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BRANCHING OF SOLUTIONS OF EQUATIONS INVOLVING MAPPINGS POSSESSING NULL LINEAR APPROXIMATIONS AND RELATED RESULTS

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Introduction. Let X be a linear normed space and let $\mathbf{R}$ be the field of real numbers. Bifurcation problems are usually concerned with equations of the form

$$\lambda x = Lx + T(\lambda, x) \tag{0.1}$$

where $L: X \to X$ is a bounded linear operator and $T(\lambda, x) = o(\| x \|)$ near $x = 0$, uniformly for λ in bounded intervals. In most of those problems it is assumed that $L \neq 0$ and the analysis of the problems depends on this fact. (See e.g., [1], [12], [13], [15]).

The bifurcation problem for equation (0.1) in the case where $L = 0$ possesses some new and interesting aspects. A related problem was considered by Cronin [3], who investigated the set of eigenvalues for equations of the form

$$\lambda x = T(x) \tag{0.2}$$

where $T(x) = o(\|x\|)$ near $x = 0$. The problem considered by Cronin was further studied in [10].

In the present paper we study (0.1) in the case $L = 0$, focusing our attention on the bifurcation problem and in particular on the existence of connected branches of solutions in $\mathbf{R} \times X$ issuing from (0,0). The methods that we use apply to other types of equations as well, and these will also be discussed.

Our first result (section 2) concerns the equation

$$\lambda x = T(\lambda, x) \tag{0.3}$$

where T is completely continuous and satisfies the condition: $T(\lambda, x) = o(\| x \|)$ near $x = 0$, uniformly for λ in bounded intervals. We also assume that for every $0 < r < R < +\infty$, $\lim_{|\lambda| \to 0+} \|\lambda^{-1}T(\lambda, x)\| = +\infty$, uniformly for $r \leq \|x\| \leq R$: Clearly (0.3) possesses the subset of solutions $\{(\lambda, 0) | \lambda \in \mathbf{R}\}$ which will be referred to as the set of *trivial solutions* of (0.3) This leads to the notion of *bifurcation*. $(\lambda, 0)$ is called a bifurcation point of (0.3) if every neighborhood of $(\lambda, 0)$ in $\mathbf{R} \times X$ contains nontrivial solu-

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tions of (0.3) (i.e., solutions (λ, x) such that $x \neq 0$).

Under the conditions on T mentioned above we show that (0, 0) is the unique bifurcation point of (0.3) and that the union of (0, 0) and the set of nontrivial solutions of (0.3) contains two unbounded connected branches issuing from (0, 0) one in $\mathbf{R}^+ \times X$ and the other in $\mathbf{R}^- \times X$ where $\mathbf{R}^+ = \{\lambda \in \mathbf{R} | \lambda \geqq 0\}$ and $\mathbf{R}^- = \{\lambda \in \mathbf{R} | \lambda \leqq 0\}$. Assuming also that $\lim_{\|x\| \to +\infty} \|x\|^{-1} \|T(\lambda, x)\| = +\infty$, uniformly for λ in bounded intervals, we show that the set of eigenvalues of T in (0.3) is $\mathbf{R} \backslash \{0\}$.

Nussbaum and Stuart **[11]** consider a bifurcation problem for a singular differential equation. This problem is the motivation for our study (in section 3) of an equation of the form

$$x = T(\lambda, x) \tag{0.4}$$

where T is a completely continuous mapping satisfying the condition $\lim_{\|x\| \to 0+} \|x\|^{-1} \|T(\lambda, x)\| = +\infty$, uniformly for λ in bounded closed intervals which do not contain $\lambda = 0$. Assuming in addition that $T(\lambda, 0) = 0$ for $\lambda \in \mathbf{R}$ and $T(0, x) = 0$ for $x \in X$, we prove that (0, 0) is the unique bifurcation point of (0.4) and that the closure of the set of nontribial solutions of (0.4) contains two unbounded connected branches issuing from (0, 0) one in $\mathbf{R}^+ \times X$ and the other in $\mathbf{R}^- \times X$. In section 4 we study (0.2) where T satisfies a set of conditions which can be considered as intermediary between those of the previous two sections.

In the remainder of this paper we study certain equations of the forms (0.2)–(0.4) involving positive mappings. The mapping T: $\mathbf{R}^+ \times K \to X$, where K is a cone in X, is called positive with respect to K if $T(\mathbf{R}^+ \times K) \subseteq K$. Bifurcation equations involving positive mappings were considered by many authors (see e.g. [4], [5], [8], [11], [16], [17]).

In section 5 we consider (0.3) where T satisfies on $\mathbf{R}^+ \times K$ conditions of the type mentioned in section 2. By an appropriate extension of T from $\mathbf{R}^+ \times K$ onto $\mathbf{R}^+ \times X$ and by using essentially the same arguments as in section 2 we prove the existence of an unbounded connected branch of nontrivial solutions issuing from (0, 0) in $\mathbf{R}^+ \times K$. We also obtain similar extensions of the results of sections 3 and 4.

The fact that our results were obtained in the framework of normed spaces rather than Banach spaces and the fact that they apply also to positive mappings which satisfy the required conditions only on the product of $\mathbf{R}^+$ with a cone is very useful in applications. Some applications to certain integral equations and certain boundary value problems for ordinary differential equations will be presented in section 6 and 7 respectively.

SECTION 1. Let X be an infinite dimensional linear normed space. We shall denote by B_R the ball $B_R = \{x \in X | \ \|x\| < R\}$. Let U be a subset of X. The boundary of U will be denoted by ∂U and its closure by $\mathrm{Cl}(U)$. The identity map on X will be denoted by I. In what follows the theory of the Leray-Schauder degree will be used. Let U be a bounded open subset of X and T: $\mathrm{Cl}(U) \to X$ be a completely continuous mapping (i.e., T is continuous and takes bounded subsets of $\mathrm{Cl}(U)$ into compact subsets of X). Suppose that $x - T(x) \neq 0$ for each $x \in \partial U$. Then the Leray-Schauder degree of the mapping $I - T$ with respect to the point $x = 0$ and relative to the subset U will be denoted by deg $(I - T, U)$. For the properties of the degree we refer the reader to [**14**]. The following definition of a cone is used throughout this paper. A set $K \subset X$ is called a cone if the following conditions are satisfied.

(i) *K is a non empty closed subset of X such that* $K \neq \{0\}$;

(ii) *if* $x_1, x_2 \in K$ *than* $\alpha_1 x_1 + \alpha_2 x_2 \in K$ *for every* $\alpha_1, \alpha_2 \geqq 0$;

(iii) *of each pair of vectors* $x, -x$ *at least one does not belong to* K *provided* $x \neq 0$.

The lemma presented below plays a key role in the sequel.

LEMMA 1.1. *Let X be an infinite dimensional linear normed space and let K be either a cone in X or the whole space X. Let U be a bounded open subset of X and let T*: $\mathrm{Cl}(U) \to K$ *be a completely continuous mapping. Suppose that there exists a positive number R such that $U \subseteq B_R$ and such that $\|T(x)\| > 2R$ for each $x \in K \cap \partial U$. Then the Leray-Schauder degree* $\deg(I - T, U)$ *is defined and* $\deg(I - T, U) = 0$.

PROOF. Consider at first the case in which K is a cone. We claim that deg $(I - T, U)$ is defined. Since T is completely continuous we have only to show that $x - T(x) \neq 0$ for every $x \in \partial U$. Let $x \in \partial U$. If $x \notin K$ then $x - T(x) \neq 0$ since $T(x) \in K$. If on the other hand $x \in K \cap \partial U$ then $x - T(x) \neq 0$ since $\|T(x)\| > 2R$ while $\|x\| \leqq R$.

Given $\delta > 0$, there exists a completely continuous *finite* dimensional mapping T_δ: $\mathrm{Cl}(U) \to X$ such that

$$\|T(x) - T_\delta(x)\| < \delta, \ \forall x \in \mathrm{Cl}(U) \tag{1.1}$$

and such that $T_\delta \mathrm{Cl}(U)$ is contained in the convex hall of $T\,\mathrm{Cl}(U)$. In view of the fact that $T\,\mathrm{Cl}(U) \subset K$ it follows that

$$T_\delta \mathrm{Cl}(U) \subset K. \tag{1.2}$$

The mapping T_δ can be constructed for instance by the method described in [9, Chap. 2, sec 3.2]. Without loss of generality we may assume that

$$T_\delta\,(x) \neq 0, \ \forall x \in \mathrm{Cl}(U). \tag{1.3}$$

Otherwise we replace T_δ by $T_\delta + k_\delta$ where $0 \neq k_\delta \in K$ and K_δ is sufficiently small so that $\|T(x) - T_\delta(x) - k_\delta\| < \delta, \forall x \in \mathrm{Cl}(U)$.

Now consider the function $F_1\colon [0, 1] \times \mathrm{Cl}(U) \to X$ given by

$$F_1(t, x) = x - t\,T(x) - (1 - t)T_R(x).$$

This function is a homotopy between $I - T$ and $I - T_R$ and it does not vanish on $[0, 1] \times \partial U$. Indeed if there exists $(t, x) \in [0, 1] \times \partial U$ such that $F_1(t, x) = 0$ then $x \in K$ since it is a convex combination of elements of K. Therefore it follows that

$$R \geqq \|x\| = \|T(x) + (1 - t)(T_R(x) - T(x))\| \geqq \|T(x)\| - (1 - t)\,\|T_R(x) - T(x)\| > 2R - (1 - t)R = (1 + t)R$$

which is impossible. Thus $\deg(I - T_R, U)$ is defined and

$$\deg(I - T, U) = \deg(I - T_R, U). \tag{1.4}$$

Let X_1 be a finite dimensional subspace of X containing $T_R\mathrm{Cl}(U)$. Such a subspace exists since T_R is finite dimensional. Let $U_1 = U \cap X_1$ and let T'_R denote the restriction of T_R to $\mathrm{Cl}(U_1)$. By the definition of the degree (see Definition 3.34 in [14]) we have

$$\deg(I - T_R, U) = \deg(I - T'_R, U_1) \tag{1.5}$$

where the degree on the right side of (1.5) is computed in X_1.

