longrightarrow & I(Q(\sqrt{2})/Q) & \longrightarrow & Q(\sqrt{2}) \otimes_Q Q(\sqrt{2}) & \longrightarrow & Q(\sqrt{2}) & \longrightarrow & 0 \end{array}$$

In (1), π is the natural projection of B onto its residue class field $Q(\sqrt{2})$. σ is the obvious map induced by π. One easily checks that σ is surjective, and that $\sigma(I(B/\hat{\mathcal{O}})) = I(Q(\sqrt{2})/Q)$. Thus, if $I(B/\hat{\mathcal{O}})$ is nilpotent, then $I(Q(\sqrt{2})/Q)$ is also nilpotent. But, $Q(\sqrt{2})$ is a separable algebraic extension of Q. Therefore, $D^1_Q(Q(\sqrt{2})) = (0)$. Thus, setting $I = I(Q(\sqrt{2})/Q)$, we have $I = I^2 = I^3 \neq \ldots$. A simple counting argument shows that $I \neq (0)$. Therefore, I is not nilpotent. Thus, $I(B/\hat{\mathcal{O}})$ cannot be nilpotent. Since

$$D^q_{\mathcal{O}}(B) = I(B/\hat{\mathcal{O}})/I(B/\hat{\mathcal{O}})^{q+1},$$

we see that $I(B/\hat{\mathcal{O}}) \neq D^q_{\mathcal{O}}(B)$ for any q.

Recall that a local domain $(\mathcal{O}, m, k)$ is said to be geometrically unibranched if $\bar{\mathcal{O}}$ contains a unique maximal ideal $\bar{m}$, and $\bar{\mathcal{O}}/\bar{m}$ is purely inseparable over k. The above example is unibranched, but not geometrically unibranched. For geometrically unibranched rings $(\mathcal{O}, m, k)$, we have the following important lemma.

LEMMA 1. *If* $(\mathcal{O}, m, k)$ *is geometrically unibranched, then* $I(\bar{\mathcal{O}}/\mathcal{O}) = D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ *for all* $q \gg 1$.

PROOF. If $(\mathcal{O}, m, k)$ is geometrically unibranched, then $\mathcal{O} \to \bar{\mathcal{O}}$ is a radical extension. Hence it follows from [**3**; p. 246, Prop. (3. 7. 1)] that $I(\bar{\mathcal{O}}/\mathcal{O})$ is a nil ideal. But, we note that $\bar{\mathcal{O}}$ is finitely generated over $\mathcal{O}$, and, therefore, $I(\bar{\mathcal{O}}/\mathcal{O})$ is finitely generated as an $\bar{\mathcal{O}}$-module. Thus, $I(\bar{\mathcal{O}}/\mathcal{O})$ is nilpotent, and the result follows.

Main Results. Throughout this section, $(\mathcal{O}, m, k)$ will denote an excellent, equicharacteristic, local domain of Krull dimension one. We further assume that $(\mathcal{O}, m, k)$ is geometrically unibranched. We shall write the blow up sequence $\mathcal{S}$ of $\mathcal{O}$ in the following form.

$$(2) \qquad \mathcal{S}: \mathcal{O} = \mathcal{O}_0 \to \mathcal{O}_1 \to \mathcal{O}_2 \to \cdots \to \mathcal{O}_n \to \bar{\mathcal{O}}.$$

In (2), the notation has been chosen so that $\mathcal{O}_n$ is the last nonregular local ring in the blow up sequence for $\mathcal{O}$ (we are temporarily ignoring the trivial case when $\mathcal{O}$ is regular). Throughout this section, we shall let $m_i(i = 0, \ldots, n)$ denote the maximal ideal of $\mathcal{O}_i$. Thus, $m = m_0$. We shall let $\bar{m}$ denote the maximal ideal of $\bar{\mathcal{O}}$. For each $i = 0, \ldots, m$, set $f_i = [\bar{\mathcal{O}}/\bar{m}\colon \mathcal{O}_i/m_i]$. We shall refer to the sequence $\{f_0, \ldots, f_n\}$ as the residue class sequence of $\mathcal{S}$. We note that if $\mathcal{O}$ is regular, then the residue class sequence for $\mathcal{S}$ can be taken to be just $\{1\}$.

