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A MAXIMUM PRINCIPLE FOR WEAKLY COUPLED SYSTEMS OF SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH NONNEGATIVE CHARACTERISTIC FORM

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ABSTRACT. The maximum principle of Fichera for a single second order partial differential equation with nonnegative characteristic form is extended to weakly coupled linear systems of such equations. A Phragmén-Lindelöf principle for such systems, giving conditions for the maximum principle to hold in unbounded domains, is proved. Comparison theorems for degenerate parabolic semilinear systems in bounded and unbounded domains are also proved.

1. In recent years there has been considerable interest in second order partial differential equations with nonnegative characteristic form. This class of equations includes elliptic and parabolic equations as special cases. Current interest in the subject began with the work of Fichera [4], [5]. Fichera stated the appropriate boundary value problem, corresponding to the Dirichlet problem, for a general second order equation with nonnegative characteristic form, and found conditions for the existence of a weak solution to that problem. Fichera also proved a maximum principle for second order equations with nonnegative characteristic form. The object of the present article is to extend Fichera's maximum principle and related results to weakly coupled systems of second order equations with nonnegative characteristic form. This is done by combining Fichera's techniques with those used by Protter and Weinberger [10] to obtain maximum principles for weakly coupled systems of elliptic and parabolic equations.

Let Ω be a bounded domain in $\mathbf{R}^n$, with piecewise C^2-boundary. Denote the boundary of Ω by Σ. Let

$$L[u] = \sum_{i,j=1}^{n} a_{ij}(x)u_{x_ix_j} \neq \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u.$$

Assume that the coefficients of L are all bounded and continuous in Ω, and that the matrix $((a_{ij}(x)))$ is symmetric and positive semi-definite for all $x \in \Omega$. Let Σ^0 be the set of points $x \in \Sigma$ so that a vector $(\nu_1, \ldots, \nu_n)$

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normal to Ω exists at x, and $\sum_{i,j=1}^{n} a^{ij}\nu_i\nu_j = 0$. Then define Σ^* to be the set of points $x_0 \in \Sigma^0$ such that Σ is given, in a neighborhood of x_0, by $F(x) = 0$, where $F(x)$ is a C^2-function with $F(x) > 0$ for $x \in \Omega$, grad $F \neq 0$, and $L(F) \geqq 0$ at x_0. Consider the equation

$$L(u) = f \quad \text{in } \Omega. \tag{1.1}$$

Let σ^* be the set of points $x_0 \in \Sigma^*$ such that in some neighborhood of x_0 in $\bar{\Omega}$, u is C^2 and (1.1) is satisfied, together with the conditions on the coefficients of L. The set σ^* is invariant under nondegenerate coordinate changes. This is proved in [9].

THEOREM (Fichera). *Suppose that* $u \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma^*)$. *Suppose that* $c < 0$ *on* $\bar{\Omega}$ *and that* (1.1) *is satisfied in* $\Omega \cup \sigma^*$. *Then*

$$|u(x)| \leqq \max\left\{\sup\left|\frac{f}{c}\right|, \max_{\Sigma\setminus\sigma^*} |u|\right\}$$

for all $x \in \bar{\Omega}$.

This theorem is proved in [9]. The primary result of §2 is a generalization of this theorem to weakly coupled systems of second order equations with non-negative characteristic form. Let

$$L^\alpha[u] = \sum_{i,j=1} a_{ij}^\alpha(x) u_{x_i x_j} + \sum_{i=1}^{n} b_i^\alpha(x) u_{x_i}, \ \alpha = 1, \ldots, N.$$

Consider the system

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta} u^\beta \geqq 0. \tag{1.2}$$

For each operator L^α, a set $\sigma_\alpha^* \subseteqq \Sigma$ may be defined in the same way that σ^* was defined for L. Suppose that for $\alpha \neq \beta$, $c^{\alpha\beta}(x) \geqq 0$ in $\Omega \cup \sigma_\alpha^*$. The main result of section 2 is the following theorem.

THEOREM 1. *Suppose that for* $\alpha = 1, \ldots, N$ *the functions* $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma_\alpha^*)$ *satisfy the inequalities*

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta} u^\beta \geqq 0 \quad \text{in } \Omega \cup \sigma_\alpha^*,$$
$$u^\alpha \leqq 0 \quad \text{on } \Sigma\setminus\sigma_\alpha^*,$$

and that for some constant $c_0 > 0$,

$$\sum_{\beta=1}^{N} c^{\alpha\beta} \leqq -c_0 < 0$$

in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$. *Then* $u^\alpha \leqq 0$ *in* $\bar{\Omega}$, $\alpha = 1, \ldots, N$.

A maximum principle of the same form as Fichera's follows from Theorem 1 as a corollary.

The main result of §3 is a Phragmén-Lindelöf principle which gives conditions under which Theorem 1 extends to the case where the domain Ω is unbounded.

In §4, results similar to those of §2 and §3 are proved for semi-linear systems of degenerate parabolic equations. Such systems, also known as reaction-diffusion equations, are important in many applications. Reaction-diffusion equations are discussed at some length by Fife in [**6**]; that article also includes an extensive bibliography. Some of the techniques used in §4 are adapted from those used by Fife in [7].

2. Let Ω be a bounded domain in $\mathbf{R}^n$, with piecewise C^2-boundary. Denote the boundary of Ω by Σ. We consider the system of differential inequalities

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta \geqq 0 \tag{2.1}$$

in Ω, $\alpha = 1, \ldots, N$, where

$$L^\alpha[v] \equiv \sum_{i,j=1}^{n} a_{ij}^\alpha(x)v_{x_ix_j} + \sum_{i=1}^{n} b_i^\alpha(x)v_{x_i}.$$

We assume throughout this section that all the coefficients of the system (2.1) are bounded and continuous in Ω. We assume that for each α, the $n \times n$ matrix $(((a_{ij}^\alpha(x)))$ is symmetric and positive semi-definite for all $x \in \Omega$, and that $c^{\alpha\beta}(x) \geqq 0$ for $\alpha \neq \beta$. For $\alpha = 1, \ldots, N$ define Σ_α^0 to be the set of points $x \in \Sigma$ such that a vector $(\nu_1, \ldots, \nu_n)$, normal to Ω, exists at x and satisfies

$$\sum_{i,j=1}^{n} a_{ij}^\alpha(x)\nu_i\nu_j = 0.$$

Then, for each α, define Σ_α^* to be the set of points $x_0 \in \Sigma_\alpha^0$ such that Σ is is given, in a neighborhood of x_0, by $F(x) = 0$, where $F(x)$ is a C^2-function with $F(x) > 0$ for $x \in \Omega$, grad $F \neq 0$, and $L^\alpha[F] \geqq 0$ at x_0. Suppose that the functions $u^1, \ldots, u^N$ satisfy (2.1) in Ω, with $u_\alpha \in C^0(\bar{\Omega}) \cup C^2(\Omega)$, $\alpha = 1, \ldots,$ N. For each α, define σ_α^* to be the set of points $x_0 \in \Sigma^*$ such that in some neighborhood of x_0 in $\mathbf{R}^n$, u^α is C^2 and the αth inequality of (2.1) is satisfied, with all the conditions on the coefficients remaining true. The sets σ_α^* are invariant under nondegenerate changes of independent variables. The proof is the same as in the case of a single equation, which is discussed in [**9**].

