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GREEN'S FUNCTIONS FOR FOCAL TYPE BOUNDARY VALUE PROBLEMS

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In this paper we will be concerned mainly with the differential equations

$$y^{(n)} = \lambda p(x)y \tag{1}$$

where $\lambda = \pm 1$, $p(x) > 0$ is continuous on $[a, b]$. Our main result is that if (1) is disfocal (see Definition 2 below) on $[a, b]$ and $1 \leqq k \leqq n - 1$, then the Green's function $G_k(x, s)$ for the k-focal point problem

$$\begin{aligned} &y^{(n)} - \lambda p(x)y = h(x) \\ &\qquad y^{(i)}(a) = 0,\ y^{(j)}(b) = 0, \\ &\qquad\qquad i = 0, \cdots, k-1,\ j = k, \cdots, n-1, \end{aligned}$$

$h \in C[a, b]$, satisfies

$$(-1)^{n-k} G_k^{(i)}(x, s) > 0$$

on $(a, b) \times (a, b)$ for $i = 0, \cdots, k-1$, where $G_k^{(i)}(x, s)$ denotes the partial derivative $\partial^i/\partial x^i\, G_k(x, s)$.

At the outset, to be more general, we consider the differential equation

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = 0 \tag{2}$$

where the coefficients $p_k(x)$, $k = 1, \cdots, n$, are assumed to be continuous on $[a, b]$. The adjoint system [2] of (2) is

$$\begin{aligned} (z^0)' &= p_1(x)z^0 + z^1 \\ (z^1)' &= -p_2(x)z^0 + z^2 \\ &\cdots \\ (z^{n-2})' &= (-1)^n p_{n-1}(x)z^0 + z^{n-1} \\ (z^{n-1})' &= (-1)^{n+1}p_n(x)z^0 \end{aligned} \tag{3}$$

Given a scalar function $z(x)$, set $z^0(x) = z(x)$, $z^1(x) = [z^0(x)]' - p_1(x)z^0(x)$, $\cdots$, $z^{n-1}(x) = [z^{n-2}(x)]' - (-1)^n p_{n-1}(x)z^0(x)$, provided $z^0(x)$, $\cdots$, $z^{n-2}(x)$ are successively differentiable on $[a, b]$. We will say that a

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scalar function $z(x)$ is a solution of (3) provided $z^0(x), \cdots, z^{n-1}(x)$ is a solution of (3). We will say a solution $z(x)$ of (3) has a zero of order k at x_0 provided $z^i(x_0) = 0$, $i = 0, \cdots, k-1$. Note that the adjoint system of (1) is equivalent to the differential equation

$$z^{(n)} = (-1)^n \lambda p(x) z \tag{4}$$

and $z^k(x) = z^{(k)}(x)$, $k = 0, \cdots, n-1$.

Let $u_k(x, x_0)$, $k = 0, \cdots, n-1$, be the solution of the initial value problem (2), $y^{(j)}(x_0) = \delta_{jk}$, $j = 0, \cdots, n-1$ (δ_{jk} is the Kronecker delta). Similarly, let the scalar function $z_k(x, x_0)$, $k = 0, \cdots, n-1$, be the solution of the initial value problem (3), $z^j(x_0) = \delta_{jk}$, $j = 0, \cdots, n-1$.

A fundamental relation between (2) and (3) is given by (see [**6**] and the reference given there to Dolan)

$$u_q^{(p)}(s, t) = (-1)^{p+q} z_{n-p-1}^{n-q-1}(t, s) \tag{5}$$

$p, q = 0, \cdots, n-1$.

Before we state some of our results we give some definitions.

Definition 1. Assume $i_1, \cdots, i_k, j_1, \cdots, j_{n-k}$ are n distinct integers with $0 \leqq i_p \leqq n-1$, $p = 1, \cdots, k$, $0 \leqq j_q \leqq n-1$, $q = 1, \cdots, n-k$. We say that (2) is $(i_1, \cdots, i_k; j_1, \cdots, j_{n-k})$-disfocal on $[a, b]$ provided there does not exist a nontrivial solution $y(x)$ and points $c < d$ in $[a, b]$ such that

$$y^{(i_p)}(c) = 0,\ p = 1, \cdots, k,\ y^{(j_q)}(d) = 0,\ q = 1, \cdots, n-k.$$

Similarly (3) is $(i_1, \cdots, i_k; j_1, \cdots, j_{n-k})$-disfocal on $[a, b]$ provided there is no nontrivial solution $z(x)$ and points $c < d$ in $[a, b]$ such that $z^{i_p}(c) = 0$, $p = 1, \cdots, k$, $z^{j_q}(d) = 0$, $q = 1, \cdots, n-k$.