We can now state the main theorem of this paper.

THEOREM. *Let* $(\mathcal{O}, m, k)$ *be an excellent, equicharacteristic, local domain of Krull dimension one. Assume* $\mathcal{O}$ *is geometrically unibranched. Let* $\mathcal{S}: \mathcal{O} = \mathcal{O}_0 \to \cdots \to \mathcal{O}_n \to \bar{\mathcal{O}}$ *be the blow up sequence for* $\mathcal{O}$, *and let* $\{f_0, \ldots, f_n\}$ *be the residue class sequence of* $\mathcal{S}$. *Then for all* $q \gg 1$, $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ *and* $\{f_0, \ldots, f_n\}$ *uniquely determine the multiplicity sequence* $\{\mu(\mathcal{O}_i)\}$ *of* $\mathcal{O}$.

Before proceeding with the proof of the theorem, we shall make three reductions which make the argument considerably easier.

REDUCTION 1. We can, without loss of generality, assume that k is an infinite field. This statement involves the usual procedure. (See [**5**; p. 17, 18] or [**6**; p. 10]) of making the flat change of rings $\mathcal{O} \to \mathcal{O}(x)$. Thus, we proceed with the proof of the theorem under the additional hypothesis that k is infinite. This is an important simplification because now it follows from [**5**; (22.1)] that every open ideal of any $\mathcal{O}_i$ has a transversal element.

REDUCTION 2. We can, without loss of generality, assume that $\mathcal{O}$ is complete. Since quadratic transforms and $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ behave nicely when passing to the completion $\hat{\mathcal{O}}$, reduction 2 is obvious. Thus, we assume $\mathcal{O}$ is complete and k is infinite.

REDUCTION 3. We can assume, without loss of generality, that $\mathcal{O}$ is Arf-closed in $\bar{\mathcal{O}}$. If $\mathcal{O}$ is not Arf-closed in $\bar{\mathcal{O}}$, then let $\mathcal{O}'$ denote the Arf-closure of $\mathcal{O}$ in $\bar{\mathcal{O}}$. Since $\mathcal{O}$ is complete and equicharacteristic, $\mathcal{O}$ contains a field. It now follows from [**4**; Cor. 4.8] that $\mathcal{O}'$ is just the strict closure of $\mathcal{O}$ in $\bar{\mathcal{O}}$. Thus, if $\delta\colon \bar{\mathcal{O}} \to I(\bar{\mathcal{O}}/\mathcal{O})$ denotes the canonical Taylor series map, then $\mathcal{O}' = \{x \in \bar{\mathcal{O}} | \delta(x) = 0\}$. Since $\mathcal{O} \subset \mathcal{O}' \subset \bar{\mathcal{O}}$, we see that $\mathcal{O}'$ is a local domain and a finitely generated $\mathcal{O}$-module. It follows from [**4**; Theorem 4.2] that $\mu(\mathcal{O}) = \mu(\mathcal{O}')$, and that the residue class field of $\mathcal{O}'$ is just k. If we let m' denote the maximal ideal of $\mathcal{O}'$, then we have $(\mathcal{O}', m', k)$ is an excellent, equicharacteristic, complete local, domain of Krull dimension one, geometrically unibranched and having infinite residue class field k. By [**4**; Theorem 3.5] Arf-closure commutes with blowing up. Thus, the blow up sequence $\mathcal{S}'$ of $\mathcal{O}'$ is given by $\mathcal{S}'\colon \mathcal{O}' \to \mathcal{O}_1' \to \cdots \to \mathcal{O}_n' \to \bar{\mathcal{O}}$. Here $\mathcal{O}_i'$ is just the Arf-closure of $\mathcal{O}_i$. It follows from the above remarks applied to each $\mathcal{O}_i$ that the multiplicity and residue class sequences for $\mathcal{S}$ and $\mathcal{S}'$ are identical.

Finally, one easily checks from the definitions that $I(\bar{\mathcal{O}}/\mathcal{O}) = I(\bar{\mathcal{O}}/\mathcal{O}')$. Thus, Lemma 1 implies that $D^q_{\mathcal{O}}(\bar{\mathcal{O}}) = D^q_{\mathcal{O}'}(\bar{\mathcal{O}})$ for all $q \gg 1$. This completes Reduction 3.