It is easy to verify that the function $F_2\colon [0, 1] \times \mathrm{Cl}(U_1) \to X_1$ given by

$$F_2(t, x) = tx - T'_R(x)$$

is a homotopy between $I - T'_R$ and $-T'_R$, which soes not vanish on $[0, 1] \times \partial U_1$. Indeed if $F_2(t, x) = 0$ for some $(t, x) \in [0, 1] \times \partial U_1$ then from (1.3) and (1.2) (for $\delta = R$) it follows that $tx = T'_R(x) = T_R(x) \in K$ and that $t > 0$. Thus $tx \in K$ and so $x \in K \cap \partial U$, a fact which yields the contradiction

$$R \geqq \|tx\| = \|T_R(x)\| \geqq \|T(x)\| - \|T_R(x) - T(x)\| > 2R - R = R.$$

Thus $\deg(-T'_R, U_1)$ is defined and

$$\deg(I - T'_R, U_1) = \deg(-T'_R, U_1). \tag{1.6}$$

From (1.3) we deduce that $0 \notin T'_R\mathrm{Cl}(U_1)$ and so from a property of the degree (see Theorem 3.16.4 in [14] it follows that

$$\deg(-T'_R, U_1) = 0. \tag{1.7}$$

The proposition of the lemma for the case in which K is a cone now follows from equations (1.4), (1.5), (1.6) and (1.7).

The proof in the case that $K = X$ is essentially as in the previous case. However some modifications are necessary. First, the finite dimensional subspace X_1 has to be chosen in such a way that $T_R\mathrm{Cl}(U)$ is contained in a proper subspace of X_1. This can be accomplished because X is infinite dimensional. Secondly, using this choice of X_1, we verify (1.7) by negation as follows. Suppose that $\deg(-T'_R, U_1) \neq 0$. Let V denote the component of $X_1 \backslash (-T'_R(\partial U_1))$ containg $x = 0$. V is a nonempty open subset of X_1. From a property of the degree (see Theorem 3.16.4 in [14]) we deduce that $V \subset -T'_R \operatorname{Cl}(U_1)$. But $T'_R \operatorname{Cl}(U_1) = T_R \operatorname{Cl}(U_1) \subseteq T_R \operatorname{Cl}(U)$ and $T_R \operatorname{Cl}(U)$ is contained in a proper subspace of X_1. Thus we conclude that V is contained in a proper subspace of X_1. But this conclusion contradicts the fact that V is a nonempty open subset of X_1. Thus (1.7) holds. The proof is now completed as before.

Let X be an infinite dimensional linear normed space and let $\mathbf{R}$ be the field of real numbers. Let $\mathbf{R} \times X$ be the linear normed space under the norm given by $\|(\lambda, x)\| = [\|x\|^2 + |\lambda|^2]^{1/2}$ where $\|\cdot\|$ denotes the norm in X. Let Λ be a closed and bounded interval in $\mathbf{R}$. Then $\Lambda \times X$ will be the topological subspace of $\mathbf{R} \times X$ equipped with the relative topology induced from $\mathbf{R} \times X$. Let W be a bounded open subset of $\Lambda \times X$. We shall use the notations

$$W_\lambda = \{x \in X \mid (\lambda, x) \in W\}; \quad (\partial W)_\lambda = \{x \in X \mid (\lambda, x) \in \partial W\}; \quad \lambda \in \Lambda.$$

Given a mapping $\Phi\colon \mathbf{R} \times X \to X$, we denote by $\Phi(\lambda, \cdot)$ the mapping $\Phi(\lambda, \cdot)\colon X \to X$ given by $\Phi(\lambda, \cdot)(x) = \Phi(\lambda, x)$.

The following version of the homotopy invariance property of the degree (for Banach Spaces) is due to Leray and Schauder (see Lemma 1.8 in [12]). The proof of the result for normed spaces is similar to the one given for Banach spaces and is omitted.

LEMMA 1.2. *Let X be an infinite dimensional linear normed space and let Λ be a nonempty closed and bounded interval in* $\mathbf{R}$. *Let W be a bounded open subset of $\Lambda \times X$. Let $\Phi(\lambda, x) = x - G(\lambda, x)$ where $G\colon \operatorname{Cl}(W) \to X$ is completely continuous. If $0 \notin \Phi(\lambda, (\partial W)_\lambda)$ for all $\lambda \in \Lambda$ then* $\deg(\Phi(\lambda, \cdot), W_\lambda) \equiv$ constant *for all $\lambda \in \Lambda$.*

SECTION 2. *We say that the mapping $T\colon \mathbf{R} \times X \to X$ satisfies the condition* $\mathbf{A}_1$ *if*:

(i) $T\colon (\mathbf{R}\backslash\{0\}) \times X \to X$ *is continuous*;

(ii) *Let Λ be any closed and bounded interval in* $\mathbf{R}$ *such that $0 \notin \Lambda$. Then the restriction of T to $\Lambda \times X$ is compact*;

(iii) *Let $0 < r < R < +\infty$. Then $\lim_{\|\lambda\| \to 0+} \|\lambda^{-1} T(\lambda, x)\| = +\infty$, uniformly with respect to $x \in \operatorname{Cl}(B_R) \backslash B_r$*;

(iv) *Let Λ be as in* (ii). *Then $\lim_{\|x\| \to 0+} \|x\|^{-1} \|T(\lambda, x)\| = 0$, uniformly for $\lambda \in \Lambda$.*

We shall consider in $\mathbf{R} \times X$ an equation of the form

$$\lambda x = T(\lambda, x) \tag{2.1}$$

where T satisfies the condition $\mathbf{A}_1$. Clearly, the set of solutions of equation (2.1) in $R \times X$ contains the subset $\{(\lambda, 0) | \lambda \in \mathbf{R}\backslash\{0\}\}$, henceforth referred to as *the set of trivial solutions.*

In the sequel we shall use the notations

$$\mathbf{R}^+ = \{\lambda \in \mathbf{R} \mid \lambda \geqq 0\}; \quad \mathbf{R}^- = \{\lambda \in \mathbf{R} \mid \lambda \leqq 0\}.$$

By a connected component for a topological space we mean a closed connected subset, maximal with respect to inclusion. By a connected component of a subset S of $\mathbf{R} \times X$ we mean a connected component of the *topological subspace* S equipped with the relative topology induced from $\mathbf{R} \times X$.

DEFINITION 2.1. *Let ζ_1 be the set of nontrivial solutions of equation* (2.1) *in* $(\mathbf{R}\backslash\{0\}) \times X$. *Let $\zeta = \zeta_1 \cup \{(0.\ 0)\}$ and let $\zeta^+ = \zeta \cap (\mathbf{R}^+ \times X)$ and $\zeta^- = \zeta \cap (\mathbf{R}^- \times X)$. Let C^+ be the connected component of ζ^+ containing* $(0, 0)$ *and let C^- be the connected component of ζ^- containing* $(0, 0)$.

We start with the following auxiliary result:

LEMMA 2.2. *Let X be an infinite dimensional linear normed space and suppose that the mapping T: $\mathbf{R} \times X \to X$ satisfies the condition* $\mathbf{A}_1$. Let *ζ^+ and ζ^- be as in definition* 2.1. *Then*:

(i) *ζ^+ and ζ^- are closed subsets of* $\mathbf{R} \times X$;

(ii) *Any closed and bounded subset of ζ^+ or of ζ^- is compact.*

PROOF OF (i). We shall prove the proposition for ζ^+. The proofs for ζ^- are similar. Let $(\lambda, x) \in \mathrm{Cl}(\zeta^+)$. Then $\lambda \geqq 0$. Let $\{(\lambda_n, x_n)\} \subset \zeta^+$ be a sequence converging to (λ, x). We shall consider first the case in which $\lambda > 0$. Let $\Lambda = [\lambda/2, 2\lambda]$. Without loss of generality we can assume that $\lambda_n \in \Lambda$ for all n. From the definition of ζ^+ it follows that $x_n \neq 0$ and that

$$\lambda_n x_n = T(\lambda_n, x_n);\ n = 1, 2, \ldots . \tag{2.2}$$

Taking the limits on both sides of the equations (2.2) and using the continuity of T we deduce that

$$\lambda x = T(\lambda, x). \tag{2.3}$$

Let us show that $x \neq 0$. Assume on the contrary that $x = 0$. Then $x_n \to 0$. From (2.2) we find that $\lambda_n = \|x_n\|^{-1}\|T(\lambda_n, x_n)\|$; $n = 1, 2, \ldots$. From these equalities and from the condition $\mathbf{A}_1$ (iv) it follows that $\lambda_n \to 0$, but this contradicts the facts that $\lambda_n \to \lambda$ and that $\lambda > 0$. Thus $x \neq 0$ and so (λ, x) is a nontrivial solution of (2.1), i.e. $(\lambda, x) \in \zeta^+$. Assume now that $\lambda = 0$. We shall show that $x = 0$. Suppose on the contrary that $x \neq 0$. Let $\|x\| = r > 0$. We can assume that $x_n \in \mathrm{Cl}(B_{2r})\backslash B_{r/2}$; $n = 1, 2, \ldots$. By the definition of ζ^+ and the fact that $\lambda = 0$, we conclude that $\lambda_n \to 0+$.

From (2.2) we find that $\|x_n\| = \|\lambda_n^{-1}T(\lambda_n, x_n)\|$; $n = 1, 2, \ldots$. These equalities and condition $\mathbf{A}_1$ (iii) imply that $\|x_n\| \to +\infty$. But this contradicts the fact that $\|x_n\| \to \|x\| = r < +\infty$. Thus $x = 0$ and so $(\lambda, x) = (0, 0) \in \zeta^+$. Hence proposition (i) holds.

PROOF OF (ii). Let A be a bounded closed subset of ζ^+. Clearly we have only to show that A is sequentially compact. Let $\{(\lambda_n, x_n)\}$ be a sequence in A.

Suppose at first that $\lambda_n \to 0$. Then we claim that $x_n \to 0$. Indeed, if this does not take place then there exists a subsequence $\{x_{n_k}\}$ and $\varepsilon > 0$ such that $\|x_{n_k}\| \geqq \varepsilon$ for all k. From this fact and from the definition of ζ^+ it follows that $\lambda_{n_k} > 0$. Using the boundedness of A and condition $\mathbf{A}_1$ (iii) we deduce, as in the proof of (i), that $\|x_{n_k}\| \to +\infty$, which contradicts the boundedness of A. Thus $(\lambda_n, x_n) \to (0, 0)$. Assume now that $\{\lambda_n\}$ does not converge to 0. Then there exists a subsequence $\{\lambda_{n_k}\}$ and $\lambda > 0$ such that $\lambda_{n_k} \to \lambda$. We can assume that $\{(\lambda_{n_k}, x_{n_k})\}$ is contained in $\Lambda \times X$ where $\Lambda = [\lambda/2, 2\lambda]$. From the compactness of T on $\Lambda \times X$ it follows that there exists a subsequence $\{\lambda_{n_{k_l}}, x_{n_{k_l}})\}$ and $x \in X$ such that $T(\lambda_{n_{k_l}}, x_{n_{k_l}}) \to x$. Using also the facts that $(\lambda_{n_{k_l}}, x_{n_{k_l}})$ satisfies equation (2.2) and that $\lambda_{n_k} \to \lambda > 0$, we deduce that $x_{n_{k_l}} \to \lambda^{-1}x$. Thus $(\lambda_{n_{k_l}}, x_{n_{k_l}}) \to (\lambda, \lambda^{-1}x)$. Hence A is sequentially compact and (ii) holds.

