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FUZZY TOPOLOGIES CHARACTERIZED BY NEIGHBORHOOD SYSTEMS

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In this paper we prove that neighborhood systems are an equivalent method for determining fuzzy topologies. This characterization of fuzzy topology uses the definition of neighborhood of a point which was given in [3] and used in [4] to describe continuity between fuzzy topological spaces. The sequence of development in this paper parallels Chapter 9 in [2].

In [1] the authors give a different definition for the neighborhood of a point and then are able to characterize a proper subclass of all fuzzy topologies on a fixed set. The difficulty with their definition is that distinct fuzzy topologies can have the same system of neighborhoods.

DEFINITION 1. Let X be a set. A *fuzzy set* in X is a function from X into $[0, 1]$, the closed unit interval. So g is a fuzzy set in X iff $g : X \longrightarrow [0, 1]$. To each set $E \subset X$ corresponds the "crisp" fuzzy set μ_E which is the characteristic function of E.

We shall continue the pattern begun in [4] of relating fuzzy sets to other fuzzy sets by the function operations of $=$, $\leqq$, $<$, $\wedge$, $\vee$, $\bigvee$, $\bigwedge$, $+$ and $-$, where $f \leqq g$ iff $f(x) \leqq g(x)$ for all $x \in X$ and $f + g$ is defined by $(f + g)(x) = f(x) + g(x)$, with similar definitions for the other operations in the list. Negation of each of the operations is determined by one $x \in X$. It is assumed that a supremum (infimum) of fuzzy sets taken over an empty index set is μ_ϕ (μ_X).

DEFINITION 2. Let X be a set and let T be a family of fuzzy sets in X. Then T is called a *fuzzy topology* on X iff it satisfies the conditions:

(a) 0 $(= \mu_\phi)$ and 1 $(= \mu_X)$ are in T;

(b) if $g_i \in T$, $i \in I$, then $\bigvee \{g_i : i \in I\} \in T$;

(c) if $g, h \in T$, then $g \wedge h \in T$.

The pair (X, T) is called a *fuzzy topological space* (abbreviated as fts). The elements of T are called *open* fuzzy sets. By a fuzzy set in a fts (X, T), we mean a fuzzy set in X.

DEFINITION 3. [3]. A fuzzy set n in a fts (X, T) is a *neighborhood of a point* $x \in X$ iff there exists $g \in T$ such that $g \leqq n$ and $n(x) = g(x) > 0$.

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By N_x we denote the family of all neighborhoods of x which are determined by the fuzzy topology T on X.

THEOREM 1. *Let* (X, T) *be a fts. Then for each* $x \in X$, N_x *satisfies*:
(i) $1 \in N_x$.
(ii) *if* $n \in N_x$, *then* $n(x) > 0$.
(iii) *if* $n \in N_x$, $n \leqq w$ *and* $n(x) = w(x)$, *then* $w \in N_x$.
(iv) *if* $n_i \in N_x$, $i \in I$, *then* $\bigvee \{n_i : i \in I\} \in N_x$.
(v) *if* $n, m \in N_x$, *then* $n \wedge m \in N_x$.
(vi) *if* $n \in N_x$, *then there exists* $g \in N_x$ *such that* $g \leqq n$, $g(x) = n(x)$ *and if* $g(y) > 0$, *then* $g \in N_y$.

PROOF. Conditions (i) through (iii) are a direct consequence of Definition 3.

(iv) Since $n_i \in N_x$, there exists $g_i \in T$ such that $g_i \leqq n_i$ and $n_i(x) = g_i(x) > 0$. Set $h = \bigvee \{g_i : i \in I\}$ and $m = \bigvee \{n_i : i \in I\}$. Then $h \in T$, $h \leqq m$ and $m(x) = h(x) > 0$. Thus $m \in N_x$.

(v) Let g, h be elements of T satisfying $g \leqq n$, $h \leqq m$, $n(x) = g(x) > 0$ and $m(x) = h(x) > 0$. Then $g \wedge h \in T$, $g \wedge h \leqq n \wedge m$ and $(n \wedge m)(x) = (g \wedge h)(x) > 0$.

(vi) Take g to be the open fuzzy set whose existence is specified in Definition 3.