PROOF OF THEOREM Let $(\mathcal{O}, m, k)$ be as in the statement of the theorem. By applying the reductions (which we could schematically indicate as $\mathcal{S} \to_1 \mathcal{S}(x) \to_2 \mathcal{S}(x) \to_3 \{\mathcal{S}(x)\}'$), we can assume that k is infinite, that $\mathcal{O}$ is complete and that $\mathcal{O}$ is Arf-closed in $\bar{\mathcal{O}}$.

Since $\bar{\mathcal{O}}$ is complete and equicharacteristic, $\bar{\mathcal{O}}$ has the form $\bar{\mathcal{O}} = F[[t]]$. Here t is a uniformizing parameter for $\bar{\mathcal{O}}$, and F is a field of representatives of $\bar{\mathcal{O}}/\bar{m}$ in $\bar{\mathcal{O}}$. Let $\nu\colon K(\bar{\mathcal{O}}) \to \mathbf{Z} \cup \{\infty\}$ denote the canonical, discrete rank one valuation given by $\nu(f) = \text{ord}_t(f)$. For each $i = 0, 1, \ldots, n$, let x_i denote a transversal for m_i. We need the following lemmas.

LEMMA 2. $\mu(\mathcal{O}_i) = f_i\nu(x_i)$ *for each* $i = 0, 1, \ldots, n$.

PROOF. For each $i = 0, 1, \ldots, n$, let $s_i = \min\{\nu(y) | y \in m_i\}$. It follows directly from [7; Vol. II, Cor 1, p. 299] that $\mu(\mathcal{O}_i) = f_i s_i$. Since x_i is a transversal for m_i, $\nu(x_i) = s_i$.

We note that Lemma 2 implies the multiplicity sequence $\{\mu(\mathcal{O}_i)\}$ of $\mathcal{O}$ has the form

$$\mu(\mathcal{O}) \geqq \mu(\mathcal{O}_1) \geqq \cdots \geqq \mu(\mathcal{O}_n) > 1.$$

Since each $\mathcal{O}_i$ in $\mathcal{S}$ is an equicharacteristic, complete local ring, we can choose a field of representative of $\mathcal{O}_i/m_i$ in $\mathcal{O}_i$. We shall call this field k_i. Thus, $k_i \subset \mathcal{O}_i$, and if $\pi\colon \bar{\mathcal{O}} \to \bar{\mathcal{O}}/\bar{m}$ denotes the natural projection, then $\pi(k_i) = \mathcal{O}_i/m_i$.

LEMMA 3. *For each* $i = 0, 1, \ldots, n$, $I(\bar{\mathcal{O}}/k_i[[x_i]])$ *is a free* $\bar{\mathcal{O}}$*-module of rank* $\mu(\mathcal{O}_i) - 1$.

PROOF Fix $i = 0, \ldots, n$, and consider the power series ring $k_i[[x_i]] \subseteq \bar{\mathcal{O}}$. One easily sees that $\bar{\mathcal{O}}$ is a free $k_i[[x_i]]$-module of rank $\mu(\mathcal{O}_i)$. In fact, if $\{z_1 = 1, z_2, \ldots, z_{f_i}\}$ are elements of $\bar{\mathcal{O}}$ which form a vector space basis of $\bar{\mathcal{O}}/\bar{m}$ over $\mathcal{O}_i/m_i$, then $\Omega = \{z_j t_\ell \mid j = 1, \ldots, f_i;\ \ell = 0, \ldots, s_i - 1\}$ is a free basis of $\bar{\mathcal{O}}$ over $k_i[[x_i]]$. Hence, Lemma 2 implies that the rank of $\bar{\mathcal{O}}$ over $k_i[[x_i]]$ is $\mu(\mathcal{O}_i)$.