In the sequel we shall use the notation $\mathscr{B}_R = \{(\lambda, x) \in \mathbf{R} \times X \mid \|(\lambda, x)\| < R\}$.

We are now ready to prove

THEOREM 2.3. *Let X be an infinite dimensional linear normed space and suppose that the mapping T: $\mathbf{R} \times X \to X$ satisfies the condition $\mathbf{A}_1$. Let C^+ and C^- be as in Definition* 2.1. *Then C^+ and C^- are not bounded.*

PROOF. We shall prove that C^+ is not bounded. The proof for C^- is similar. Assume on the contrary that C^+ is bounded. Let R be a positive number such that $C^+ \subset \mathscr{B}_R$. Let $A = \zeta^+ \cap \mathrm{Cl}(\mathscr{B}_R)$. From Lemma 2.2 it follows that A is a compact metric space under the induced topology from $\mathbf{R} \times X$. By construction $C^+ \cap \partial\mathscr{B}_R = \varnothing$, and thus C^+ and $\zeta^+ \cap \partial\mathscr{B}_R$ are disjoint closed subsets of A. Moreover, there does not exist a connected subset of A meeting both of these sets. Indeed, such a subset would be contained in C^+ and this would imply the contradiction $C^+ \cap \partial\mathscr{B}_R \neq \varnothing$. By a theorem of Whyburn [18, Chap. 1. (9.3)] there exist two disjoint compact subsets $A_1, A_2 \subset A$ such that $C^+ \subset A_1$, $\zeta^+ \cap \partial\mathscr{B}_R \subset A_2$ and $A = A_1 U A_2$. We claim that $A_1 \cap [(\mathbf{R} \times X)\backslash\mathscr{B}_R] = \varnothing$. Indeed (by the very definitions)

$$A_1 \cap [(\mathbf{R} \times X)\backslash\mathscr{B}_R] \subseteq A_1 \cap \partial\mathscr{B}_R \subseteq A_1 \cap \zeta^+ \cap \partial\mathscr{B}_R \subseteq A_1 \cap A_2 = \varnothing.$$

Let us choose a fixed $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{3} \min (\mathrm{dist}\,(A_1, A_2); \mathrm{dist}\,(A_1, (\mathbf{R} \times X)\backslash \mathscr{B}_R)). \tag{2.4}$$

Let U be the ε-neighborhood of A_1 in $\mathbf{R} \times X$. From the definitions of A and U it follows that $\mathrm{Cl}(U) \subset \mathscr{B}_R$ and that $\zeta^+ \cap \partial U = \varnothing$.

By using the condition A_1 (iii) we now choose a fixed $\lambda_0 > 0$ such that

$$0 < \lambda_0 < \varepsilon/2 \tag{2.5}$$

and such that

$$\|\lambda_0^{-1} T(\lambda_0, x)\| > 2R \text{ for each } x \,\varepsilon\, \mathrm{Cl}(B_R)\backslash B_{\varepsilon/2}. \tag{2.6}$$

Let $\Lambda = [\lambda_0, R]$. By using condition $\mathbf{A}_1$(iv) we choose a fixed $r \in (0, \varepsilon/2)$ such that $\|x\|^{-1}\|T(\lambda, x)\| < \lambda_0/2$ for each $x \in \mathrm{Cl}(B_r)\backslash\{0\}$ and for all $\lambda \in \Lambda$. In particular

$$\|\lambda^{-1}T(\lambda, x)\| < \|x\|/2 \text{ for each } x \in \mathrm{Cl}(B_r)\backslash\{0\} \text{ and for all } \lambda \in \Lambda. \tag{2.7}$$

Let $V = \Lambda \times \mathrm{Cl}(B_r)$ and let $W = [U \cap (\Lambda \times X)]\backslash V$. Then W is a bounded open subset of $\Lambda \times X$. Let G: $\mathrm{Cl}(W) \to X$ be defined by $G(\lambda, x) = \lambda^{-1}T(\lambda, x)$ and let $\Phi(\lambda, x) \equiv x - G(\lambda, x)$. From the fact that $0 \notin \Lambda$ and from condition $\mathbf{A}_1$ it follows that G is completely continuous on $\mathrm{Cl}(W)$. We claim that $0 \notin \Phi(\lambda, (\partial W)_\lambda)$ for all $\lambda \in \Lambda$. Indeed assume on the contrary that there exist $\lambda \in \Lambda$ and $x \in (\partial W)_\lambda$ such that $\Phi(\lambda, x) = 0$. Then $(\lambda, x) \in \partial W$ and (λ, x) is a solution of equation (2.1). From the definitions of W and V we deduce that $\|x\| \geqq r$ and so (λ, x) is a nontrivial solution of (2.1), i.e. $(\lambda, x) \in \zeta^+$. From the fact that $\zeta^+ \cap \partial U = \varnothing$ it follows that $(\lambda, x) \in \partial V$ and so $\|x\| = r$. Moreover, $x = \lambda^{-1}T(\lambda, x)$ and thus $\|\lambda^{-1}T(\lambda, x)\| = \|x\|$. The last equality clearly contradicts (2.7). Hence, $0 \notin \Phi(\lambda, (\partial W)_\lambda)$ for all $\lambda \in \Lambda$. Applying Lemma 1.2 we have

$$\deg\,(\Phi(\lambda, \cdot), W_\lambda) \equiv \gamma = \text{constant for each } \lambda \in \Lambda. \tag{2.8}$$

It is clear that $W_R = \varnothing$ since $W \subset U \subset \mathscr{B}_R$. From this fact and (2.8) it follows that $\gamma = 0$ and in particular that $\deg\,(\Phi(\lambda_0, \cdot), W_{\lambda_0}) = 0$. But $W_{\lambda_0} = U_{\lambda_0}\backslash \mathrm{Cl}(B_r)$ and so

$$\deg\,(\Phi(\lambda_0, \cdot), U_{\lambda_0}\backslash \mathrm{Cl}(B_r)) = 0. \tag{2.9}$$

Let us show that $\mathrm{Cl}(B_{\varepsilon/2}) \subset U_{\lambda_0}$. Let $x \in \mathrm{Cl}(B_{\varepsilon/2})$, i.e. $\|x\| \leqq \varepsilon/2$. Since $0 < \lambda_0 < \varepsilon/2$ we have $(\lambda_0, x) \in \mathscr{B}_\varepsilon$. From the definition of U and the fact that $(0, 0) \in C^+ \subset A_1$ it follows that $\mathscr{B}_\varepsilon \subset U$. In particular $(\lambda_0, x) \in U$ and thus $x \in U_{\lambda_0}$.

The set U_{λ_0} is a bounded open subset of X and $\mathrm{Cl}(B_{\varepsilon/2}) \subset U_{\lambda_0} \subset B_R$; therefore $\partial U_{\lambda_0} \subset \mathrm{Cl}(B_R)\backslash B_{\varepsilon/2}$. From this fact and (2.6) we deduce that

$$\|\lambda_0^{-1}T(\lambda_0, x)\| > 2R \text{ for each } x \in \partial U_{\lambda_0}. \tag{2.10}$$

Moreover, $\lambda_0^{-1}T(\lambda_0, \cdot)$: $\mathrm{Cl}(U_{\lambda_0}) \to X$ is completely continuous and

$\Phi(\lambda_0, \cdot) = I - \lambda_0^{-1}T(\lambda_0, \cdot)$. Hence the conditions of Lemma 1.1 are satisfied for the subset U_{λ_0} and the mapping $\lambda_0^{-1}T(\lambda_0, \cdot)$. Applying that lemma we have

$$\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}) = 0. \tag{2.11}$$

Next we show that $\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}) = 1$, thus reaching a contradiction. From (2.7) (for λ_0 and $\|x\| = r$) it follows that $\deg(\Phi(\lambda_0, \cdot), B_r)$ is defined and that $\Phi(\lambda_0, \cdot)$ and I are homotopic on ∂B_r. Hence

$$\deg(\Phi(\lambda_0, \cdot), B_r) = 1. \tag{2.12}$$

From (2.9), (2.11) and the fact the $\mathrm{Cl}(B_r) \subset U_{\lambda_0}$ (recall that $\mathrm{Cl}(B_{\varepsilon/2}) \subset U_{\lambda_0}$ and $r < \varepsilon/2$) it follows that

$$\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}) = \deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}\backslash \mathrm{Cl}(B_r)) + \deg(\Phi(\lambda_0, \cdot), B_r) = 1.$$

The last equality contradicts (2.11). This contradiction concludes the proof that C^+ is not bounded.

We say that λ is an eigenvalue of the mapping T if there exists $x \neq 0$ such that $\lambda x = T(\lambda, x)$. As a consequence of the theorem we have:

COROLLARY 2.4. *Suppose that the conditions of Theorem* 2.3 *are satisfied. Then*:

(i) (0, 0) *is the unique bifurcation point of equation* (2.1);

(ii) *There exist δ_1 and δ_2 such that $0 < \delta_1, \delta_2 \leqq +\infty$ and such that the set of eigenvalues of T in equation* (2.1) *contains the subset* $(-\delta_2, 0) \cup (0, \delta_1)$;

(iii) *Suppose also that T satisfies the following condition: for each $a > 0$; $\lim_{\|x\|\to+\infty}\|x\|^{-1}\|T(\lambda, x)\| = +\infty$, uniformly with respect to $0 < |\lambda| \leqq a$. Then the set of eigenvalues of T is $\mathbf{R}\backslash\{0\}$.*

PROOF OF (i). The fact that (0, 0) is a bifurcation point of equation (2.1) is a direct consequence of the fact that $C^+ \setminus \{(0, 0)\} \neq \emptyset$. From the definitions of ζ^+ and ζ^- and the fact that these sets are closed we conclude that (0, 0) is the unique bifurcation point of (2.1).