THEOREM 1. *Suppose that for* $\alpha = 1, \ldots, N$ *the functions* $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cap \sigma_\alpha^*)$ *satisfy the inequalities*

$$\begin{aligned} &L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta \geqq 0 \quad \text{in } \Omega \cup \sigma_\alpha^*, \\ &u^\alpha \leqq 0 \quad \textit{on } \Sigma\backslash\sigma_\alpha^*. \end{aligned} \tag{2.2}$$

Suppose also that for some constant $c_0 > 0$,

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x) \leqq -c_0 \tag{2.3}$$

in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$. *Then* $u^\alpha \leqq 0$ *in* $\bar{\Omega}$, $\alpha = 1, \ldots, N$.

PROOF. Suppose that for some α, $u^\alpha(x) > 0$ at some point $x \in \bar{\Omega}$. Then $\sup\{u^\alpha(x): \alpha = 1, \ldots, N, x \in \bar{\Omega}\} = M > 0$. Since $\bar{\Omega}$ is compact, for some γ and for some $x_0 \in \bar{\Omega}$, $u^\gamma(x_0) = M$. By the definition of M, $u^\gamma(x)$ has its maximum on $\bar{\Omega}$ at x_0. Since $u^\gamma \leqq 0$ on $\Sigma \backslash \sigma_\alpha^*$ by hypothesis, it follows that $x_0 \in \Omega \cup \sigma_\gamma^*$. We consider the cases where $x_0 \in \Omega$ and where $x_0 \in \sigma_\gamma^*$ separately.

Case I. $x_0 \in \Omega$. In this case u^γ must have a local maximum at x_0, so $u_{x_i}(x_0) = 0$ for $i = 1, \ldots, n$ and the matrix $((u^\gamma_{x_i x_j}(x_0)))$ (i, $j = 1, \ldots, n$) must be negative semi-definite. Since the matrix $((a^\gamma_{ij}))$ is assumed to be positive semi-definite and symmetric,

$$\sum_{i,j=1}^{n} a^\gamma_{ij} u^\gamma_{x_i x_j} \leqq 0$$

at x_0. Further, since $c^{\alpha\beta} \geqq 0$ for $\alpha \neq \beta$, and $u^\beta \leqq M$ for each β by the definition of M,

$$\sum_{\beta=1}^{N} c^{\gamma\beta} u^\beta \leqq c^{\gamma\gamma} u^\gamma + \sum_{\substack{\beta=1 \\ \beta\neq\gamma}}^{N} c^{\gamma\beta} M.$$

At x_0, $u^\gamma = M$, so

$$\begin{aligned} \sum_{\beta=1}^{N} c^{\alpha\beta}(x_0) u^\beta(x_0) &\leqq c^{\gamma\gamma}(x_0) M + \sum_{\substack{\beta=1 \\ \beta\neq\gamma}} c^{\alpha\beta}(x_0) M \\ &= M \sum_{\beta=1}^{N} c^{\gamma\beta}(x_0) \\ &\leqq -Mc_0 < 0 \end{aligned}$$

by assumption (2.3). Hence, at x_0, $L^\gamma[u^\gamma] + \sum_{\beta=1} c^{\gamma\beta} u^\beta \leqq -Mc_0 < 0$, which contradicts hypothesis (2.2). Thus we cannot have $x_0 \in \Omega$.

Case II. $x_0 \in \sigma_\gamma^*$. Since $\sigma_\gamma^* \subseteqq \Sigma_\gamma^*$, there exists a function $F(x)$ such that Σ is given, in a neighborhood of x_0, by $F(x) = 0$, with grad $F \neq 0$ and $F(x) > 0$ for $x \in \Omega$. We can change coordinates from $x = (x_1, \ldots, x_n)$ to $y = (y_1, \ldots, y_n)$ with $y_k = F^k(x)$ for $k = 1, \ldots, n$ and $y_n \equiv F^n(x) = F(x)$. Then Σ can be written, in a neighborhood of x_0, as $\{y: y_n = 0\}$. In terms of the new coordinates, we have

$$L^\gamma[u] = \sum_{i,j=1}^{n} \tilde{a}^\gamma_{ij} u_{y_i y_j} + \sum_{i=1}^{n} \tilde{b}^\gamma_i u_{y_i}, \tag{2.4}$$

where

$$\tilde{a}^{\gamma}_{ij} = \sum_{k,\ell=1}^{n} a^{\gamma}_{ij} F^{i}_{x_k} F^{j}_{x_l}$$

for $i, j = 1, \ldots, n$. An inward normal to Ω at x_0 is given by grad $F = (F^n_{x_1}, \ldots, F^n_{x_n})$; since $x_0 \in \sigma^{\gamma}_{*}$, it follows that

$$\sum_{k,\ell=1}^{n} a^{\gamma}_{kl} F^{n}_{x_k} F^{n}_{x_l} = 0$$

at x_0. Thus, since $((a_{ij}))$ is positive semi-definite and symmetric, $\sum_{k=1}^{n} a^{\gamma}_{k,l} F^{n}_{x_k} = 0$ at x_0 for $\ell = 1, \ldots, n$. Hence, at x_0,

$$\tilde{a}^{\gamma}_{jn} = \tilde{a}^{\gamma}_{nj} = \sum_{k,\ell=1}^{n} a^{\gamma}_{k\ell} F^{n}_{x_k} F^{j}_{x_l} = \sum_{\ell=1}^{n} \Big(\sum_{k=1} a^{\gamma}_{k\ell} F^{n}_{x_k}\Big) F^{j}_{x_l} = 0 \tag{2.5}$$

for $j = 1, \ldots, n$. Now, $F^n_{y_i} \equiv 0$ and thus $F^n_{y_i y_j} \equiv 0$ for $i, j \neq n$, and $F^n_{y_n} = 1$, so it follows from (2.4) and (2.5) that at x_0, $L^{\gamma}[F^n] = \tilde{b}^{\gamma}_{n}$. Since $x_0 \in \sigma^{*}_{\gamma}$, $0 \leqq L^{\gamma}[F] \equiv L^{\gamma}[F^n]$ at x_0, so $\tilde{b}^{\gamma}_{n} \geqq 0$ at x_0. We may now change coordinates again, leaving the y_n-axis fixed, so that the change diagonalizes the matrix $((\tilde{a}^{\gamma}_{ij}))$, $i, j = 1, \ldots, n-1$, at x_0. Call the new coordinates $(z_1, \ldots, z_{n-1}, y_n)$. In this last coordinate system we have, at the point x_0,