Now consider the following short exact sequence.

$$0 \to I(\bar{\mathcal{O}}/k_i[[x_i]]) \to \bar{\mathcal{O}} \otimes_{k_i[[x_i]]} \bar{\mathcal{O}} \xrightarrow{\ \mathcal{O}\ } \bar{\mathcal{O}} \to 0. \tag{3}$$

Since $\bar{\mathcal{O}}$ is a free $k_i[[x_i]]$-module of rank $\mu(\mathcal{O}_i)$,

$$\bar{\mathcal{O}} \otimes_{k_i[[x_i]]} \bar{\mathcal{O}}$$

is a free $\bar{\mathcal{O}}$-module of rank $\mu(\mathcal{O}_i)$. Since (3) splits as $\bar{\mathcal{O}}$-modules, we conclude that $I(\bar{\mathcal{O}}/k_i[[x_i]])$ is a free $\bar{\mathcal{O}}$-module of rank $\mu(\mathcal{O}_i) - 1$.

Now for each $i = 0, 1, \ldots, n$, set $A_i = k_i[[x_i]]$. Then we have $A_i \subset \mathcal{O}_i \subset \mathcal{O}_{i+1}$ ($\mathcal{O}_{i+1}$ is just $\bar{\mathcal{O}}$ if $i = n$). These inclusions lead to the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I(\bar{\mathcal{O}}/A_i) & \longrightarrow & \bar{\mathcal{O}} \otimes_{A_i} \bar{\mathcal{O}} & \longrightarrow & \bar{\mathcal{O}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_1} & & \| & & \\
0 & \longrightarrow & I(\bar{\mathcal{O}}/\mathcal{O}_i) & \longrightarrow & \bar{\mathcal{O}} \otimes_{\mathcal{O}_i} \bar{\mathcal{O}} & \longrightarrow & \bar{\mathcal{O}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} & & \| & & \\
0 & \longrightarrow & I(\bar{\mathcal{O}}/\mathcal{O}_{i+1}) & \longrightarrow & \bar{\mathcal{O}} \otimes_{\mathcal{O}_{i+1}} \bar{\mathcal{O}} & \longrightarrow & \bar{\mathcal{O}} & \longrightarrow & 0.
\end{array} \tag{4}$$

In (4), φ_1 and φ_2 are the obvious maps induced by the inclusions $A_i \subset \mathcal{O}_i$, and $\mathcal{O}_i \subset \mathcal{O}_{i+1}$. Clearly, both φ_1 and φ_2 are surjective, and one easily checks that $\varphi_1(I(\bar{\mathcal{O}}/A_i)) = I(\bar{\mathcal{O}}/\mathcal{O}_i)$, and $\varphi_2(I(\bar{\mathcal{O}}/\mathcal{O}_i)) = I(\bar{\mathcal{O}}/\mathcal{O}_{i+1})$. Thus, the diagram in (4) gives us the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N(\mathcal{O}_i) & \longrightarrow & I(\bar{\mathcal{O}}/A_i) & \xrightarrow{\varphi_1} & I(\bar{\mathcal{O}}/\mathcal{O}_i) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow{\scriptstyle \varphi_2} & & \\
0 & \longrightarrow & N(\mathcal{O}_{i+1}) & \longrightarrow & I(\bar{\mathcal{O}}/A_i) & \xrightarrow{\varphi_2 \circ \varphi_1} & I(\bar{\mathcal{O}}/\mathcal{O}_{i+1}) & \longrightarrow & 0.
\end{array} \tag{5}$$

In (5), $N(\mathcal{O}_i)$ is the kernel of φ_1, and $N(\mathcal{O}_{i+1})$ is the kernel of $\varphi_2 \circ \varphi_1$. Thus, the vertical map on the left in (5) is just inclusion.

We need one last lemma before proving the theorem.

LEMMA 4. *Using the notation in diagram* (5), *we have*

$$x_i N(\mathcal{O}_{i+1}) = N(\mathcal{O}_i).$$

PROOF. This argument is exactly the same as in [2; p. 514], and therefore we omit it.

We can now proceed with the proof of the theorem. Since $I(\bar{\mathcal{O}}/\mathcal{O})$ (which equals $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ for $q \gg 1$) is a finitely generated module over the principal ideal domain $\bar{\mathcal{O}}$, it has a natural set of invariant factors associated with it. The idea of the proof is to write down what the invariant factors of $I(\bar{\mathcal{O}}/\mathcal{O})$ must be, given that we know $\mathcal{S}$, and then reverse this procedure.