PROOF OF (ii). Let C_p^+ and C_p^- be the projections of C^+ and C^- on $\mathbf{R}$ respectively. C^+ is connected and so C_p^+ is a connected subset of $\mathbf{R}^+$. In addition $0 \in C_p^+$ (since $(0, 0) \in C^+$) and $C_p^+\backslash\{0\} \neq \emptyset$ (by the definition of ζ^+ and the fact that $C^+\backslash\{(0, 0)\} \neq \emptyset$). Consequently there exist $\delta_1 \in (0, +\infty]$ such that $C_p^+ = [0, \delta_1)$ or $C_p^+ = [0, \delta_1]$. Clearly $C_p^+\backslash\{0\}$ is contained in the set of eigenvalues of T. A similar argument concerning C_p^- completes the proof of (ii).

PROOF OF (iii). Let C_p^+ and C_p^- be as in the proof of (ii). Clearly we have only to show that $C_p^+ = \mathbf{R}^+$ and $C_p^- = \mathbf{R}^-$. Suppose for example that $C_p^+ \subseteq [0, \delta_1]$ where $\delta_1 < +\infty$. Then C^+ contains a sequence $\{(\lambda_n, x_n)\}$

such that $\lambda_n \in (0, \delta_1]$ and $\|x_n\| > n$; $n = 1, 2, \ldots$ (since C^+ is not bounded). From equation (2.1) it follows that $\|x_n\|^{-1}\|T(\lambda_n, x_n)\| = \lambda_n \leqq \delta_1$; $n = 1, 2, \ldots$ and thus $\limsup_{n\to\infty} \|x_n\|^{-1}\|T(\lambda_n, x_n)\| \leqq \delta_1 < +\infty$. This contradicts the condition given in (iii) since $\|x_n\| \to +\infty$ and $\lambda_n \in (0, \delta_1]$. Hence $C_p^+ = \mathbf{R}^+$. Similarly one can show that $C_p^- = \mathbf{R}^-$.

SECTION 3. *We say that the mapping T: $\mathbf{R} \times X \to X$ satisfies the condition* $\mathbf{A}_2$ *if*:

(i) *T is completely continuous*;

(ii) *$T(\lambda, 0) = 0$ for each $\lambda \in \mathbf{R}$*;

(iii) *$T(0, x) = 0$ for each $x \in X$*;

(iv) *Let Λ be any closed and bounded interval in $\mathbf{R}$ such that $0 \notin \Lambda$. Then* $\lim_{\|x\|\to 0+}\|x\|^{-1}\|T(\lambda, x)\| = +\infty$, *uniformly for $\lambda \in \Lambda$.* We shall consider in $\mathbf{R} \times X$ and equation of the form

$$x = T(\lambda, x) \tag{3.1}$$

where T satisfies the condition $\mathbf{A}_2$. From $\mathbf{A}_2$ (ii) it follows that the set of solutions of equation (3.1) contains the subset of the trivial solutions.

The next theorem provides a result similar to that of Theorem 2.3.

THEOREM 3.1. *Let X be an infinite dimensional linear normed space and suppose that the mapping T: $\mathbf{R} \times X \to X$ satisfies the condition $\mathbf{A}_2$. Let ζ_1 be the set of nontrivial solutions of equation* (3.1) *in $\mathbf{R} \times X$. Let ζ^+, ζ^-, C^+ and C^- be defined as in Defintion* 2.1 *with $\zeta = \zeta_1 \cup \{(0, 0)\}$. Then C^+ and C^- are not bounded.*

PROOF. We shall prove that C^+ is not bounded. The proof for C^- is similar. Assume on the contrary that C^+ is bounded. Let $\mathscr{B}_R$ be a ball containing C^+ and let U be an open set constructed as in the proof of Theorem 2.3. Let λ_0 be a positive number such that $[0, \lambda_0] \times \mathrm{Cl}(B_{\lambda_0}) \subset U$. Such a number exists because $(0, 0) \in C^+ \subset U$. Using condition $\mathbf{A}_2$ (iv) we choose $r \in (0, \lambda_0)$ such that

$$\|x\|^{-1}\|T(\lambda, x)\| > 2 \quad \forall(\lambda, x) \in [\lambda_0, R] \times \mathrm{Cl}(B_r)\backslash\{0\}). \tag{3.2}$$

Now let Φ: $\mathbf{R}^+ \times X \to X$ be defined by $\Phi(\lambda, x) = x - T(\lambda, x)$. As in the proof of Theorem 2.3 we show that

$$\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}\backslash \mathrm{Cl}(B_r)) = 0. \tag{3.3}$$

Let $\Lambda = [0, \lambda_0]$. Clearly $\mathrm{Cl}(B_r) \subset U_\lambda$ for all $\lambda \in \Lambda$. From this fact and the fact that $\zeta^+ \cap \partial U = \varnothing$ we deduce that $0 \notin \Phi(\lambda, (\partial U)_\lambda)$ for all $\lambda \in \Lambda$. Applying Lemma 1.2 we obtain in particular

$$\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}) = \deg(\Phi(0, \cdot), U_0) = 1. \tag{3.4}$$

The second equality in (3.4) holds because $\Phi(0, \cdot) = I$ and $0 \in U_0$. From

(3.2) it follows that the conditions of Lemma 1.1 are satisfied for the subset B_r and the mapping $T(\lambda_0, \cdot)$. Consequently

$$\deg(\Phi(\lambda_0, \cdot), B_r) = 0. \tag{3.5}$$

Combining (3.3), (3.5) and the fact that $\mathrm{Cl}(B_r) \subset U_{\lambda_0}$, we conclude that

$$\deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}) = \deg(\Phi(\lambda_0, \cdot), U_{\lambda_0}\backslash \mathrm{Cl}(B_r)) + \deg(\Phi(\lambda_0, \cdot), B_r) = 0.$$

which contradicts (3.4). Thus C^+ is not bounded.

REMARK. Under the conditions of Theorem 3.1 one can obtain a corollary similar to Corollary 2.4. Details are omitted.

SECTION 4. We shall now consider an equation of the form

$$\lambda x = T(x) \tag{4.1}$$

where T satisfies a condition which can be considered as intermediary between $\mathbf{A}_1$ and $\mathbf{A}_2$.

We say that the mapping $T: X \to X$ satisfies the condition $\mathbf{A}_3$ *if*

(i) *T is completely continuous*;

(ii) *Let* $0 < r < R < +\infty$. *Then* $\inf\{\|T(x)\| \mid r \leqq \|x\| \leqq R\} > 0$;

(iii) $\limsup_{\|x\| \to 0+} \|x\|^{-1}\|T(x)\| = M$ *where* $0 < M < +\infty$.

From $\mathbf{A}_3$ (iii) it follows that the set of solutions of equation (4.1) in $\mathbf{R} \times X$ contains the subset of the trivial solutions.

DEFINITION 4.1. *Let ξ_1 be the set of nontrivial solutions of* (4.1) *in* $\mathbf{R} \times X$. *Let $\xi = \xi_1 U([-M, M] \times \{0\})$ where M is given in the condition* $\mathbf{A}_3$ (iii). *Let $\xi^+ = \xi \cap (\mathbf{R}^+ \times X)$ and $\xi^- = \xi \cap (\mathbf{R}^- \times X)$. Let D^+ be the connected component of ξ^+ containing* $(0, 0)$ *and let D^- be the connected component of ξ^- containing* $(0, 0)$.

Let T be a mapping satisfying the condition $\mathbf{A}_3$ and let $T': \mathbf{R} \times X \to X$ be the mapping defined by

$$T'(\lambda, x) = \|x\| T(x). \tag{4.2}$$

Together with equation (4.1) we consider the auxiliary equation

$$\lambda x = T'(\lambda, x). \tag{4.3}$$

One can easily verify that T' satisfies the condition $\mathbf{A}_1$ when T satisfies the condition $\mathbf{A}_3$. Let ζ^+, ζ^-, C^+ and C^- be as in Definition 2.1 with respect to (4.3) (instead of (2.1) there). $C^+\backslash\{(0, 0)\}$ and $C^-\backslash\{(0, 0)\}$ (by their very definitions) contain only nontrivial solutions of (4.3) and thus the sets

$$A^+ = \{(\lambda\|x\|^{-1}, x) \mid (\lambda, x) \in C^+\backslash\{(0, 0)\}\}; \tag{4.4}$$
$$A^- = \{(\lambda\|x\|^{-1}, x) \mid (\lambda, x) \in C^-\backslash\{(0, 0)\}\}$$

are well defined. We shall need the following

LEMMA 4.2. *Let X be an infinite dimensional linear normed space. Let $T: X \to X$ be a mapping satisfying the condition* $\mathbf{A}_3$ *and let T' be defined by* (4.2). *Let D^+ and D^- be as in Definition* 4.1 *and let C^+ and C^- be as in*

DEFINITION 2.1 (*for equation* (4.3)). *Let A^+ and A^- be given by* (4.4). *Then*

$$D^+ = A^+ \cup ([0, M] \times \{0\}) \text{ and } D^- = A^- \cup ([-M, 0] \times \{0\}).$$

PROOF. We shall give the proof for D^+. The proof for D^- is similar. Let

$$A \equiv A^+ \cup ([0, M] \times \{0\}).$$

Proof that A is connected. Suppose on the contrary that A is not connected. Then there exist two disjoint nonempty subsets A_1 and A_2 such that $A = A_1 \cup A_2$ and such that A_1 and A_2 are closed in the topological subspace A equipped with the relative topology induced from $R \times X$. $[0, M] \times \{0\}$ is a connected subset of A and thus it is contained in one of these subsets, say in A_1. Then $A_2 \subset A^+$. Clearly $(\lambda, x) \in A^+$ if and only if $(\lambda\|x\|, x) \in C^+\backslash\{(0, 0)\}$.

Let us show that

$$\inf\{\|x\| \,|(\lambda, x) \in A_2\} \equiv \delta > 0. \tag{4.5}$$

If $\delta = 0$ then there exists a sequence $\{(\lambda_n, x_n)\} \subset A_2$ such that $\|x_n\| \to 0$. $A_2 \subset A^+$ and thus $\{(\lambda_n\|x_n\|, x_n)\} \subset C^+\backslash\{(0, 0)\}$. From the definition of C^+ it follows that $x_n \neq 0$ and that $\lambda_n\|x_n\|x_n = T'(\lambda_n, x_n) = \|x_n\|\, T(x_n)$. Consequently $\lambda_n x_n = T(x_n)$ and so $\lambda_n = \|x_n\|^{-1}\|T(x_n)\|$. From condition $\mathbf{A}_3$ (iii) we deduce that $0 \leqq \limsup_{n\to\infty} \lambda_n \leqq M$. Without loss of generality we can thus assume that $(\lambda_n, x_n) \to (\lambda, 0)$ where $(\lambda, 0) \in [0, M] \times \{0\} \subset A_1$. But then we arrive at the contradiction: $(\lambda, 0) \in A_2 \cap A_1 = \varnothing$. Thus $\delta > 0$.