$$L^{\gamma}[u] = \sum_{i,j=1}^{n-1} \hat{a}^{\gamma}_{ij} u_{z_i z_j} + \sum_{i=1}^{n-1} \hat{b}^{\gamma}_{i} u_{z_i} + \tilde{b}_{n} u_{y_n} \tag{2.6}$$

with $\hat{a}^{\gamma}_{ij} = 0$ for $i \neq j$, $\hat{a}_{ii} \geqq 0$ for $i = 1, \ldots, n-1$. Now, u^{γ} attains its maximum on $\bar{\Omega}$ at x_0. Since $x_0 \in \Sigma$, $u^{\gamma}(x)$ need not have a local maximum at x_0; however, the function $u^{\gamma}(z_1, \ldots, z_{n-1}, 0)$ with argument restricted to Σ must have a local maximum with respect to $z_1, \ldots, z_{n-1}$ at x_0. Thus, at x_0, $u^{\gamma}_{z_i} = 0$ and $u^{\gamma}_{z_i z_i} \leqq 0$ for $i = 1, \ldots, n-1$. Also, $u_{y_n} \leqq 0$ at x_0 since otherwise $u^{\gamma}(x)$ would increase as x moved from x_0 into Ω. Hence, it follows from (2.6) that at x_0, $L^{\gamma}[u^{\gamma}] \leqq 0$. We have, at x_0,

$$L^{\gamma}[u^{\gamma}] + \sum_{\beta=1}^{N} c^{\alpha\beta} u^{\beta} \leqq \sum_{\beta=1}^{N} c^{\gamma\beta} u^{\beta} \leqq -Mc_0 < 0$$

as in Case I, which contradicts (2.3). Thus x_0 cannot belong to σ^{*}_{γ}; this completes the proof of Theorem 1.

REMARK. Suppose that for each α, τ^{*}_{α} is an open subset of σ^{*}_{α}. If for each α, $u^{\alpha} \leqq 0$ on $\Sigma \backslash \tau^{*}_{\alpha}$ and the inequality

$$L^{\alpha}[u^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta} u^{\beta} \geqq 0$$

holds on $\Omega \cup \tau^{*}_{\alpha}$, then if $u^{\alpha} > 0$ in Ω for some α, a contradiction results just as in the proof of Theorem 1. Thus, Theorem 1 still holds if σ^{*}_{α} is replaced by τ^{*}_{α}.

EXAMPLE 1. Suppose $n = 2$. Let $\Omega = (0, 1) \times (0, 1)$. Define a function $f(\eta)$ by

$$f(\eta) = \begin{cases} 0, & 0 \leqq \eta \leqq 1/2, \\ 2\eta - 1, & 1/2 \leqq \eta \leqq 1. \end{cases}$$

Suppose that for $\alpha = 1, 2$, $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ and the functions u^α satisfy the following system in Ω:

$$\begin{aligned} &u^1_{x_1x_1} + f(x_2)u^1_{x_2x_2} - 2u^1 + u^2 \geqq 0, \\ &u^2_{x_1x_1} + u^2_{x_2} + u^1 - 3u^2 \geqq 0. \end{aligned} \tag{2.7}$$

For the domain Ω and the system (2.7), $\Sigma = \partial\Omega$, $\Sigma^0_1 = (0, 1) \times \{0\}$, $\Sigma^0_2 = (0, 1) \times \{0\} \cup (0, 1) \times \{1\}$, and $\Sigma^*_1 = \Sigma^*_2 = (0, 1) \times \{0\}$. Since we assume nothing about the behavior of u^1 and u^2 outside Ω, we cannot determine σ^*_1 and σ^*_2. However, suppose that for $\alpha = 1, 2$, $u^\alpha \leqq 0$ on $\Sigma\backslash[(0, 1) \times \{0\}]$. If $x_0 \in \Omega$, there exists a number ε satisfying $0 < \varepsilon < 1/2$ such that $x_0 \in (0, 1) \times (\varepsilon, 1)$. Let $\Omega_\varepsilon = (0, 1) \times (\varepsilon, 1)$, and consider the system (2.7) in Ω_ε. For the system considered in Ω_ε, $\Sigma^*_1 = \Sigma^*_2 = (0, 1) \times \{\varepsilon\}$. Now, the functions u^1 and u^2 were assumed to satisfy (2.7) in Ω, and Ω contains $\Omega_\varepsilon \cup [0, 1] \times \{\varepsilon\}]$, so for the system considered in Ω_ε, $\sigma^*_1 = \sigma^*_2 = (0, 1) \times \{\varepsilon\}$. Since $\partial\Omega_\varepsilon\backslash\sigma^*_\beta \subseteqq \Sigma\backslash[(0, 1) \times \{0\}]$ for $\alpha = 1, 2$, the functions u^α satisfy $u^\alpha \leqq 0$ on $\partial\Omega_\varepsilon\backslash\sigma^*_\alpha$. The other hypotheses of Theorem 1 are satisfied, so $u^\alpha \leqq 0$ on $\bar{\Omega}_\varepsilon$ for $\alpha = 1, 2$. In particular, $u^\alpha \leqq 0$ at x_0 for each α. Since x_0 was an arbitrary point in Ω, $u^\alpha \leqq 0$ in Ω, and hence by continuity in $\bar{\Omega}$, for each α. Thus, for system (2.7) we may omit data on $(0, 1) \times \{0\}$ and still conclude that $u^\alpha \leqq 0$ on $\bar{\Omega}$ for $\alpha = 1, 2$, although we cannot determine σ^*_1 and σ^*_2 for (2.7) in Ω.

Theorem 1 immediately yields the following results.

COROLLARY 2.1. *Suppose that for $\alpha = 1, \ldots, N$, the functions $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma^*_\alpha)$ satisfy the inequalities*

$$\begin{aligned} &L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta \leqq 0 \quad \text{in } \Omega \cup \sigma^*_\alpha, \\ &u^\alpha \geqq 0 \quad \text{on } \Sigma\backslash\sigma^*_\alpha, \end{aligned} \tag{2.8}$$

and hypothesis (2.3) *of Theorem* 1 *is satisfied. Then $u^\alpha \geqq 0$ on $\bar{\Omega}$.*

PROOF. Apply Theorem 1 to the functions $-u^\alpha$, $\alpha = 1, \ldots, N$.