We now define disfocal as Nehari did in [3].

Definition 2. We say that (2) is disfocal on $[a, b]$ provided there is no nontrivial solution $y(x)$ of (2) such that each of $y^{(i)}(x)$, $i = 0, \cdots, n-1$, vanishes at least once on $[a, b]$.

Note that if (1) is disfocal on $[c, d]$, then (1) is disfocal on $(c - \epsilon, d + \epsilon)$ for some $\epsilon > 0$. For if not, then for each k there is a solution $y_k(x)$ of (1) and points x_{ik}, $i = 0, \cdots, n-1$, in $(c - (1/k), d + (1/k))$ such that $y_k^{(i)}(x_{ik}) = 0$. By normalizing the coefficients of y_k with respect to a basis, there is a subsequence of $\{y_k\}$ which converges uniformly on compact subintervals to a nontrivial solution y. Then there are points t_i, $i = 0, \cdots, n-1$ in $[a, b]$ such that $y^{(i)}(t_i) = 0$, $i = 0, \cdots, n-1$ which is a contradiction. Nehari proved [3] that (1) is disfocal on $[a, b]$ iff (4) is disfocal on $[a, b]$. Another result due to Nehari [3] to keep in mind while reading this paper is that (1) is disfocal on $[a, b]$ iff it is $(i_1, \cdots, i_k; j_1, \cdots, j_{n-k})$-disfocal on $[a, b]$ for all possible $(i_1, \cdots, i_k; j_1, \cdots, j_{n-k})$, $k = 1, \cdots, n-1$.

A relationship between (2) and (3) that we will use repeatedly is given by the following theorem.

THEOREM 1. *The differential equation* (2) *is* $(i_1, \cdots, i_k; j_1, \cdots, j_{n-k})$-*disfocal on* $[a, b]$ *iff the adjoint system* (3) *is* $(n - j_1 - 1, \cdots, n - j_{n-k} - 1; n - i_1 - 1, \cdots, n - i_k - 1)$-*disfocal on* $[a, b]$.

PROOF. The proof follows easily from the following equation which we obtain by use of (5) and properties of determinants.

$$\begin{vmatrix} u_{j_1}^{(j_1)}(s, t) & \cdots & u_{j_{n-k}}^{(j_1)}(s, t) \\ \cdots & & \cdots \\ u_{j_1}^{(j_{n-k})}(s, t) & \cdots & u_{j_{n-k}}^{(j_{n-k})}(s, t) \end{vmatrix} = \begin{vmatrix} z_{n-j_1-1}^{n-j_1-1}(t, s) & \cdots & z_{n-j_{n-k}-1}^{n-j_1-1}(t, s) \\ \cdots & & \cdots \\ z_{n-j_1-1}^{n-j_{n-k}-1}(t, s) & \cdots & z_{n-j_{n-k}-1}^{n-j_{n-k}-1}(t, s) \end{vmatrix}$$

$a \leqq t < s \leqq b$ (note there are no minus signs in this last determinant).

An interesting question is how are the various types of disfocalness of (1) related to each other. To give a result of this nature we first give some definitions.

DEFINITION 3. We say that $y(x)$ is a focal solution of (2) on $[c, d] \subset [a, b]$ provided $y(x)$ is a nontrivial solution of (2) with $y^{(i)}(x_i) = 0$, $i = 0, \cdots, n-1$, where $c \leqq x_i \leqq d$, $i = 0, \cdots, n-1$.

In the next two definitions assume that $1 \leqq k \leqq n-1$ and that $i_1, \cdots, i_k$ are distinct integers with $0 \leqq i_j \leqq n-1$.

DEFINITION 4. We say that (2) is $(i_1, \cdots, i_k)$-disfocal on $[a, b]$ provided there is no $\alpha \in [a, b)$ such that there is a focal solution $y(x)$ with $y^{(j)}(x_j) = 0$, $j = 0, \cdots, n-1$, where $x_j = \alpha$ if $j \in \{i_1, \cdots, i_k\}$ and $\alpha \leqq x_j \leqq b$, $j = 0, \cdots, n-1$.

DEFINITION 5. We say that (2) is two point $(i_1, \cdots, i_k)$-disfocal on $[a, b]$ provided there are not points $\alpha < \beta$ in $[a, b]$ such that there is a focal solution $y(x)$ such that $y^{(i)}(x_i) = 0$, $i = 0, \cdots, n-1$ with $x_i = \alpha$ for $i \in \{i_1, \cdots, i_k\}$ and $x_i \in \{\alpha, \beta\}$, $0 \leqq i \leqq n-1$.