Let $A_i' \equiv \{(\lambda\|x\|, x)|(\lambda, x) \in A_i\}$; $i = 1, 2$. Using (4.5) and the fact that A_1 and A_2 are disjoint nonempty subsets we deduce that A_1' and A_2' are disjoint nonempty subsets. Let us show that $C^+ = A_1' \cup A_2'$. Let $(\lambda, x) \in A_1 \cup A_2 = A$. If $x = 0$ then $(\lambda\|x\|, x) = (0, 0) \in C^+$. If $x \neq 0$ then $(\lambda, x) \in A^+$ and so $(\lambda\|x\|, x) \in C^+\backslash\{(0, 0)\}$. Thus if $(\lambda, x) \in A_1 \cup A_2$ then $(\lambda\|x\|, x) \in C^+$. From this fact it follows that $A_1' \cup A_2' \subseteq C^+$. Assume now that $(\lambda, x) \in C^+$. If $x = 0$ then $\lambda = 0$ and so $(\lambda, x) = (0, 0) \in A_1'$. If $x \neq 0$ then $(\lambda, x) \in C^+\backslash\{(0, 0)\}$ and so $(\lambda\|x\|^{-1}, x) \in A^+ \subset A_1 \cup A_2$. Thus $(\lambda, x) \in A_1' \cup A_2'$. Hence $C^+ \subseteq A_1' \cup A_2'$. Consequently $C^+ = A_1' \cup A_2'$.

We shall now show that A_1' and A_2' are closed subsets of the topological subspace C^+. Let $(\lambda, x) \in C^+$ and let $\{(\lambda_n, x_n)\} \subset A_1'$ such that $(\lambda_n, x_n) \to (\lambda, x)$. If $(\lambda, x) = (0, 0)$ then $(\lambda, x) \in A_1'$. On the other hand if $(\lambda, x) \in C^+\backslash\{(0,0)\}$ then in particular $x \neq 0$ and so we can assume that $x_n \neq 0$ for all n. Then $\{(\lambda_n\|x_n\|^{-1}, x_n)\} \subset A_1$ and $(\lambda\|x\|^{-1}, x) \in A^+ \subset A$. In addition

$(\lambda_n\|x_n\|^{-1}, x_n) \to (\lambda\|x\|^{-1}, x)$. But A_1 is closed in A and so $(\lambda\|x\|^{-1}, x) \in A_1$. Thus $(\lambda, x) \in A_1'$. Consequently A_1' is closed in C^+. Let now $(\lambda, x) \in C^+$ and let $\{(\lambda_n, x_n)\} \subset A_2'$ be such that $(\lambda_n, x_n) \to (\lambda, x)$. From (4.5) and the definition of A_2' it follows that $\|x_n\| \geqq \delta$ and so $\|x\| \geqq \delta$. Thus $\{(\lambda_n\|x_n\|^{-1}, x_n)\} \subset A_2$ and $(\lambda\|x\|^{-1}, x) \in A^+ \subset A$. In addition $(\lambda_n\|x_n\|^{-1}, x_n) \to (\lambda\|x\|^{-1}, x)$ and A_2 is closed in A. Thus $(\lambda\|x\|^{-1}, x) \in A_2$ and so $(\lambda, x) \in A_2'$. Consequently A_2' is closed in C^+. We have thus proved that if A is notconnected then C^+ is not connected But C^+ is connected and so A is connected.

Proof that $D^+ = A$: Let ξ_1, ξ and ξ^+ be as in Definition 4.1. One can easily verify that (λ, x) is a nontrivial solution of equation (4.3) if and only if $(\lambda\|x\|^{-1}, x)$ is a nontrivial solution of equation (4.1). From this fact and the definitions of A^+ and C^+ it follows that $A^+ \subset \xi_1 \cap (\mathbf{R}^+ \times X)$. Thus $A \subset \xi^+$. Moreover A is connected and $(0, 0) \in A$. Hence $A \subseteq D^+$.

Let G: $\mathbf{R} \times X \to \mathbf{R} \times X$ be defined by $G(\lambda, x) = (\lambda\|x\|, x)$. Clearly $G(\xi^+) \subseteq \zeta^+$ and in particular $G(D^+) \subseteq \zeta^+$. G is continuous and so $G(D^+)$ is a connected subset of ζ^+. Moreover $(0, 0) \in G(D^+)$ and thus $G(D^+) \subseteq C^+$. Let $(\lambda, x) \in D^+$. If $x = 0$ then $(\lambda, x) = (\lambda, 0) \in [0, M] \times \{0\} \subset A$. If on the other hand $x \neq 0$ then $(\lambda\|x\|, x) = G(\lambda, x) \in C^+\backslash\{(0, 0)\}$ and thus $(\lambda, x) \in A^+ \subset A$. Hence $D^+ \subseteq A$. Consequently $D^+ = A$.

Now an application of Theorem 2.3 yields

THEOREM 4.3. *Let X be an infinite dimensional linear normed space. Suppose that the mapping T: $X \to X$ satisfies the condition* $\mathbf{A}_3$. *Let D^+ and D^- be as in Definition* 4.1. *Then D^+ and D^- are not bounded.*

PROOF. We shall prove that D^+ is not bounded. The proof for D^- is similar. We have only to show that $D^+\backslash\mathscr{B}_R \neq \emptyset$ where R is an arbitrary positive number. Let then R be an arbitrary positive number. Let T' be defined by (4.2). Then T' satisfies the condition $\mathbf{A}_1$. Let, C^+, A^+ and A be as in Lemma 4.2. From Theorem 2.3 it follows that C^+ is not bounded. Thus there exists (λ, x) such that $(\lambda, x) \in C^+\backslash([0, R^2] \times \mathrm{Cl}(B_R))$. From the definition of A^+ it follows that $(\lambda\|x\|^{-1}, x) \in A^+ \subset A$. From Lemma 4.2 it follows that $(\lambda\|x\|^{-1}, x) \in D^+$. One can easily verify that $\|(\lambda\|x\|^{-1}, x)\| > R$. Thus $(\lambda\|x\|^{-1}, x) \in D^+\backslash\mathscr{B}_R$ and so $D^+\backslash\mathscr{B}_R \neq \emptyset$.

COROLLARY 4.4. *Suppose that the conditions of Theorem* 4.3 *are satisfied. Then equation* (4.1) *has at least one bifurcation point in* $[0, M] \times \{0\}$ *where M is given in condition* $\mathbf{A}_3$ (iii).

PROOF. Suppose that equation (4.1) has no bifurcation points in $[0, M] \times \{0\}$. Then each point in $[0, M] \times \{0\}$ has a bounded open neighborhood which does not contain nontrivial solutions of equation (4.1). By using standard compactness arguments we can thus prove the existence of a bounded open subset U such that $[0, M] \times \{0\} \subset U$ and such that $\mathrm{Cl}(U)$

does not contain nontrivial solutions of (4.1). From this fact and the definition of D^+ it follows that $D^+ \cap \mathrm{Cl}(U) = [0, M] \times \{0\} = D^+ \cap U \neq \varnothing$. Moreover $D^+ \backslash \mathrm{Cl}(U) \neq \varnothing$ since U is bounded and D^+ is not bounded. Thus D^+ is not connected. We have arrived at a contradiction which concludes the proof.

SECTION 5. Let X be an infinite dimensional linear normed space. Let K be a cone in X. *The mapping* T: $\mathbf{R}^+ \times K \to X$ *is called a positive mapping with respect to the cone* K (*or briefly a positive mapping*) *if* $T(\mathbf{R}^+ \times K) \subset K$. In the sequel we shall consider equations involving mappings which are positive with respect to some cones in X. Our treatment of such equations will be based on the results of the previous sections and on the following extension lemma.

LEMMA 5.1. *Let X be a linear normed space and let K be a cone in X. Then there exists a continuous retraction f: $X \to K$ satisfying*:

(i) $f(x) = x$ *for each* $x \in K$;

(ii) $\|f(x)\| \leqq 3\|x\|$ *for each* $x \in X$.

PROOF. Using Dugundji's extension theorem [6] it can be shown (see for example [2, Chap. II Corollary 3.4]) that there exists a continuous retraction f: $X \to K$ satisfying (i) and such that

$$\|f(x) - x\| \leqq 2 \inf_{k \in K} \|x - k\| \text{ for each } x \in X.$$

Condition (ii) follows from the last inequality and the fact that $0 \in K$.

In the sequel we shall use the notation K-lim to indicate that the corresponding limit is taken only with respect to elements of the cone K.

We say that the positive mapping T: $\mathbf{R}^+ \times K \to K$ *satisfies the condition* $\mathbf{K}_1$ *if*:

(i) T: $(\mathbf{R}^+ \backslash \{0\}) \times K \to K$ *is continuous*;

(ii) *Let Λ be a closed and bounded interval in* $\mathbf{R}^+$ *such that* $0 \notin \Lambda$. *Then* T: $\Lambda \times K \to K$ *is compact*;

(iii) *Let* $0 < r < R < +\infty$. *Then* $\lim_{\lambda \to 0+} \|\lambda^{-1} T(\lambda, x)\| = +\infty$, *uniformly for* $x \in K \cap (\mathrm{Cl}(B_R) \backslash B_r)$;

(iv) *Let Λ be as in* (ii). *Then* $K\text{-}\lim_{\|x\| \to 0+} \|x\|^{-1} \|T(\lambda, x)\| = 0$, *uniformly for* $\lambda \in \Lambda$.

We shall now consider the equation

$$\lambda x = T(\lambda, x) \tag{5.1}$$

in which T is a positive mapping satisfying the condition $\mathbf{K}_1$.