COROLLARY 2.2. *Suppose that for $\alpha = 1, \ldots, N$, the functions $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma^*_\alpha)$ satisfy the equations*

$$\begin{aligned} &L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta = f^\alpha \quad \text{in } \Omega \cup \sigma^*_\alpha \\ &u^\alpha = g^\alpha \quad \text{on } \Sigma\backslash\sigma^*_\alpha, \end{aligned} \tag{2.9}$$

where f^α and g^α are bounded continuous functions on $\Omega \cup \sigma_\alpha^*$ and $\Sigma\backslash\sigma_\alpha^*$ respectively for $\alpha = 1, \ldots, N$. Suppose that hypothesis (2.3) *of Theorem* 1 *holds for some constant c_0. Then for each α, u^α satisfies*

$$|u^\alpha| \leqq \max\left[\sup_{\substack{\alpha=1,\ldots,N\\ x\in\Omega\cup\sigma_\alpha^*}} (|f^\alpha(x)|/c_0), \sup_{\substack{\alpha=1,\ldots,N\\ x\in\Sigma\backslash\sigma_\alpha^*}} |g^\alpha(x)|\right] \tag{2.10}$$

in $\bar{\Omega}$. Further, if the functions v^α satisfy the system (2.9), *with $v^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma_\alpha^*)$, for $\alpha = 1, \ldots, N$, then $v^\alpha = u^\alpha$ in $\bar{\Omega}$ for each α.*

Proof. Let

$$M = \max\left[\sup_{\substack{\alpha=1,\ldots,N\\ x\in\Omega\cup\sigma_\alpha^*}} (|f^\alpha(x)|/c_0), \sup_{\substack{\alpha=1,\ldots,N\\ x\in\Sigma\backslash\sigma_\alpha^*}} |g^\alpha(x)|\right].$$

Then for each α, $u^\alpha - M = g^\alpha - M \leqq 0$ on $\Sigma\backslash\sigma_\alpha^*$. Also,

$$\begin{aligned} &L^\alpha[u^\alpha - M] + \sum_{\beta=1}^N c^{\alpha\beta}[u^\beta - M] \\ &= L^\alpha[u^\alpha] + \sum_{\beta=1}^N c^{\alpha\beta}u^\beta - M\sum_{\beta=1}^N c^{\alpha\beta} \\ &\geqq f^\alpha + c_0 M \geqq 0 \end{aligned}$$

in $\Omega \cup \sigma_\alpha^*$. Thus Theorem 1 applies to the functions $u^\alpha - M$, and hence $u^\alpha - M \leqq 0$ or $u^\alpha \leqq M$ in $\bar{\Omega}$. Similarly, Corollary 2.1 applies to the functions $u^\alpha + M$, so $u^\alpha \geqq -M$ in $\bar{\Omega}$. It follows that $|u^\alpha| \leqq M$ in $\bar{\Omega}$ for $\alpha = 1, \ldots, N$, which is precisely (2.10). If the functions v^α also satisfy (2.9), then the functions $w^\alpha = u^\alpha - v^\alpha$ satisfy

$$\begin{aligned} &L^\alpha[w^\alpha] + \sum_{\beta=1}^N c^{\alpha\beta}w^\beta = 0 \quad \text{on } \Omega \cup \sigma_\alpha^*, \\ &w^\alpha = 0 \quad \text{on } \Sigma\backslash\sigma_\alpha^*. \end{aligned} \tag{2.11}$$

Theorem 1 and Corollary 2.1 both apply to (2.11); so $w^\alpha \leqq 0$ in $\bar{\Omega}$ and $w^\alpha \geqq 0$ in $\bar{\Omega}$ for each α. Thus $u^\alpha - v^\alpha = 0$ or $u^\alpha = v^\alpha$ in $\bar{\Omega}$ for each α.

Theorem 1 is a weak maximum principle; it asserts that if the functions u^α satisfy the system

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^N c^{\alpha\beta}u^\beta \geqq 0$$

in $\Omega \cup \sigma_\alpha^*$ and u^γ has a positive maximum at a point of $\Omega \cup \sigma_\gamma^*$, then for some α, u^α must be positive at some point of $\Sigma\backslash\sigma_\alpha^*$. Theorem 1 says nothing about the behavior of u^γ elsewhere in $\Omega \cup \sigma_\gamma^*$. In contrast, suppose that u satisfies the uniformly elliptic equation

$$\sum_{i,j=1}^n a_{ij}^{ij}(x)u_{x_ix_j} + \sum_{i=1}^n b^i(x)u_{x_i} + c(x)u = 0$$

in Ω with $u \leqq 0$ on $\partial\Omega$, where $c(x) \leqq -c_0 < 0$. The strong maximum principle for elliptic equations asserts that if u has a positive maximum at $x_0 \in \Omega$, with $u(x_0) = M$, then $u(x) \equiv M$ in Ω. Strong maximum principles for weakly coupled systems of elliptic and parabolic equations have been proved by Protter and Weinberger **[10]**. Their proof is accomplished by showing that each component u^α of the solution of the system must itself satisfy a differential inequality. Then, the usual strong maximum principles for a single equation may be applied. For general second order equations with nonnegative characteristic form, the question of strong maximum principles for a single equation is much more complicated than in the elliptic or parabolic case. However, some very general strong maximum principles have been proved by Bony and Aleksandrov; these are discussed in **[9]**. The following corollary permits the extension to systems of whatever strong maximum principles are available for a single differential equation with nonnegative characteristic form.

COROLLARY 2.3. *Under the hypotheses of Theorem* 1, *each of the functions* u^α *satisfies the inequality*

$$L^\alpha[u^\alpha] + c^{\alpha\alpha}u^\alpha \geqq 0 \quad \text{in } \Omega \cup \sigma_\alpha^*.$$

PROOF. By assumption, $L^\alpha[u^\alpha] + \sum_{\beta=1}^N c^{\alpha\beta}u^\beta \geqq 0$ in $\Omega \cup \sigma_\alpha^*$. By Theorem 1, $u^\beta \leqq 0$ in $\bar{\Omega}$ for $\beta = 1, \ldots, N$; since we assume that $c^{\alpha\beta} \geqq 0$ for $\alpha \neq \beta$, it follows that

$$L^\alpha[u^\alpha] + c^{\alpha\alpha}u^\alpha \geqq -\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N} c^{\alpha\beta}u^\beta \geqq 0,$$

which proves the corollary.

It should be noted that condition (2.3) implies that $c^{\alpha\alpha} \leqq -c_0 < 0$.

REMARKS. Suppose that the operators L^α in (2.2) are degenerate parabolic, that is,

$$L^\alpha[u] = \sum_{i,j=1}^{n-1} a_{ij}^\alpha(x)u_{x_i x_j} + \sum_{i=1}^{n-1} b_i^\alpha(x)u_{x_i} - u_{x_n}$$

for $\alpha = 1, \ldots, N$, with $((a_{ij}^\alpha(x)))$ $(i, j = 1, \ldots, n-1)$ positive semidefinite. If Ω is a cylindrical domain, that is, $\Omega = \omega \times (0, T)$ where ω is a smooth, bounded domain in $\mathbf{R}^{n-1}$, then for each α, Σ_α^* contains all of $\omega \times \{T\}$ and none of $\omega \times \{0\}$. Which parts of $\partial\omega \times (0, T)$ lie in Σ_α^* will vary, depending on L^α. If all the operators L^α are degenerate parabolic, data may always be omitted on $\omega \times \{T\}$, even though σ_α^* may not be known. Theorem 1 may be applied by using the same trick as in example 1. Further, if the operators L^α are degenerate parabolic, condition (2.3) of Theorem 1 may be omitted. The functions $c^{\alpha\beta}$ are assumed to be bounded;