In terms of our new terminology Nehari [3] proved that (1) is two point $(i_1, \cdots, i_k)$-disfocal for all $(i_1, \cdots, i_k)$, $1 \leqq k \leqq n-1$ iff (1) is disfocal on $[a, b]$. Using the same techniques we can prove the following generalization.

THEOREM 2. *Equation* (1) *is two point* $(i_1, \cdots, i_k)$*-disfocal on* $[a, b]$ *iff* (1) *is* $(i_1, \cdots, i_k)$*-disfocal on* $[a, b]$.

PROOF. Assume (1) is not $(i_1, \cdots, i_k)$-disfocal on $[a, b]$, but is two point $(i_1, \cdots, i_k)$-disfocal on $[a, b]$. Then there is an $\alpha \in [a, b]$ and a focal solution $y(x)$ of (1) with $y^{(i)}(t_i) = 0$, $i = 0, \cdots, n-1$ where $t_i = \alpha$ for $i \in \{i_1, \cdots, i_k\}$ and $\alpha \leqq t_i \leqq b$, $i = 0, \cdots, n-1$. Let β be the infimum of points $c \in (\alpha, b]$ such that there is a focal solution $y(x)$ and points t_i, $0 \leqq i \leqq n-1$, such that $y^{(i)}(t_i) = 0$, $0 \leqq i \leqq n-1$, where $t_i = \alpha$ for $i \in \{i_1, \cdots, i_k\}$ and $\alpha \leqq t_i \leqq c$ for $0 \leqq i \leqq n-1$. By a standard compactness argument we get that there is a focal solution $u(x)$ and points x_i, $0 \leqq i \leqq n-1$, such that $u^{(i)}(x_i) = 0$, $i = 0, \cdots, n-1$, $x_i = \alpha$ for $i \in \{i_1, \cdots, i_k\}$, $\alpha \leqq x_i \leqq \beta (\alpha < \beta)$, $0 \leqq i \leqq n-1$, and $\beta \in \{x_0, \cdots, x_{n-1}\}$. Of all such solutions $u(x)$, let N be the maximum number of x_i's such that $x_i \in \{\alpha, \beta\}$. Without loss of generality the above $u(x)$ is such a solution corresponding to N. Since (1) is two point $(i_1, \cdots, i_k)$-disfocal, $1 \leqq N < n$. Choose $p \in \{0, \cdots, n-1\}$ such that $x_p \in (\alpha, \beta)$. Let $u_1(x), \cdots, u_n(x)$ be a fundamental set of solutions of (1), and set

$$\omega(x) = \begin{vmatrix} u_1(x) & \cdots & u_n(x) \\ u_1(x_0) & \cdots & u_n(x_0) \\ \cdots & & \cdots \\ u_1^{(p-1)}(x_{p-1}) & \cdots & u_n^{(p-1)}(x_{p-1}) \\ u_1^{(p+1)}(x_{p+1}) & \cdots & u_n^{(p+1)}(x_{p+1}) \\ \cdots & & \cdots \\ u_1^{(n-1)}(x_{n-1}) & \cdots & u_n^{(n-1)}(x_{n-1}) \end{vmatrix}.$$

If $\omega^{(p)}(\beta) = 0$, then there is a focal solution of (1) contradicting the maximality of N, i.e., a nontrivial solution y so that $y^{(i)}(x_i) = 0$, $i \neq p$, $y^{(p)}(\beta) = 0$. Since $\omega^{(p)}(\beta) \neq 0$, ω is a nontrivial solution of (1). Also note that $\omega^{(i)}(x_i) = 0$, $i \neq p$.

For $\epsilon > 0$, sufficiently small, set

$$v_\epsilon(x) = \begin{vmatrix} u_1(x) & \cdots & u_n(x) \\ u_1(t_0) & \cdots & u_n(t_0) \\ \cdots & & \cdots \\ u_1^{(p-1)}(t_{p-1}) & \cdots & u_n^{(p-1)}(t_{p-1}) \\ u_1^{(p+1)}(t_{p+1}) & \cdots & u_n^{(p+1)}(t_{p+1}) \\ \cdots & & \cdots \\ u_1^{(n-1)}(t_{n-1}) & \cdots & u_n^{(n-1)}(t_{n-1}) \end{vmatrix},$$