THEOREM 5.2. *Let X be an infinite dimensional linear normed space and let K be a cone in X. Let the mapping* T: $\mathbf{R}^+ \times K \to K$ *satisfy the condition*

$\mathbf{K}_1$. *Let S_1 be the set of nontrivial solutions of* (5.1) *in* $(\mathbf{R}^+\backslash\{0\}) \times K$ *and let* $S = S_1 \cup \{(0, 0)\}$. *Let C_K be the connected component of S containing* (0, 0). *Then C_K is not bounded.*

PROOF. Let $f\colon X \to K$ be as in Lemma 5.1 and let $T'\colon \mathbf{R}^+ \times X \to K$ be defined by

$$T'(\lambda, x) = T(\lambda, f(x)). \tag{5.2}$$

Together with equation (5.1) we consider the auxiliary equation

$$\lambda x = T'(\lambda, x). \tag{5.3}$$

Let ζ^+ and C^+ be as in Definition 2.2 for (5.3) instead of (2.1) there and let S and C_K be as in the formulation of the Theorem. We claim that $C_K = C^+$. In order to prove this claim we have only to show that $S = \zeta^+$. Let $(\lambda, x) \in \zeta^+\backslash\{(0, 0)\}$. Then $(\lambda, x) \in (\mathbf{R}^+\backslash\{0\}) \times X$ and it is a nontrivial solution of (5.3). Hence

$$\lambda x = T'(\lambda, x) = T(\lambda, f(x)) \in K,$$

since from its very definition $T'(\lambda, x) \in K$. But $\lambda > 0$ and so $x \in K$. Thus $f(x) = x$ and so $\lambda x = T(\lambda, f(x)) = T(\lambda, x)$. Hence $(\lambda, x) \in S$. Thus $\zeta^+ \subseteq S$ (clearly $(0, 0) \in \zeta^+ \cap S$).

Let now $(\lambda, x) \in S\backslash\{(0, 0)\}$. Then $(\lambda, x) \in (\mathbf{R}^+\backslash\{0\}) \times K$ and $\lambda x = T(\lambda, x)$. But $x \in K$ and so $f(x) = x$ and thus $T(\lambda, x) = T'(\lambda, x)$. Hence $(\lambda, x) \in \zeta^+$. Thus $S \subseteq \zeta^+$. Hence $S = \zeta^+$ and in particular $C_K = C^+$.

In view of the above considerations it is sufficient to show that C^+ is not bounded. The proof of this fact is the same as the proof ot Theorem 2.3 except for some minor modifications which we now describe. (We use the notations introduced in the proof of Theorem 2.3). The compactness of $\zeta^+ \cap \mathrm{Cl}(\mathscr{B}_R)$ now follows (by means of Lemma 2.2) from condition $\mathbf{K}_1$ and the fact that $\zeta^+ = S \subset \mathbf{R}^+ \times K$.

Let $G'\colon \Lambda \times X \to K$ be given by $G'(\lambda, x) = \lambda^{-1} T'(\lambda, x) \equiv \lambda^{-1} T(\lambda, f(x))$, and let $\Phi'(\lambda, x) \equiv x - G'(\lambda, x)$. The fact that G' is completely continuous on $\Lambda \times X$ follows from $\mathbf{K}_1$ and Lemma 5.1 which shows that the retraction f is continuous and bounded.

The proof of (2.9) (for Φ' instead of Φ) is the same as before since $\zeta^+ = S$ and thus $G'|\,\zeta^+ = G|\zeta^+$. Conditions (2.6), (2.7) and (2.10) hold in the present case only under the additional assumption that $x \in K$. Therefore here we obtain (2.11) (for $\Phi'(\lambda_0, \cdot)$) by means of Lemma 1.1 for the cone rather than for the entire space.

Finally the homotopy of $\Phi'(\lambda_0, \cdot)$ and I on ∂B_r is decuced in the present case from the following observation. The function $F'\colon [0, 1] \times \partial B_r \to X$ defixed by $F'(t, x) = x - t\lambda_0^{-1}T'(\lambda_0, x)$ is a homotopy which does not vanish on $[0, 1] \times \partial B_r$. Indeed $F'(t, x)$ can vanish only if $x \in K$. But for

$x \in K \cap \partial B_r$ and $t \in [0, 1]$, $F'(t, x) \neq 0$ because of (2.7). Equality (2.12) (for $\Phi'(\lambda_0, \cdot)$) now follows from this homotopy.

Using the extension of T given by (5.2) we obtain results for positive mappings parallel to Theorem 3.2 and 4.3 by introducing similar modifications in the proofs of the related theorems. These results will be formulated below.

Let X be an infinite dimensional linear normed space and let K be a cone in X. *We say that the mapping* T: $\mathbf{R}^+ \times K \to K$ *satisfies the condition* $\mathbf{K}_2$ *if*:

(i) T is *completely continuous*;
(ii) $T(\lambda, 0) = 0$ *for each* $\lambda \in \mathbf{R}^+$;
(iii) $T(0, x) = 0$ *for each* $x \in K$;
(iv) *Let* Λ *be a bounded closed interval in* $\mathbf{R}^+$ *such that* $0 \notin \Lambda$. *Then* $K\text{-}\lim_{\|x\| \to 0+} \|x\|^{-1} \|T(\lambda, x)\| = +\infty$, *uniformly for* $\lambda \in \Lambda$.

Consider now the equation

$$x = T(\lambda, x) \tag{5.4}$$

where T satisfies the condition $\mathbf{K}_2$. We formulate the following result concerning equation (5.4).

THEOREM 5.3. *Let* X *be an infinite dimensional linear normed space and let* K *be a cone in* X. *Let* T: $\mathbf{R}^+ \times K \to K$ *satisfy the condition* $\mathbf{K}_2$. *Let* S_1 *be the set of nontrivial solutions of* (5.4) *in* $\mathbf{R}^+ \times K$ *and let* $S = S_1 \, U\{(0, 0)\}$. *Let* C_K *be the connected component of* S *containing* (0, 0). *Then* C_K *is not bounded.*

Let X *be an infinite dimensional linear normed space and let* K *be a cone in* X. *We say that the mapping* T: $K \to K$ *satisfies the condition* $\mathbf{K}_3$ *if*:

(i) T *is completely continuous*;
(ii) *Let* $0 < r < R < +\infty$. *Then* $\inf\{\|T(x)\| \,|\, x \in K \cap (\mathrm{Cl}(B_R) \backslash B_r)\} > 0$;
(iii) $K\text{-}\lim\sup_{\|x\| \to 0+} \|x\|^{-1} \|T(x)\| = M$ *where* $0 < M < +\infty$.

We finally consider the equation

$$\lambda x = T(x) \tag{5.5}$$

where T satisfies the condition $\mathbf{K}_3$. The following result holds concerning equation (5.5).

THEOREM 5.4. *Let* X *be an infinite dimensional linear normed space and let* K *be a cone in* X. *Let* T: $K \to K$ *satisfy the condition* $\mathbf{K}_3$. *Let* S_1 *be the set of nontrivial solutions of* (5.5) *in* $\mathbf{R}^+ \times K$.

Let M *be the number given in condition* $\mathbf{K}_3$(iii) *and let* $S = S_1 \cup ([0, M] \times \{0\})$. *Let* D_K *be the connected component of* S *containing* (0, 0). *Then* D_K *is not bounded.*

Remark. Under the conditions of Theorems 5.2, 5.3, and 5.4 respectively one can obtain corollaries similar respectively to these mentioned for Theorems 2.3, 3.1 and 4.3. Details are omitted.

SECTION 6. We now present some applications of the results obtained in section 5 for equations involving positive mappings. Our fixst example is concerned with an integral equation of the form

$$\lambda x(s) = \int_G g(s, t)F(t, \lambda, x(t))\,dt \tag{6.1}$$

where G is a bounded closed subset of $\mathbf{R}^n$ having a positive Lebesgue measure $m(G)$, and where g and F are continuous functions satisfying conditions described below.

We denote by $C(G)$ the space of continuous real valued functions on G, equipped with the standard supremum norm. Let x_0 be a nonnegative function in $C(G)$ such that x_0 is not identically zero and let k be a fixed number, $k \geqq 1$. We denote by $K_{x_0;k}$ the following cone in K:

$$K_{x_0;k} = \{x \in X \mid \alpha\, x_0(t) \leqq x(t) \leqq k\alpha x_0(t),\ \forall t \in G, \text{ for some } \alpha \geqq 0\}.$$

With the above notations we have

THEOREM 6.1. *Let x_0 and y_0 be nonnegative functions in $C(G)$ such that x_0 and y_0 are not identically zero and let g: $G \times G \to \mathbf{R}^+$ be a continuous nonnegative function satisfying*

$$x_0(s)y_0(t) \leqq g(s, t) \leqq kx_0(s)y_0(t) \text{ for all } s, t \in G$$

where k is a fixed number, $k \geqq 1$. Let F: $G \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ be a continuous nonnegative real function satisfying:

(i) $F(t, \lambda, u_1) \leqq F(t, \lambda, u_2)$ *for all* $(t, \lambda) \in G \times \mathbf{R}^+$ *and all* $0 \leqq u_1 < u_2$;

(ii) $\lim_{\lambda \to 0+} \lambda^{-1} \int_G y_0(t)F(t, \lambda, \eta\, x_0(t))\,dt = +\infty$ *for all* $\eta > 0$;

(iii) $\lim_{u \to 0+} u^{-1}F(t, \lambda, u) = 0$, *uniformly for* $(t, \lambda) \in G \times \Lambda$ *where* Λ *is a bounded closed interval in* $\mathbf{R}^+$ *such that* $0 \notin \Lambda$.

Let S_1 be the set of nontrivial solutions of (6.1) in $\mathbf{R}^+ \times K_{x_0;k}$ *and let C_K be the connected component of $S_1 \cup \{(0, 0)\}$ containing* $(0, 0)$. *Then C_K is not bounded.*

PROOF. Note first that $K_{x_0;k}$ is a nonempty cone in X. (For the proof of this fact see e.g. [**9**, Chap. V, p. 250]). Let now T: $\mathbf{R}^+ \times K_{x_0;k} \to X$ be defined by

$$T(\lambda, x)(s) = \int_G g(s, t)F(t, \lambda, x(t))\,dt.$$

Eq. (6.1) can then be represented in the form $\lambda x = T(\lambda, x)$. Clearly T maps $\mathbf{R}^+ \times K_{x_0;k}$ into X. Moreover for $(\lambda, x) \in \mathbf{R}^+ \times K_{x_0;k}$ we have

$$(6.2) \qquad x_0(s)\int_G y_0(t)F(t, \lambda, x(t))\,dt \leqq T(\lambda, x)(s)$$
$$\leqq kx_0(s)\int_G y_0(t)F(t, \lambda, x(t))\,dt,$$

which shows that T is positive with respect ot $K_{x_0;k}$.

We shall show that T satisfies condition $\mathbf{K}_1$. Condition $\mathbf{K}_1$(i)-(ii) follow from the continuity of g and F by using Arzela's theorem.