so even without condition (2.3), $\sum_{\beta=1}^{N} c^{\alpha\beta} \leqq M$ in Ω for some constant M, $\alpha = 1, \ldots, N$. If we let $w^\alpha = u^\alpha \exp(-(M+1)x_n)$, and the functions u^α satisfy the system

$$(2.13) \qquad \sum_{i,j=1}^{n-1} a_{ij}^\alpha u_{x_i x_j}^\alpha + \sum_{i=1}^{n-1} b_i^\alpha u_{x_i}^\alpha - u_{x_n}^\alpha + \sum_{\beta=1}^{N} c^{\alpha\beta} u^\beta \geqq 0$$

in $\Omega \cup \sigma_\alpha^*$, then the functions w satisfy the system

$$(2.13) \qquad \sum_{i,j=1}^{n-1} a_{ij}^\alpha w_{x_i x_j}^\alpha + \sum_{i=1}^{n-1} b_i^\alpha w_{x_i}^\alpha - w_{x_n}^\alpha + \sum_{\beta=1}^{n} \bar{c}^{\alpha\beta} w^\beta \geqq 0$$

in $\Omega \cup \sigma_\alpha^*$, where $\bar{c}^{\alpha\beta} = c^{\alpha\beta}$ for $\alpha \neq \beta$ and $\bar{c}^{\alpha\alpha} = c^{\alpha\alpha} - M - 1$. Thus, $\sum_{\beta=1}^{N} \bar{c}^{\alpha\beta} \leqq -1$, so condition (2.3) holds for system (2.13). If $u^\alpha \leqq 0$ on $\Sigma \backslash \sigma_\alpha^*$, then $w^\alpha \leqq 0$ on $\Sigma \backslash \sigma_\alpha^*$. Thus, Theorem 1 applies to the functions w^α, so $w^\alpha \leqq 0$ in $\bar{\Omega}$, and thus $u^\alpha \leqq 0$ in $\bar{\Omega}$, $\alpha = 1, \ldots, N$. Hence, condition (2.3) may be omitted and Theorem 1 will still apply, in the degenerate parabolic case.

3. In this section, Ω will be an unbounded domain in $\mathbf{R}^n$, with $\partial\Omega$ piecewise C^2. The operators L^α are as in §2, but their coefficients and the functions $c^{\alpha\beta}$ are only assumed to be bounded on each bounded subdomain of Ω. Thus the coefficients may grow as $|x| \to \infty$. Suppose that the functions u^α, $\alpha = 1, \ldots, N$, satisfy the system

$$(3.1) \qquad \begin{aligned} &L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta} u^\beta \geqq 0 \quad \text{in } \Omega \cup \sigma_\alpha^*, \\ &u^\alpha \leqq 0 \quad \text{on } \Sigma \backslash \sigma_\alpha^*. \end{aligned}$$

THEOREM 2. *Suppose that the functions u^α, $\alpha = 1, \ldots, N$, satisfy the system* (3.1), *with $u^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma_\alpha^*)$. Suppose that there exists a function $H(x)$ such that for $\alpha = 1, \ldots, N$.*

$$(3.2) \qquad L^\alpha[H] + \sum_{\beta=1}^{N} c^{\alpha\beta} H \leqq 0$$

in $\Omega \cup \sigma_\alpha^*$, with $H(x) > 0$ in $\bar{\Omega}$. Assume that for each $R > 0$ there is a constant $c_0(R) > 0$ so that for $\alpha = 1, \ldots, N$,

$$(3.3) \qquad \sum_{\beta=1}^{N} c^{\alpha\beta}(x) \leqq -c_0(R)$$

for all $x \in \Omega \cup \sigma_\alpha^*$ with $|x| \leqq R$. If

$$(3.4) \qquad \liminf_{R\to\infty} \left[\sup_{\substack{x \in \bar{\Omega}, |x| = R \\ \alpha = 1, \ldots, N,}} (u^\alpha(x)/H(x)) \right] \leqq 0$$

then $u^\alpha \leqq 0$ in $\bar{\Omega}$, $\alpha = 1, \ldots, N$.

PROOF. Suppose $x_0 \in \Omega$. Given any $\varepsilon > 0$, condition (3.4) implies that there exists some $R > |x_0|$ such that for $\alpha = 1, \ldots, N$, $u^\alpha(x) - \varepsilon H(x) \leqq 0$ for all $x \in \bar{\Omega}$ with $|x| = R$. Since $H > 0$ in $\bar{\Omega}$ and $u^\alpha \leqq 0$ on $\Sigma \backslash \sigma_\alpha^*$, $u^\alpha - \varepsilon H \leqq 0$ on $\Sigma \backslash \sigma_\alpha^*$ for each α. Let $\Omega_R = \Omega \cap \{x\colon |x| \leqq R\}$. Let $\sigma_{R\alpha}^*$ be defined for Ω_R in the same way that σ_α^* was defined for Ω in §2. Then $\sigma_\alpha^* \cap \{x\colon |x| < R\} \subseteqq \sigma_{R\alpha}^*$. Let $\tau_\alpha^* = \sigma_\alpha^* \cap \{x\colon |x| < R\}$. Since $u^\alpha - \varepsilon H \leqq 0$ for $|x| = R$ and $x \in \Sigma \backslash \sigma_\alpha^*$,

$$u^\alpha - \varepsilon H \leqq 0 \quad \text{on } \partial\Omega_R \backslash \tau_\alpha^*,\ \alpha = 1, \ldots, N. \tag{3.5}$$

(By the definition of Ω_R, $\partial\Omega_R = [\bar{\Omega} \cap \{x\colon |x| = R\}] \cup [\Sigma \cap \{x\colon |x| < R\}]$, so the only part of $\partial\Omega_R$ where $u^\alpha - \varepsilon H$ is not known to be nonpositive is $\sigma_\alpha^* \cap \{x\colon |x| < R\} = \tau_\alpha^*$). For each α, $\tau_\alpha^* \subseteqq \sigma_\alpha^*$, so by (3.1) and (3.2),

$$L^\alpha[u^\alpha - \varepsilon H] + \sum_{\beta=1}^{N} c^{\alpha\beta}(u^\beta - \varepsilon H) \leqq 0 \tag{3.6}$$

in $\Omega_R \cup \tau_\alpha^*$, $\alpha = 1, \ldots, N$. Condition (3.3) insures that hypothesis (2.3) of Theorem 1 holds in Ω_R, so it follows from the remark at the end of the proof of Theorem 1 that (3.5) and (3.6) imply $u^\alpha - \varepsilon H \leqq 0$ in Ω_R for $\alpha = 1, \ldots, N$. In particular, $u^\alpha(x_0) \leqq \varepsilon H(x_0)$. Since $\varepsilon > 0$ was arbitrary, $u^\alpha(x_0) \leqq 0$ for each α. Since $x_0 \in \Omega$ was arbitrary, $u^\alpha \leqq 0$ in Ω, and hence by continuity in $\bar{\Omega}$, for $\alpha = 1, \ldots, N$.