where $t_j = x_j$ if $x_j < \beta$ and $t_j = \beta - \epsilon$ if $x_j = \beta$. Note that $v_\epsilon^{(i)}(t_i) = 0$, $i \neq p$, and $v_\epsilon(x) \to \omega(x)$ in $C^n[\alpha, \beta]$ space as $\epsilon \to 0^+$. Because of the existence of the focal solution $u(x)$ ($u^{(i)}(x_i) = 0$, $i = 0, \cdots, n-1$) we have that $\omega^{(p)}(x_p) = 0$. If x_p is an odd ordered zero of $\omega^{(p)}(x)$, then for $\epsilon > 0$, sufficiently small, $v_\epsilon^{(p)}(x)$ has a zero near x_p. This contradicts the definition of β. Hence, $\omega^{(p+1)}(x_p) = 0$. If $p = n-1$, then $\omega^{(n)}(x_p) = 0$ which by (1) implies $\omega(x_p) = 0$. Therefore we have that $\omega^{(p+1)}(x_p) = 0$ where the order of the derivative is interpreted modulo n. Pick $1 \leqq \ell < n$ such that $\omega^{(p+\ell)}(x_p) = 0$ ($p + \ell$ interpreted modulo n) but $\omega^{(p+\ell+1)}(x_p) \neq 0$. If $x_{p+\ell+1}$ ($p + \ell + 1$ interpreted modulo n) $\notin \{\alpha, \beta\}$ we get a contradiction as above with p in the definition of $v_\epsilon(x)$ replaced by $p + \ell + 1$ (modulo n). Therefore either $\omega^{(p+\ell+1)}(\alpha) = 0$ or $\omega^{(p+\ell+1)}(\beta) = 0$. By Rolle's theorem and (1) we get that $\omega^{(p+\ell+2)}(x)$ ($p+\ell+2$ interpreted $\bmod n$) has an odd ordered zero in (α, β). If $x_{p+\ell+2} \notin \{\alpha, \beta\}$ we get a contradiction as above. Hence, either $\omega^{(p+\ell+2)}(\alpha) = 0$ or $\omega^{(p+\ell+2)}(\beta) = 0$, so we can apply Rolle's theorem. Continuing like this we finally conclude that $\omega^{(p+n)}(x) = \omega^{(p)}(x)$ (because of our interpretation of $p + n$) has an odd ordered zero in (α, β). But $\omega^{(p)}(\alpha) \neq 0$ and $\omega^{(p)}(\beta) \neq 0$ so we get out final contradiction as above.

Before we prove our main result we state without proof the following lemma. One can easily check this result by showing that the basic properties ([1], p. 105) which uniquely determine the Green's function are satisfied. See [5] and [6] for the method of construction for this form of the Green's function. The reader can easily verify the inequality in this lemma using the $(0, \cdots, k-1; k, \cdots, n-1)$-disfocalness of (2) and a continuity argument.

LEMMA 3. *Let $G_k(x, s)$, $1 \leqq k \leqq n-1$, be the Green's function for the k-focal point problem*

$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_n(x)y = h(x)$, $y^{(i)}(a) = 0$,

$y^{(j)}(b) = 0$, $i = 0, \cdots, k-1$, $j = k, \cdots, n-1$ ($h \in C[a, b]$).

If (2) *is* $(0, \cdots, k-1; k, \cdots, n-1)$-*disfocal on* $[a, b]$, *then* $G_k(x, s)$ *exists and*

$$G_k(x, s) = \frac{1}{D} \begin{vmatrix} 0 & u_k(x, a) & \cdots\ u_{n-1}(x, a) \\ u_{n-1}^{(k)}(b, s) & u_k^{(k)}(b, a) & \cdots\ u_{n-1}^{(k)}(b, a) \\ \cdots & & \cdots \\ u_{n-1}^{(n-1)}(b, s) & u_k^{(n-1)}(b, a) & \cdots\ u_{n-1}^{(n-1)}(b, a) \end{vmatrix}$$

on the triangle $a \leqq x \leqq s \leqq b$ *and*

$$G_k(x, s) = \frac{1}{D} \begin{vmatrix} u_{n-1}(x, s) & u_k(x, a) & \cdots\ u_{n-1}(x, a) \\ u_{n-1}^{(k)}(b, s) & u_k^{(k)}(b, a) & \cdots\ u_{n-1}^{(k)}(b, a) \\ \cdots & & \cdots \\ u_{n-1}^{(n-1)}(b, s) & u_k^{(n-1)}(b, a) & \cdots\ u_{n-1}^{(n-1)}(b, a) \end{vmatrix}$$

on the triangle $a \leqq s \leqq x \leqq b$, *where*

$$D = \begin{vmatrix} u_k^{(k)}(b, a) & \cdots\ u_{n-1}^{(k)}(b, a) \\ \cdots & \cdots \\ u_k^{(n-1)}(b, a) & \cdots\ u_{n-1}^{(n-1)}(b, a) \end{vmatrix} > 0.$$

We are now ready for the main result of this paper.