Next we prove condition $\mathbf{K}_1$(iii). Let $0 \neq x \in K_{x_0;k}$. Then there exists $\alpha > 0$ such that $\alpha x_0(t) \leqq x(t) \leqq k\alpha x_0(t)$. Choose $t_0 \in G$ such that $\|x\| = x(t_0)$. Then $\|x\| = x(t_0) \leqq k\alpha x_0(t_0) \leqq k\alpha\|x_0\|$ and so

$$(6.3) \qquad x(t) \geqq k^{-1}\|x_0\|^{-1}\|x\|\, x_0(t) \text{ for all } t \in G.$$

Let now $r > 0$ be fixed and let $\|x\| \geqq r$. From (6.2) (6.3) and (i) it follows that

$$T(\lambda, x)(s) \geqq x_0(s)\int_G y_0(t)F(t, \lambda, k^{-1}\|x_0\|^{-1}\|x\|x_0(t))\,dt.$$

Thus $\|\lambda^{-1}T(\lambda, x)\| \geqq \|x_0\|\lambda^{-1}\int_G y_0(t)F(t, \lambda, k^{-1}\|x_0\|^{-1}rx_0(t))\,dt$. Condition $\mathbf{K}_1$(iii) follows from the last inequality and (ii).

Finally we prove condition $\mathbf{K}_1$(iv). Let Λ be a bounded closed interval in $\mathbf{R}^+$ such that $0 \notin \Lambda$. Let $\varepsilon > 0$ be given. From (iii) and the continuity if F it follows that there exists $\delta \equiv \delta(\varepsilon) > 0$ such that for $0 \leqq u \leqq \delta$ we have $F(t, \lambda, u) \leqq \varepsilon u$ for all $(t, \lambda) \in G \times \Lambda$. Let $x \in K_{x_0;k}$ such that $0 < \|x\| \leqq \delta$. Then $0 \leqq x(t) \leqq \delta$ and so $F(t, \lambda, x(t)) \leqq \varepsilon x(t)$. Hence

$$T(\lambda, x)(s) \leqq kx_0(s)\int_G y_0(t)F(t, \lambda, x(t))\,dt \leqq kx_0(s)\|y_0\|\,\|x\|m(G)\,\varepsilon.$$

Therefore $\|x\|^{-1}\|T(\lambda, x)\| \leqq k\|x_0\|\,\|y_0\|m(G)\,\varepsilon$. Condition $\mathbf{K}_1$(iv) follows from this inequality. Now an application of Theorem 5.2 completes the proof.

COROLLARY 6.2. *Suppose that the conditions of Theorem* 6.1. *are satisfied. Let* $H(\eta, \lambda) \equiv \int_G y_0(t)F(t, \lambda, \eta x_0(t))dt$. *If for any* $a > 0$ *we have* $\lim_{\eta\to+\infty}\eta^{-1}H(\eta, \lambda) = +\infty$, *uniformly for* $\lambda \in (0, a)$, *then the set of eigenvalues of the integral operator in* (6.1) *contains* $\mathbf{R}^+\backslash\{0\}$.

PROOF. Let $x \in K_{x_0;k}$, $x \neq 0$. By using (6.2) and (6.3) we can deduce that

$$\|x\|^{-1}\|T(\lambda, x)\| \geqq \|x\|^{-1}H(k^{-1}\|x_0\|^{-1}\|x\|, \lambda)\|x_0\|$$

and so $K\text{-}\lim_{\|x\|\to+\infty}\|x\|^{-1}\|T(\lambda, x)\| = +\infty$, uniformly for $\lambda \in (0, a]$. Our result follows from a corollary of Theorem 5.2 which is similar to Corollary 2.4.

Our next example is concerned with an integral equation of Lyapunov type. Let G be as before and let $\{g_i: \overbrace{G \times \cdots \times G}^{i+1 \text{ times}} \to \mathbf{R}^+\}_{i=2}^{\infty}$ be a sequcnce of continuous positive functions such that $0 < m_i \leqq g_i(s, t_1, \ldots, t_i) \leqq M_i < +\infty$ for all $s, t_1, \ldots, t_i \in G$; $i = 2, 3, \ldots$. Assume also that there exists $k \geqq 1$ such that $M_i/m_i \leqq k$; $i = 2, 3, \ldots$ and that $\lim_{i\to\infty} M_i^{1/i} = 0$. Let $T: C(G) \to C(G)$ be defined by

$$T(x)(s) = \sum_{i=2}^{\infty} \int_G \cdots \int_G g_i(s, t_1, \ldots, t_i)\, x(t_1) \ldots x(t_i)\, dt_1 \ldots dt_i.$$

and consider the Lyapunov type integral equation

$$\lambda x = T(x). \tag{6.4}$$

Let $x_0 \equiv 1$. Let S_1 be the set of nontrivial solutions of (6.4) in $\mathbf{R}^+ \times K_{x_0;k}$ and let C_K be the connected component of $S_1 \cup \{(0, 0)\}$ containing $(0, 0)$. Under the above assumptions we have

Theorem 6.3.
(i) *C_K is not bounded.*
(ii) *The set of eigenvalues of T in* (6.4) *contains* $\mathbf{R}^+\backslash\{0\}$

Proof. Krasnoselskii [**9**, Chap. V, Ex. 2, pp. 253] shows that T is positive with respect to the cone $K_{x_0;k}$ and that for $0 \neq x \in K_{x_0;k}$, $\|T(x)\| \geqq \sum_{i=2}^{\infty} m_i(k^{-1}\|x\| \cdot m(G))^i$. On the other hand it is clear that $\|T(x)\| \leqq \sum_{i=2}^{\infty} M_i(\|x\| \cdot m(G))^i$. From these inequalities it follows that T satisfies condition $\mathbf{K}_1$ and that $K\text{-}\lim_{\|x\|\to+\infty} \|x\|^{-1}\|T(x)\| = +\infty$. Now assertion (i) of the theorem follows from Theorem 5.2 while assertion (ii) is proved by an argument similar to that of Corollary 2.4 (iii).

Remark. If the series defining T contains also a term of the form $\int_G g_1(s, t_1)\, x(t_1)\, dt_1$ then one can show that

$$0 < M_1 k^{-1} m(G) \leqq K\text{-}\limsup_{\|x\|\to 0+} \|x\|^{-1}\|T(x)\| \leqq M_1 m(G) < +\infty.$$

From Theorem 5.4 it then follows that the connected component of $S_1 \cup ([0, M_1 \cdot m(G)] \times \{0\})$ containing $(0, 0)$ is not bounded and that equation (6.4) has at least one bifurcation point with respect to $\mathbf{R}^+ \times K_{x_0;k}$ in $[0, M_1 \cdot m(G)] \times \{0\}$.

Section 7. In this section we present applications to some boundary value problems of ordinary differential equations (abbreviated b.v.p.).

We consider a b.v.p. of the form

$$\begin{cases} x''(t) + F(t, \lambda, x(t), x'(t)) = 0; & t \in (0, 1) \\ x(0) = x(1) = 0 \end{cases} \tag{7.1}$$

where F satisfies certain conditions that are described below. Before stating our results we introduce some notations. Let $C^J[0, 1]$; $J = 0, 1, \ldots$

be the space of real valued functions defined on [0, 1] and possessing on [0, 1] continuous derivatives up to order J. $C^J[0, 1]$ is a Banach space under the *standard norm*. $\|x\|_J \equiv \sum_{i=0}^{J} \sup_{t\in[0,1]} |x^{(i)}(t)|$. We denote by X^J the subspace of $C^J[0, 1]$ consisiting of these functions which satisfy the boundary conditions given in the b.v.p. (7.1). In some cases we shall use in a given normed space X another norm $\|\cdot\|$, different from the one that we call the standard norm. We indicate this fact by using the notation $(X, \|\cdot\|)$.

We need the following auxiliary result.

LEMMA 7.1. *Let X be the normed space $X = \{x \in C^1[0, 1] \mid x(0) = x(1) = 0\}$ equipped with the norm defined by $\|x\| \equiv \int_0^1 |x'(t)|\, dt$. Let $K = \{x \in X | x(\mu t_1 + (1-\mu)t_2) \geqq \mu x(t_1) + (1-\mu)x(t_2),\ \forall t_1, t_2, \mu \in [0, 1]\}$. Then*:

(i) $\|x\|_0 \leqq \|x\|$ *for each* $x \in X$;
(ii) $\|x\| = 2\|x\|_0$ *for each* $x \in K$;
(iii) *K is a cone in X.*

PROOF OF (i). For any $t \in [0, 1]$ we have $x(t) = \int_0^t x'(\tau)\, d\tau$ and so $|x(t)| \leqq \int_0^t |x'(\tau)|\, d\tau \leqq \|x\|$. Thus $\|x\|_0 \leqq \|x\|$.

PROOF OF (ii). Let $x \in K$. There exists $t_0 \in (0, 1)$ such that $\|x\|_0 = x(t_0)$. Clearly $x'(t_0) = 0$. Being an element of K, x is a concave function on [0, 1] and so x' is a monotonic decreasing function on [0, 1]. From these facts it is clear that

$$|x'(t)| = \begin{cases} x'(t); & 0 \leqq t \leqq t_0 \\ -x'(t); & t_0 \leqq t \leqq 1. \end{cases}$$

Thus

$$\begin{aligned} \|x\| &\equiv \int_0^1 |x'(t)|\, dt = \int_0^{t_0} x'(t)\, dt - \int_{t_0}^1 x'(t)\, dt \\ &= x(t_0) - x(0) - x(1) + x(t_0) = 2x(t_0) = 2\|x\|_0. \end{aligned}$$

PROOF OF (iii). Clearly K is a cone in $(X, \|\cdot\|_0)$. From this fact and from (i) it follows that K is closed in X. Therefore K is a cone in X.

For the b.v.p. (7.1) we obtain

THEOREM 7.2. *Let F in* (7.1) *be a real nonnegative continuous function defined on* $[0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}$ *such that*:

(i) $F(t, \lambda, 0, 0) = 0$ *for all* $(t, \lambda) \in [0, 1] \times \mathbf{R}^+$;
(ii) $F(t, 0, u, v) = 0$ *for all* $(t, u, v) \in [0, 1] \times \mathbf{R}^+ \times \mathbf{R}$;
(iii) *There exists a continuous function* F_1: $[0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $F(t, \lambda, u, v) \leqq F_1(t, \lambda, u)$ *for all* $(t, \lambda, u, v) \in [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}$;
(iv) *There exist two continuous functions* F_2; $[0, 1] \times \mathbf{R}^+ \to \mathbf{R}^+$ *and*

$F_3 : \mathbf{R}^+ \to \mathbf{R}^+$ such that $F(t, \lambda, u, v) \geqq F_2(t, \lambda)F_3(u)$ for all $(t, \lambda, u, v) \in [0, 1] \times \mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}$ and such that $\int_0^1 F_2(t, \lambda)dt > 0$ for any $\lambda > 0$ and $\lim_{u \to 0+} u^{-1}F_3(u) = +\infty$.