Remarks. Theorem 2 is a Phragmén-Lindelöf principle for weakly coupled systems of second order equations with nonnegative characteristic form. Various types of Phragmén-Lindelöf principles have been proved for elliptic and parabolic equations. Some of these results are discussed in **[10]**. In the case of elliptic equations, the condition (3.3) may be modified or eliminated; how this can be done is discussed in **[10]**. However, the techniques used in **[10]** for elliptic equations cannot be used in equations with nonnegative characteristic form without modification. (The arguments allowing data to be omitted on σ_α^* are rather delicate, since they consider the behavior of both the first and second order terms in L^α.)

The usefulness of a result such as Theorem 2 depends on whether or not one can construct an appropriate comparison function $H(x)$. In general, finding the proper $H(x)$ may be difficult or impossible. However, if the operators L^α are degenerate parabolic and Ω is cylindrical, then comparison functions $H(x)$ can be constructed under a rather wide range of conditions.

Suppose that for $\alpha = 1, \ldots, N$.

$$L^\alpha[u^\alpha] \equiv \sum_{i,j=1}^{n-1} a_{ij}^\alpha(x) u_{x_i x_j} + \sum_{i=1}^{n-1} b_i^\alpha(x) u_{x_i} - u_{x_n},$$

and $\Omega = \omega \times (0, T)$ where ω is an unbounded domain in $\mathbf{R}^{n-1}$, with $\partial\omega$ piecewise C^2. The proof of Theorem 2 may be modified slightly by replacing $\Omega_R = \Omega \cap \{x: |x| < R\}$ with $\omega_R \times (0, T)$ where $\omega_R = \omega \cap \{(x_1, \ldots, x_{n-1}): (\sum_{i=1}^{n-1} x_i^2)^{1/2} < R\}$, and replacing (3.4) with the condition

$$\liminf_{r \to \infty} \left[\sup_{\substack{x \in \bar{\Omega},\ |(x_1, \ldots, x_{n-1})| = r \\ \alpha = 1, \ldots, N}} [u(x)/H(x)] \right] \leqq 0.$$

The proof of Theorem 2 remains essentially the same, but since $\omega_R \times (0, T)$ is cylindrical and the operators L^α degenerate parabolic, hypothesis (2.3) of Theorem 1 may be omitted, and hence condition (3.3) may be omitted in Theorem 2. Further, suppose that the coefficients of (3.1) satisfy, for each α, the conditions

$$\sum_{i,j=1}^{n-1} a_{ij}^\alpha(x) \xi_i \xi_j \leqq K_1(1 + r^2)^{1-\lambda} |\xi|^2,$$

$$|b_1^\alpha| \leqq K_2(1 + r^2)^{1/2}, \quad i = 1, \ldots, n-1,$$

and

$$\sum_{\beta=1}^{N} c^{\alpha\beta} \leqq K_3(1 + r^2)^\lambda$$

for some, λ, K_1, K_2, $K_3 > 0$, where $r = (\sum_{i=1}^{n-1} x_i^2)^{1/2}$. Then if k and β are properly chosen positive constants, the function $H(x) = \exp\{k(1 + r^2)^\lambda e^{\beta x_n}\}$ will satisfy (3.2) in $\omega \times (0, T_1)$ for some $T_1 > 0$. This particular $H(x)$ was introduced by Bodanko [**1**], and used to prove a Phragmén-Lindelöf principle for systems of parabolic equations. Bodanko's arguments extend to the degenerate case, but do not allow for the omission of data on $[\partial\omega \times (0, T)] \cap \sigma_\alpha^*$ even if the form $\sum_{i=1}^{n-1} a_{ij}^\alpha(x)\xi_i\xi_j$ degenerates. Comparison functions similar to Bodanko's but allowing a wider range of growth conditions on the coefficients of (3.1) have been constructed by Kusano, Kuroda, and Chen [**8**], Chabrowski [**2**], and the author [**3**]. A detailed analysis of the degenerate parabolic case is given in [**3**]. The Phragmén-Lindelöf principle, together with comparison functions, can be used to study the asymptotic behavior of solutions of weakly coupled systems of degenerate parabolic equations. This application is discussed in [**3**].

4. The object of this section is to extend the results of §2 and §3 to certain semi-linear systems of degenerate parabolic equations. Let ω be a domain in $\mathbf{R}^n$, either bounded or unbounded, with $\partial\omega$ piecewise C^2. Let $\Omega = \omega \times (0, T)$. In this section the operators L^α are assumed to have the form

$$(4.1) \qquad L^\alpha[u] = \sum_{i,j=1}^{n} a_{ij}^\alpha(x, t) u_{x_i x_j} + \sum_{i=1}^{n} b_i^\alpha(x, t) u_{x_i} + c^\alpha(x, t) u - u_t,$$

where $\sum_{i,j=1}^{n} a_{ij}^\alpha(x, t)\xi_i\xi_j \geqq 0$ for $(x, t) \in \Omega$, $\xi \in \mathbf{R}^n$, and $\alpha = 1, \ldots, N$.

The type of operator in (4.1) is a special case of the types of operators considered in §2 and §3. For the operator in (4.1) and the domain Ω, it is easy to verify that Σ_α^* contains all of $\omega \times \{T\}$, none of $\omega \times \{0\}$, and may or may not contain points of $\partial\omega \times (0, T)$. Further, the device of Example 1 in §2 can often be used to omit data on $\omega \times \{T\}$.

Let $\mathbf{u} = (u^1, \ldots, u^N)$ and let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^n$. Let $\mathbf{F}(x, t, \mathbf{u})$ and $\boldsymbol{\mathfrak{F}}(x, t, \mathbf{u})$ be functions from $\Omega \times \mathbf{R}^n$ to $\mathbf{R}^n$, with components F^α and $\mathfrak{F}^\alpha$ respectively, such that $F^\alpha(x, t, \mathbf{u}) \geqq \mathfrak{F}^\alpha(x, t, \mathbf{u})$ for $(x, t) \in \Omega$, $\mathbf{u} \in \mathbf{R}^N$, and $\alpha = 1, \ldots, N$. Assume that $\boldsymbol{\mathfrak{F}}(x, t, \mathbf{u})$ is Lipschitz in $\mathbf{u}$, uniformly for $(x, t) \in \bar{\Omega}$, with respect to the norm $|\mathbf{u}| = \sup_{\alpha=1,\ldots,N}|u^\alpha|$. Assume also that for each α, $\boldsymbol{\mathfrak{F}}^\alpha(x, t, \mathbf{u})$ is nondecreasing in each component u^β of $\mathbf{u}$ with $\beta \neq \alpha$.

In the following theorem, the operators L^α are assumed to satisfy the same general conditions as in §2.