DEFINITION 4. Let X be a set and let θ be a function from X into the power set of $[0, 1]^X$. Then θ is called a *fuzzy neighborhood system* on X iff θ satisfies:
N1: $1 \in \theta(x)$ for each $x \in X$;
N2: if $n \in \theta(x)$, then $n(x) > 0$;
N3: if $n \in \theta(x)$, $n \leqq w$ and $n(x) = w(x)$, then $w \in \theta(x)$;
N4: if $n_i \in \theta(x)$, $i \in I$, *then* $\bigvee \{n_i : i \in I\} \in \theta(x)$;
N5: if $n, m \in \theta(x)$, then $n \wedge m \in \theta(x)$;
N6: if $n \in \theta(x)$, then there exists $g \in \theta(x)$ such that $g \leqq n$, $g(x) = n(x)$ and if $g(y) > 0$, then $g \in \theta(y)$.

If θ is a fuzzy neighborhood system on X, then we define $T(\theta)$ as the family of all fuzzy sets g in X with the property that if $g(x) > 0$, then $g \in \theta(x)$.

THEOREM 2. *If* θ *is a fuzzy neighborhood system on* X, *then* $(X, T(\theta))$ *is a fts.*

PROOF. Clearly 0 and 1 are in $T(\theta)$. For each $i \in I$ let $g_i \in T(\theta)$. Set $h = \bigvee \{g_i : i \in I\}$. If $h(x) > 0$, then there exists nonempty $J_x \subset I$ such that $g_i(x) > 0$ iff $i \in J_x$. By the definition of $T(\theta)$, if $i \in J_x$, then $g_i \in \theta(x)$. From N4 we conclude that $\bigvee \{g_i : i \in J_x\} \in \theta(x)$. By N3 $h \in \theta(x)$. Therefore $h \in T(\theta)$.

Let g, $h \in T(\theta)$ and suppose that $(g \wedge h)(x) > 0$. Then $g(x) > 0$ and $h(x) > 0$. Hence, by the definition of $T(\theta)$ g and h are in $\theta(x)$. It follows from N5 that $g \wedge h \in \theta(x)$. Thus $g \wedge h \in T(\theta)$.

THEOREM 3. *For every $x \in X$, the family N_x of all neighborhoods of x with respect to the fuzzy topology $T(\theta)$ is exactly $\theta(x)$.*

PROOF. If $n \in N_x$, then there exists $g \in T(\theta)$ such that $g \leqq n$ and $n(x) = g(x) > 0$. By the definition of $T(\theta)$, $g \in \theta(x)$. It then follows from N3 that $n \in \theta(x)$.

If $n \in \theta(x)$, then by N2 $n(x) > 0$ and by N6 there exists $g \in \theta(x)$ such that $g \leqq n$, $g(x) = n(x)$ and if $g(y) > 0$, then $g \in \theta(y)$. It follows from the definition of $T(\theta)$ that $g \in T(\theta)$. Consequently $n \in N_x$.

THEOREM 4. *Let (X, T) be a fts and let θ_T be the function $\{(x, N_x) : x \in X$ and N_x is the family of all neighborhoods of x* with respect to $T\}$. *Then θ_T is a fuzzy neighborhood system on X and $T(\theta_T) = T$.*

PROOF. As a result of Theorem 1, θ_T satisfies conditions N1 through N6 and therefore is a fuzzy neighborhood system on X. By Theorem 2 $T(\theta_T)$ is a fuzzy topology on X. From Theorem 3 we conclude that for each $x \in X$ the neighborhoods of x with respect to $T(\theta_T)$ are exactly the members of N_x. Since a fuzzy set is open iff it is a neighborhood of each point at which it assumes a positive value, it follows that $T(\theta_T) = T$.

REMARK. Note that axiom N6 was built from axiom N_4' in [2]. By the following example we shall show that it is not possible to replace N6 with an axiom paralleling N_4 in [2] such as:

if $n \in \theta(x)$, then there exists $g \in \theta(x)$ such that $g \leqq n$,

$g(x) = n(x)$ *and if* $g(y) > 0$, then $n \in \theta(y)$.

Let X contain at least two elements and choose $x \in X$. Define fuzzy sets g and n in X as follows:

$$g(x) = n(x) = 3/4,$$

$$g(y) = 1/2 \text{ if } y \in X - \{x\},$$

$$n(y) = 3/4 \text{ if } y \in X - \{x\}.$$

Let the fuzzy topology on X be $\{0, 1, g\}$. Then $n \in N_x$, but $n \notin N_y$ for each $y \in X - \{x\}$.

It would be useful to know if there are neighborhood systems of fuzzy sets which yield the open neighborhoods or a local base for the fuzzy topology. See Remark 9.4 in [2] for a discussion of these concepts in general topology.

References

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