We consider diagram (5) with $i = n$. In this case, (5) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(\mathcal{O}_n) & \longrightarrow & I(\bar{\mathcal{O}}/A_n) & \longrightarrow & I(\bar{\mathcal{O}}/\mathcal{O}_n) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & N(\bar{\mathcal{O}}) & \longrightarrow & I(\bar{\mathcal{O}}/A_n) & \longrightarrow & I(\bar{\mathcal{O}}/\bar{\mathcal{O}}) & \longrightarrow & 0. \end{array} \tag{6}$$

By Lemma 3, $I(\bar{\mathcal{O}}/A_n)$ is a free $\bar{\mathcal{O}}$-module of rank $\mu(\mathcal{O}_n) - 1$. Since $I(\bar{\mathcal{O}}/\bar{\mathcal{O}}) = (0)$, $N(\bar{\mathcal{O}}) = I(\bar{\mathcal{O}}/A_n)$. By Lemma 4, $x_n I(\bar{\mathcal{O}}/A_n) = N(\mathcal{O}_n)$. We can now conclude from the first row of (6) that a set Γ_1 of invariant factors for the $\bar{\mathcal{O}}$-module $I(\bar{\mathcal{O}}/\mathcal{O}_n)$ is given by

$$\Gamma_1 = \{\underbrace{x_n, \ldots, x_n}_{\mu(\mathcal{O}_n) - 1}\}. \tag{7}$$

Now we consider diagram (5) with $i = n - 1$. In this case, (5) becomes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N(\mathcal{O}_{n-1}) & \longrightarrow & I(\bar{\mathcal{O}}/A_{n-1}) & \longrightarrow & I(\bar{\mathcal{O}}/\mathcal{O}_{n-1}) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & N(\mathcal{O}_n) & \longrightarrow & I(\bar{\mathcal{O}}/A_{n-1}) & \longrightarrow & I(\bar{\mathcal{O}}/\mathcal{O}_n) & \longrightarrow & 0. \end{array} \tag{8}$$

We warn the reader that the notation is a bit ambiguous at this point. The module $N(\mathcal{O}_n)$ in diagram (8) is not the same as $N(\mathcal{O}_n)$ in (6). (As long as we keep track of the rank of $I(\bar{\mathcal{O}}/A_i)$ no real confusion will arise.) Now, again by Lemma 3, $I(\bar{\mathcal{O}}/A_{n-1})$ is a free $\bar{\mathcal{O}}$-module of rank $\mu(\mathcal{O}_{n-1}) - 1$. Thus, the $N(\mathcal{O}_n)$ appearing in diagram (8) can be generated by $\mu(\mathcal{O}_{n-1}) - 1$ elements ([7; vol. I, p. 246]). Therefore, $I(\bar{\mathcal{O}}/\mathcal{O}_n)$ can be represented by a relations matrix of size $(\mu(\mathcal{O}_{n-1}) - 1) \times (\mu(\mathcal{O}_{n-1}) - 1)$. Since a set of invariant factors for $I(\bar{\mathcal{O}}/\mathcal{O}_n)$ is given by (7), and $x_{n-1}(N(\mathcal{O}_n)) = N(\mathcal{O}_{n-1})$ by Lemma 4, we conclude from (8) that a set Γ_2 of invariant factors for $I(\bar{\mathcal{O}}/\mathcal{O}_{n-1})$is given by

$$\Gamma_2 = \{\underbrace{x_{n-1}, \ldots, x_{n-1}}_{\mu(\mathcal{O}_{n-1}) - \mu(\mathcal{O}_n)}, \underbrace{x_{n-1}\, x_n, \ldots, x_{n-1}\, x_n}_{\mu(\mathcal{O}_n) - 1}\} \tag{9}$$

This argument can obviously be repeated until we obtain a set of invariant factors for $I(\bar{\mathcal{O}}/\mathcal{O})$. What we get is the following result. If the blow up sequence for $\mathcal{O}$ is given by (2), then a set Γ of invariant factors for $I(\bar{\mathcal{O}}/\mathcal{O})$ is given by

$$(10)\quad \Gamma = \{\underbrace{x_0, \ldots, x_0}_{\mu(\mathcal{O}_0) - \mu(\mathcal{O}_1)}, \underbrace{x_0x_1, \ldots, x_0x_1}_{\mu(\mathcal{O}_1) - \mu(\mathcal{O}_2)}, \ldots, \underbrace{x_0x_1 \ldots x_n, \ldots, x_0x_1 \ldots x_n}_{\mu(\mathcal{O}_n) - 1}\}$$