THEOREM 3. *Suppose that ω is bounded and $\Omega = \omega \times (0, T)$. Suppose also that the vector functions* $\mathbf{u}$ *and* $\boldsymbol{\mathfrak{u}}$ *satisfy*

$$
(4.2) \qquad \begin{aligned} L^\alpha[u^\alpha] + F^\alpha(x, t, \mathbf{u}) &\leqq 0 \\ L^\alpha[\mathfrak{u}^\alpha] + \mathfrak{F}^\alpha(x, t, \boldsymbol{\mathfrak{u}}) &\geqq 0 \end{aligned}
$$

in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$. with u^α, $\mathfrak{u}^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma_\alpha^*)$ for each α, and that for each α,

$$
(4.3) \qquad u^\alpha \geqq \mathfrak{u}^\alpha \quad on\ \Sigma \backslash \sigma_\alpha^*.
$$

Then for each α, $u^\alpha \geqq \mathfrak{u}^\alpha$ in $\bar{\Omega}$, $\alpha = 1, \ldots, N$.

PROOF. The proof is adapted from that used in [7] for the uniformly parabolic case. Let $\mu = \sup_{(x,t)\in\bar{\Omega}}|c^\alpha(x, t)| + M$, where M is the Lipschitz constant for $\boldsymbol{\mathfrak{F}}$. Then for $\varepsilon = 0$, define $\mathbf{v}$ by

$$
(4.4) \qquad \mathbf{v} = \mathbf{u} - \varepsilon e^{\mu t}\, \mathbf{1}.
$$

For each α, $v^\alpha = \mathfrak{u}^\alpha - \varepsilon e^{\mu t}$; so

$$
(4.5) \qquad \begin{aligned} L^\alpha[v^\alpha] &= L^\alpha[\mathfrak{u}^\alpha] - \varepsilon L^\alpha[e^{\mu t}] \\ &\geqq -\mathfrak{F}^\alpha(x, t, \boldsymbol{\mathfrak{u}}) + \varepsilon(\mu - c^\alpha)e^{\mu t} \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v} + \varepsilon e^{\mu t}\mathbf{1}) + \varepsilon(\mu - c^\alpha)e^{\mu t} \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) - M\varepsilon e^{\mu t} + \varepsilon(\mu - c^\alpha)e^{\mu t} \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) + \varepsilon(\mu - c^\alpha - M)e^{\mu t} \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) \end{aligned}
$$

for $(x, t) \in \Omega \cup \sigma_\alpha^*$. Now let $\mathbf{w} = \mathbf{u} - \mathbf{v}$. Define $\tau \in [0, T]$ by $\tau = \sup\{t \in [0, T]\colon w^\alpha \geqq 0$ in $\bar{\omega} \times [0, t]$ for $\alpha = 1, \ldots, N\}$. Since $w^\alpha \geqq \varepsilon$ for $t = 0$, $\alpha = 1, \ldots, N$, it follows by continuity that $\tau > 0$. Theorem 3 will be proved by showing that if $\tau < T$, a contradiction results.

Let $\hat{u}^\beta$ denote the $N - 1$ vector whose components are the functions u^α, $\alpha \neq \beta$. Then write $\mathfrak{F}^\alpha(x, t, \mathbf{u}) = \mathfrak{F}^\alpha(x, t, \hat{u}^\alpha, u^\alpha)$. Since by hypothesis $\mathfrak{F}^\alpha(x, t, \mathbf{u})$ is nondecreasing with respect to u^β, $\beta \neq \alpha$, it follows that on $(\Omega \cup \sigma_\alpha^*) \cap \{(x, t)\colon t \in [0, \tau]\}$,

$$
\begin{aligned}
-\mathfrak{F}^\alpha(x, t, \mathbf{u}) &= -\mathfrak{F}^\alpha(x, t, \hat{u}^\alpha, u^\alpha) \\
&\leqq -\mathfrak{F}^\alpha(x, t, \hat{\mathrm{v}}^\alpha, u^\alpha) \\
&\leqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) + M(u^\alpha - \mathrm{v}^\alpha).
\end{aligned} \tag{4.6}
$$

Combining (4.5) and (4.6) and using the definition of $\mathbf{w}$ yields, for each α,

$$
\begin{aligned}
L^\alpha[w^\alpha] &= L^\alpha[u^\alpha] - L^\alpha[\mathrm{v}^\alpha] \\
&\leqq -F^\alpha(x, y, \mathbf{u}) + \mathfrak{F}^\alpha(x, t, \mathbf{v}) \\
&\leqq -\mathfrak{F}^\alpha(x, t, \mathbf{u}) + \mathfrak{F}^\alpha(x, t, \mathbf{v}) \\
&\leqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) + Mw^\alpha + \mathfrak{F}^\alpha(x, t, \mathbf{v})
\end{aligned} \tag{4.7}
$$

in $(\Omega \cup \sigma_\alpha^*) \cap \{(x, t)\colon t \in [0, \tau]\}$. Thus, for each α,

$$
(L^\alpha - M)\,[w^\alpha] \leqq 0 \tag{4.8}
$$

in $(\Omega \cup \sigma_\alpha^*) \cap \{(x, t)\colon t \in [0, \tau]\}$. Now, $w^\alpha \geqq \varepsilon e^{\mu t} \geqq \varepsilon e^{-\mu t}$ on $\Sigma \backslash \sigma_\alpha^*$; so $w^\alpha - \varepsilon e^{-\mu t} \geqq 0$ on $\Sigma \backslash \sigma_\alpha^*$. Also,

$$
\begin{aligned}
(L^\alpha - M)[w^\alpha - \varepsilon e^{-\mu t}] &= (L^\alpha - M)[w^\alpha] - (L^\alpha - M)[\varepsilon e^{-\mu t}] \\
&\leqq -(\mu + c^\alpha - M)e^{-\mu t} \\
&\leqq 0
\end{aligned} \tag{4.9}
$$

on $(\Omega \cup \sigma_\alpha^*) \cap \{(x, t)\colon t \in [0, \tau]\}$. It follows from corollary (2.1) (or in fact from Fichera's original maximum principle, see **[4]**, **[5]**) that for each α, $w^\alpha - \varepsilon e^{-\mu t} \geqq 0$ or $w^\alpha \geqq \varepsilon e^{-\mu t} \geqq \varepsilon e^{-\mu T} > 0$ on $\bar{\omega} \times [0, \tau]$. (It is easy to check that boundary data may be omitted on $\omega \times \{\tau\}$, and condition (2.3) of Theorem 1 is unnecessary since L^α is degenerate parabolic; see the remarks at the end of §2.) But if $w^\alpha \geqq \varepsilon_0 = \varepsilon e^{-\mu t} > 0$ for all α on $\bar{\omega} \times [0, \tau)$, and $\tau \neq T$, then it follows by continuity that for some $\delta > 0$, $w^\alpha > 0$ on $\bar{\omega} \times [0, \tau + \delta]$, contradicting the definition of τ. Thus $\tau = T$, so for each α, $w^\alpha \geqq 0$ on $\bar{\Omega}$, or $u^\alpha - \mathrm{u}^\alpha + \varepsilon e^{\mu t} \geqq 0$ on $\bar{\Omega}$. Since $\varepsilon > 0$ was arbitrary, $u^\alpha \geqq \mathrm{u}^\alpha$ on $\bar{\Omega}$ for each α, as desired.