Let K be as in Lemma 2.4. Let S_1 be the set of nontrivial solutions of the b.v.p. (7.1) in $\mathbf{R}^+ \times (X^2 \cap K)$ and let C_K be the connected component (in the topology induced from $\mathbf{R}^+ \times X^2$) of $S_1 \cup \{(0, 0)\}$ containing $(0, 0)$. Then C_K is not bounded in $\mathbf{R}^+ \times X^2$ as well as in $R^+ \times X^0$.

PROOF. Let $g(s, t)$ be the Green function of the linear operator $\mathscr{L}$ given by $\mathscr{L}x \equiv -x''$ with respect to the boundary conditions $x(0) = x(1) = 0$. $g(s, t)$ is given by

$$g(s, t) = \begin{cases} s(1 - t); & 0 \leqq s \leqq t \leqq 1 \\ t(1 - s); & 0 \leqq t \leqq s \leqq 1. \end{cases} \tag{7.2}$$

Let X and K be as in Lemma 6.4. Let T: $\mathbf{R}^+ \times K \to X^2$ be defined by

$$T(\lambda, x)(s) \equiv \int_0^1 g(s, t)F(t, \lambda, x(t), x'(t))\, dt. \tag{7.3}$$

We replace the b.v.p. (7.1) by the equivalent integral equation

$$x = T(\lambda, x) \tag{7.4}$$

in $\mathbf{R}^+ \times K$ where T is given by (7.3). We shall show that T satisfies condition $\mathbf{K}_2$ with respect to the cone K in X.

Positivity of T. Let L be the linear operator defined by $Lx(s) \equiv \int_0^1 g(s, t)\, x(t)\, dt$. It is well known that L maps $C^0[0, 1]$ into X^2. For $(\lambda, x) \in \mathbf{R}^+ \times K$ we have $T(\lambda, x) = L(F(\cdot, \lambda, x(\cdot), x'(\cdot)))$. Therefore $T(\lambda, x) \in X^2 \subset X$ since $F(\cdot, \lambda, x(\cdot), x'(\cdot)) \in C^0[0, 1]$. Moreover $T(\lambda, x)''(s) = -F(s, \lambda, x(s), x'(s)) \leqq 0$ and so $T(\lambda, x)$ is concave on $[0, 1]$, i.e. $T(\lambda, x) \in K$. Thus T is positive with respect to K.

Compactness of T. Let $\{(\lambda_n, x_n)\}$ be a bounded sequence in $\mathbf{R}^+ \times K$. From Lemma 7.1 (i) it follows that $\{(\lambda_n, x_n)\}$ is bounded in $\mathbf{R}^+ \times X^0$. Therefore $\{F_1(\cdot, \lambda_n, x_n(\cdot))\}$ is bounded in $C^0[0, 1]$. Let $y_n(t) \equiv F(t, \lambda_n, x_n(t), x_n'(t))$; $n = 1, 2, \ldots$. Then $0 \leqq y_n(t) \leqq F_1(t, \lambda_n, x_n(t))$ and so $\{y_n\}$ is bounded in $C^0[0, 1]$. In additon $T(\lambda_n, x_n) = Ly_n$. The compactness of T now follows from the above argument by using the fact that L: $C^0[0, 1] \to (X, \|\cdot\|_1)$ is compact. (We also used the obvious fact that $\|x\| \leqq \|x\|_1$ for each $x \in X$).

Continuity of T. We shall really prove the stronger result that $T : R^+ \times K \to (K, \|\cdot\|_1)$ is continuous. Let $\{(\lambda_n, x_n)\}$ be a sequence in $R^+ \times K$ converging to (λ_0, x_0) and let $y_n(t) = F(t, \lambda_n, x_n(t), x_n'(t))$; $n = 0, 1, \ldots$. Clearly $\{(\lambda_n, x_n)\}$ is bounded and thus, as before, $\{y_n\}$ is bounded in $C^0[0, 1]$. Let us show that $\{y_n\}$ converges to y_0 in $L_1[0, 1]$. From the fact that $\|x_n - x_0\| \to 0$ it follows that $\|x_n - x_0\|_0 \to 0$ and that $\{x_n'\}$ converges to x_0' in $L_1[0, 1]$. From these facts and the continuity of F we deduce that $\{y_n\}$ converges to y_0 in (Lebesgue) measure. Moreover $\{y_n\}$ is bounded in $C^0[0, 1]$ and thus $\lim_{m(E)\to 0}\int_E|y_n(t)|\,dt = 0$, uniformly for n (where $m(E)$ denotes the Lebesgue measure of E). Applying Theorem III.3.6 in [7, pp. 122] we deduce that

$$\lim_{n\to\infty}\int_0^1|y(t)_n - y_0(t)|\,dt = 0. \tag{7.5}$$

Using the fact that $0 \leqq g(s, t) \leqq 1$ one can easily verify that

$$\|T(\lambda_n, x_n) - T(\lambda_0, x_0)\|_1 \leqq 2\int_0^1 |y_n(t) - y_0(t)|\,dt.$$

The continuity of T follows from this inequality and (7.5). Thus $\mathbf{k}_2(i)$ holds. It is easy to verify that $T(\lambda, 0) = 0$ for $\lambda \in \mathbf{R}^+$ and that $T(0, x) = 0$ for $x \in K$. We have only to show that $\mathbf{k}_2$(iv) holds too. Let Λ be a bounded closed interval in $\mathbf{R}^+$ such that $0 \notin \Lambda$. From our assumptions concerning F_2 it follows that there exist $\varepsilon_0, \alpha_0 > 0$ such that

$$\int_{\varepsilon_0}^{1-\varepsilon_0} F_2(t, \lambda)\,dt \geqq \alpha_0 \text{ for all } \lambda \in \Lambda. \tag{7.6}$$

Now let M be an arbitrary positive number. From (iv) it follows that there exists $\delta > 0$ such that for $0 < u \leqq \delta$ we have $F_3(u) \geqq Mu$. Let $(\lambda, x) \in \Lambda \times K$ such that $0 < \|x\| \leqq \delta$. For any $t \in [\varepsilon_0, 1 - \varepsilon_0]$ we have $\varepsilon_0\|x\|_0 \leqq x(t) \leqq \|x\|_0 \leqq \|x\| \leqq \delta$ and so $F_3(x(t)) \geqq Mx(t) \geqq M\varepsilon_0\|x\|_0$. From this inequality and (7.6) it follows that for any $s \in [\varepsilon_0, 1 - \varepsilon_0]$

$$\begin{aligned} T(\lambda, x)(s) = \textstyle\int_0^1 g(s, t)F(t, \lambda, x(t), x'(t))\,dt &\geqq \varepsilon_0^2 \textstyle\int_{\varepsilon_0}^{1-\varepsilon_0} F_2(t, \lambda)F_3(x(t))\,dt \\ &\geqq M\varepsilon_0^3\alpha_0\|x\|_0. \end{aligned}$$

Thus

$$\|T(\lambda, x)\| = 2\|T(\lambda, x)\|_0 \geqq 2M\varepsilon_0^3\alpha_0\|x\|_0 = M\varepsilon_0^3\alpha_0\|x\|$$

for any $(\lambda, x) \in \Lambda \times K$ for which $0 < \|x\| \leqq \delta$. $\mathbf{K}_2$(iv) follows from this inequality and the arbitrariness of M.

Let S_1' be the set of nontrivial solutions of (7.4) in $\mathbf{R}^+ \times K$ and let C_K' be the connected component (in the topology induced from $\mathbf{R}^+ \times X$) of $S_1' \cup \{(0, 0)\}$ containing $(0, 0)$. Applying Theorem 5.3 we thus deduce that C_K' is not bounded in $\mathbf{R}^+ \times K$. But then clearly C_k' is also not bounded in $\mathbf{R}^+ \times X^0$ (since $\|x\| = 2\|x\|_0$ for $x \in K$). Our theorem would follow once we show that $C_K = C_K'$.

It is easy to verify that $S_1 \subseteq S_1'$. On the other hand if $(\lambda, x) \in S_1'$ then $x = T(\lambda, x) \in X^2$ and so $(\lambda, x) \in S_1$. Thus $S_1 = S_1'$. The subset C_K is connected in $\mathbf{R}^+ \times X^2$ and so C_K is also connected in $\mathbf{R}^+ \times X$. Hence $C_K \subseteq C_K'$.

Let $\{(\lambda_n, x_n)\}$ be a sequence in C'_K converging to (λ_0, x_0) in $\mathbf{R}^+ \times K$ and let $\{y_n\}$ be defined as before. Then $x_n = T(\lambda_n, x_n)$; $n = 0, 1, 2, \ldots$ and so $\|x_n - x_0\|_1 \to 0$ (since $T: \mathbf{R}^+ \times K \to (K, \|\cdot\|_1)$ is continuous). But then $\|y_n - y_0\|_0 \to 0$ and therefore $\|x_n - x_0\|_2 \to 0$ (since $x_n = T(\lambda_n, x_n) = Ly_n$ and $L: C^0[0, 1] \to X^2$ is continuous). From the above argument it follows that C'_K is connected in $\mathbf{R}^+ \times X^2$, and thus $C'_K \subseteq C_K$. Hence $C_K = C'_K$.

Using Theorem 5.3 one can prove also

COROLLARY 7.3. *Suppose that the conditions of Theorem 7.2 are satisfied. Assume also that there exist two continuous founctions* $F_4: [0, 1] \times \mathbf{R}^+ \to \mathbf{R}^+$ *and* $F_5: \mathbf{R}^+ \to \mathbf{R}^+$ *such that* $F_1(t, \lambda, u) \leqq F_4(t, \lambda)F_5(u)$ *and so that* $\lim_{u \to +\infty} u^{-1}F_5(u) = 0$. *Then the projection of* C_K *on* $\mathbf{R}^+$ is $\mathbf{R}^+$.

REMARK: In a similar way we can apply Theorem 5.2 to the study of the existence of connected branches of solutions of a b.v.p. of the form

$$\begin{cases} \lambda x''(t) + F(t, \lambda, x(t), x'(t)) = 0; \ t \in (0, 1) \\ x(0) = x(1) = 0 \end{cases}$$

under appropriate conditions on F. Details are omitted.

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