Now the invariant factors of $I(\bar{\mathcal{O}}/\mathcal{O})$ are unique up to units in $\bar{\mathcal{O}}$. Hence the above procedure can be reversed in the following manner. Suppose we are given a set $\Gamma = \{e_1, \ldots, e_N\}$ of invariant factors of $I(\bar{\mathcal{O}}/\mathcal{O})$ (wihch equals $D_0^q(\bar{\mathcal{O}})$for all $q \gg 1$). If every e_j in Γ is a unit in $\bar{\mathcal{O}}$, then $I(\bar{\mathcal{O}}/\mathcal{O}) = (0)$. It is well known that this implies $\bar{\mathcal{O}} = \bar{\mathcal{O}}$. Thus, $\mathcal{O}$ is regular, and the multiplicity sequence is trivial.

Let us assume that some e_j in Γ is not a unit in $\bar{\mathcal{O}}$. Casting out all units from Γ, we can, without loss of generality, assume that no e_j is a unit. Now suppose e is an element of Γ of maximum ν-value. Then by equation (10), e must occur (up to unit factors) exactly $\mu(\mathcal{O}_n) - 1$ times in Γ. Thus, $\mu(\mathcal{O}_n)$ is determined. Now by Lemma 2, if x_n is any transversal of m_n, then $\mu(\mathcal{O}_n) = f_n\nu(x_n)$. Since we are assuming the residue class sequence $\{f_0, \ldots, f_n\}$ is known, we have determined $\nu(x_n)$. We next consider the element $e/t^{\nu(x_n)} = e_1$. Again by equation (10), there must be precisely $\mu(\mathcal{O}_{n-1}) - \mu(\mathcal{O}_n)$ elements of Γ which have ν-value $\nu(e_1)$. If no term in Γ has value $\nu(e_1)$, then clearly $\mu(\mathcal{O}_{n-1}) = \mu(\mathcal{O}_n)$. At any rate $\mu(\mathcal{O}_{n-1})$ is determined. Again using the fact that $\mu(\mathcal{O}_{n-1}) = f_{n-1}\nu(x_{n-1})$ (x_{n-1}) is any transversal of $\mathcal{O}_{n-1}$), we determine $\nu(x_{n-1})$. We can obviously continue this procedure until we have computed every term in $\{\mu(\mathcal{O}_i)\}$. Thus, for all $q \gg 1$, $D_{\mathcal{O}}^q(\bar{\mathcal{O}})$ and $\{f_0, \ldots, f_n\}$ uniquely determine the multiplicity sequence of $\mathcal{O}$. This completes the proof of the Theorem.

We conclude this paper with an example which shows that $D_{\mathcal{O}}^q(\bar{\mathcal{O}})$ alone will not determine the multiplicity sequence of $\mathcal{O}$ is general.

EXAMPLE 2. Let F denote the Galois field consisting of to elements, and let x be an indeterminate over F. Set $k = F(x)$, and let $\sqrt{x}$ and $\sqrt[4]{x}$ denote the square root and fourth root of x in some algebraic closure $\bar{k}$ of k. Let t be an indeterminate over $\bar{k}$, and consider the discrete rank one valuation ring $\bar{\mathcal{O}} = k(\sqrt[4]{x})[[t]]$. Let P be the subring of $\bar{\mathcal{O}}$ defined by $P = k[[t, \sqrt[4]{x}\, t^2]]$. If we set $u = t$, and $v = \sqrt[4]{x}\, t^2$, then $v/u^2 = \sqrt[4]{x}$. Thus, $K(P) = K(\bar{\mathcal{O}})$. It follows that $(P, M = (u, v), k)$ is an excellent, equicharacteristic, complete local domain of dimension one. $\bar{\mathcal{O}}$ is the integral closure of P in $K(P)$, and, thus, P is geometrically unibranched.