REMARK. The remarks following the proof of Theorem 1 also apply here; so σ_α^* may be replaced by any open subset of σ_α^* in (4.2) and (4.3).

The following theorem extends Theorem 3 to unbounded domains, just as Theorem 2 extends Theorem 1. Here the operators L^α are assumed to satisfy the same general conditions as in §3; in particular, their coefficients are assumed to be bounded on any bounded subset of $\bar{\Omega}$. The functions $\mathbf{F}$ and $\mathfrak{F}$ are assumed to satisfy the same hypotheses as for Theorem 3.

THEOREM 4. *Suppose that ω is unbounded, and $\Omega = \omega \times (0, T)$. Suppose that the vector functions $\mathbf{u}$ and $\mathfrak{u}$ satisfy the inequalites* (4.2) *and* (4.3), *with u^α, $\mathrm{u}^\alpha \in C^0(\bar{\Omega}) \cap C^2(\Omega \cup \sigma_\alpha^*)$ for each α. Let $M > 0$ be the Lipschitz constant for $\mathfrak{F}$. Assume that the operators L^α are such that there exists a function $H(x, t) > 0$ in $\bar{\Omega}$ such that for each α,*

$$(L^\alpha + M)[H] \leqq 0 \tag{4.10}$$

in $\Omega \cup \sigma_\alpha^*$, and

$$\limsup_{R\to\infty}\left[\inf_{\substack{(x,t)\in\bar{\Omega},\,|x|=R\\ \alpha=1,\ldots,N}}\{(u^\alpha(x, t) - \mathrm{u}^\alpha(x, t))/H(x ,t)\}\right] \geqq 0. \tag{4.11}$$

Then $u^\alpha \geqq \mathrm{u}^\alpha$ in $\bar{\Omega}$, $\alpha = 1, \ldots, N$.

PROOF. Theorem 4 follows from Theorem 3 essentially as Theorem 2 follows from Theorem 1. Some of the notation developed in the proof of Theorem 2 will be used here. Given $\varepsilon > 0$, let $\mathfrak{v} = \mathfrak{u} - H\mathbf{1}$. Then it follows as in (4.5) that for each α,

$$\begin{aligned} L^\alpha[v^\alpha] &= L^\alpha[\mathrm{u}^\alpha] - \varepsilon L^\alpha[H] \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{u}) - \varepsilon L^\alpha[H] \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) - M\varepsilon H - \varepsilon L^\alpha[H] \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) - \varepsilon(L^\alpha + M)[H] \\ &\geqq -\mathfrak{F}^\alpha(x, t, \mathbf{v}) \end{aligned} \tag{4.12}$$

in $\Omega \cup \sigma_\alpha^*$. (The last step follows from (4.10).) Thus, the vector functions $\mathbf{u}$ and $\mathbf{v}$ satisfy the system

$$\begin{aligned} L^\alpha[u^\alpha] + F^\alpha(x, t, \mathbf{u}) &\leqq 0 \\ L^\alpha[v^\alpha] + \mathfrak{F}^\alpha(x, t, \mathbf{v}) &\geqq 0 \end{aligned} \tag{4.13}$$

in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$. Now, suppose that $(x_0, t_0) \in \Omega$. Condition (4.11) implies that there exists some $R > |x_0|$ such that for $\alpha = 1, \ldots, N$, $u^\alpha - v^\alpha = u^\alpha - \mathrm{u}^\alpha + \varepsilon H \geqq 0$ for all $(x, t) \in \bar{\Omega}$ with $|x| = R$. Let $\Omega_R = \Omega \cap \{x \in \mathbf{R}^n \colon |x| < R\}$, and let $\sigma_{R\alpha}^*$ be defined for Ω_R as σ_α^* was for Ω. Then, by the same reasoning as in the proof of Theorem 2, $u^\alpha - v^\alpha \geqq 0$ on a set containing $\partial\Omega_R \backslash \sigma_{R\alpha}^*$, and (4.13) holds on the remainder of $\bar{\Omega}_R$, for each α. Thus it follows from Theorem 3 (see also the remarks at the end of the proof of Theorem 1) that $u^\alpha - v^\alpha \geqq 0$ on $\bar{\Omega}_R$, and in particular at (x_0, t_0). Thus, at (x_0, t_0), $u^\alpha - v^\alpha = u^\alpha - \mathrm{u}^\alpha - \varepsilon H \geqq 0$, for $\alpha = 1, \ldots, N$. Since $\varepsilon > 0$ was arbitrary, $u^\alpha - \mathrm{u}^\alpha \geqq 0$ at (x_0, t_0). Since $(x_0, t_0) \in \Omega$ was aribitrary, $u^\alpha - \mathrm{u}^\alpha \geqq 0$ in Ω and thus by continuity in $\bar{\Omega}$.

REMARKS. If Theorem 3 or Theorem 4 is to be used to analyze the behavior of solutions of a given system $L^\alpha[u^\alpha] + F^\alpha(x, t, \mathbf{u}) \geqq 0$ in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$, it is necessary to construct another system, $L^\alpha[\mathrm{u}^\alpha] -$

$\mathfrak{F}^\alpha(x, t, \mathfrak{u}) \leqq 0$ in $\Omega \cup \sigma_\alpha^*$, $\alpha = 1, \ldots, N$, such that $\mathbf{u}$ and $\mathfrak{u}$ may be compared. Furthermore, enough must be known about $\mathfrak{u}$ so that the comparison actually yields some information. Methods of constructing such systems are discussed in [7]. Also, for Theorem 4 to apply, it is necessary to construct the function H. In the remarks at the end of §3, some conditions are given under which such a function can be constructed. The function H may only satisfy the required conditions in a strip, $\bar{\omega} \times [0, T_1]$ but Theorem 4 can be applied first in $\bar{\omega} \times [0, T_1]$, then again with a new H in $\bar{\omega} \times [T_1, T_2]$, and so on. Conditions under which this process exhausts Ω and more general ways of constructing H are discussed in [3]. Condition (4.11) is of course satisfied if the inequalities

$$\lim_{R\to\infty}\left[\sup_{(x,t)\in\bar{\Omega}, |x|=R} (|u^\alpha(x, t)|/H(x, t))\right] = 0, \quad \alpha = 1, \ldots, N$$

and the corresponding inequalities for the functions $\mathfrak{u}^\alpha$ hold; however, (4.11) is a weaker condition. Finally, the same sort of analysis used in Theorems 3 and 4 can be used to obtain comparison results in which the Lipschitz condition is imposed on $\mathbf{F}$ rather than $\mathfrak{F}$; such a variation is useful in some applications.

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