By Lemma 2, $\mu(P) = [k(\sqrt[4]{x})\colon k] = 4$. We next note that $u = t$ is a transversal for M. To see this, we have

$$(11)\qquad \lambda_P(P/uP) = \lambda_P(\bar{\mathcal{O}}/u\bar{\mathcal{O}}) = \lambda_P(k(\sqrt[4]{x})) = 4.$$

The first equality sign on the left in equation (11) follows from [**4**; (b) p.

657]. It now follows from [**4**; (a) p. 657] that u is a transversal for M. Since u is a transversal for M, the first blow up P_1 of P is given by $P_1 = P[v/u] = k[[t, \sqrt[4]{x}\,t]]$.

We again check by Lemma 2, that $\mu(P_1) = 4$. Following the same procedure as in equation (11), we see that u is again a transversal for the maximal ideal of P_1, and that the blow up P_2 of P_1 is just $P_2 = \bar{\mathcal{O}}$. Thus, the blow up sequence, residue class sequence and multiplicity sequence for (P, M, k) are as follows.

$$\begin{aligned} &\mathcal{S}\colon P = P_0 \to P_1 \to \bar{\mathcal{O}}, \\ &\{f_0, f_1\} = \{4, 4\}, \\ &\{\mu(P_0), \mu(P_1)\} = \{4, 4\}. \end{aligned} \tag{12}$$

It follows from the discussion immediately preceding equation (10) that a set of invariant factors for $I(\bar{\mathcal{O}}/P)$ is $\{t^2, t^2, t^2\}$.

Now, in $\bar{\mathcal{O}}$, consider a second subring $\mathcal{O}$ given by

$$\mathcal{O} = k(\sqrt{x})[[t^2, \sqrt[4]{x}\,t^2, t^3]].$$

Again we have $K(\mathcal{O}) = K(\bar{\mathcal{O}})$. We easily see that $\bar{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in $K(\mathcal{O})$. Thus, if we set $u = t^2$, $v = \sqrt[4]{x}\,t^2$ and $w = t^3$, then we have $(\mathcal{O}, m = (u, v, w), k(\sqrt{x}))$ is an excellent, equicharacteristic, complete, local domain of dimension one which is geometrically unibranched. Again applying Lemma 2, we see that $\mu(\mathcal{O}) = 4$. Using the same methods as in equation (11), we see that $u = t^2$ is a transversal for m. Thus, the blow up $\mathcal{O}_1$ of $\mathcal{O}$ is given by

$$\mathcal{O}_1 = \mathcal{O}[v/u, w/u] = \mathcal{O}[\sqrt[4]{x}, t] = \bar{\mathcal{O}}.$$

Consequently, the blow up sequence, residue class sequence and multiplicity sequence for $(\mathcal{O}, m, k(\sqrt{x}))$ are as follows.

$$\begin{aligned} &\mathcal{S}\colon \mathcal{O} = \mathcal{O}_0 \to \bar{\mathcal{O}}, \\ &\{f_0\} = \{2\}, \\ &\{\mu(\mathcal{O}_0)\} = \{4\}. \end{aligned} \tag{13}$$

Again our discussion preceding equation (10) implies that a set of invariant factors for $I(\bar{\mathcal{O}}/\mathcal{O})$ is $\{t^2, t^2, t^2\}$.

Since the $\bar{\mathcal{O}}$-modules $I(\bar{\mathcal{O}}/P)$ and $I(\bar{\mathcal{O}}/\mathcal{O})$ have the same set of invariant factors, we conclude that they are isomorphic. Hence, by Lemma 1, $D^q_{\mathcal{O}}(\bar{\mathcal{O}}) \cong D^q_P(\bar{\mathcal{O}})$ for all $q \gg 1$. However, as equations (12) and (13) show, the multiplicity sequences for $\mathcal{O}$ and P are not the same. Thus, if $\bar{\mathcal{O}}/\bar{m} \neq k$, then the module $D^q_{\mathcal{O}}(\bar{\mathcal{O}})$ does not determine the multiplicity sequence of $\mathcal{O}$. One must also know the residue class sequence $\{f_0, \ldots, f_n\}$ before $\{\mu(\mathcal{O}_i)\}$ can be computed.

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