THEOREM 4. *If* (1) *is disfocal on* $[a, b]$, *then*

$$(-1)^{n-k} G_k^{(i)}(x, s) > 0$$

on $(a, b) \times (a, b)$ *for* $i = 0, \cdots, k-1$.

PROOF. Set, for $s \in (a, b)$,

$$v(x) = \begin{vmatrix} u_{n-1}(x, s) & u_k(x, a) & \cdots\ u_{n-1}(x, a) \\ u_{n-1}^{(k)}(b, s) & u_k^{(k)}(b, a) & \cdots\ u_{n-1}^{(k)}(b, a) \\ \cdots & & \cdots \\ u_{n-1}^{(n-1)}(b, s) & u_k^{(n-1)}(b, a) & \cdots\ u_{n-1}^{(n-1)}(b, a) \end{vmatrix}$$

and let $u(x)$ be the above determinant with $u_{n-1}(x, s)$ replaced by zero. By Lemma 3, it suffices to show that $(-1)^{n-k}u^{(i)}(x) > 0$ on $(a, s]$, $(-1)^{n-k}v^{(i)}(x) > 0$ on $[s, b)$ for $i = 0, \cdots, k-1$. We will first show that $(-1)^{n-k}u^{(i)}(x) > 0$ on $(a, s]$ for $i = 0, \cdots, k-1$. Note that

$$v(x) - u(x) = Du_{n-1}(x,s) \tag{6}$$

and so $u(x) = v(x) - Du_{n-1}(x, s)$ is a linear combination of the $k + 1$ solutions $u_0(x, b), \cdots, u_{k-1}(x, b)$, and $u_{n-1}(x, s)$. Set

$$W_{k+1}(x) = W[u_0(x, b), \cdots, u_{k-1}(x, b), u_{n-1}(x, s)], \tag{7}$$

the Wronskian of $u_0(x, b), \cdots, u_{k-1}(x, b), u_{n-1}(x, s)$. By use of (5) and using properties of determinants we get that

$$W_{k+1}(x) = (-1)^{k-2} \begin{vmatrix} z_{n-1}^{(n-1)}(b, x) \cdots z_{n-1}(s, x) \\ z_{n-2}^{(n-1)}(b, x) \cdots z_{n-2}(s, x) \\ \cdots \qquad \cdots \\ z_{n-k-1}^{(n-1)}(b, x) \cdots z_{n-k-1}(s, x) \end{vmatrix}.$$

It follows from this last expression for $W_{k+1}(x)$ that $W_{k+1}(x_0) = 0$ at $x_0 \in [a, s)$ iff there is a nontrivial solution $z(x)$ of (4) such that $z^{(i)}(x_0) = 0$, $i = 0, \cdots, n - k - 2$, $z(s) = 0$, and $z^{(j)}(b) = 0$, $j = n - k, \cdots, n - 1$. By Rolle's theorem $z(x)$ is a focal solution of (4) on $[x_0, b]$. This contradicts the fact that (4) is disfocal. Hence, $W_{k+1}(x) \neq 0$ for $a \leqq x < s$. This holds for each $s \in (a, b]$. Let $s = b$ in (7) to obtain

$$(-1)^{n-k-1}W_{k+1}(x)|_{s=b} > 0.$$

Hence, $(-1)^{n-k-1}W_{k+1}(x) > 0$ for $a \leqq x < s$. Now define the kth order differential operator ℓ_k by

$$\ell_k[y(x)] = \frac{W[u_0(x, b), \cdots, u_{k-1}(x, b), y(x)]}{W_k(x)}$$

where $W_k(x) = W[u_0(x, b), \cdots, u_{k-1}(x, b)]$. By the $(0, \cdots, k - 1; k, \cdots, n - 1)$-disfocalness of (1) and since $W_k(b) = 1$ we have that $W_k(x) > 0$ for $a \leqq x \leqq b$. By use of (6) we have that

$$\ell_k[u(x)] = -D\frac{W_{k+1}(x)}{W_k(x)}.$$

Since $u^{(i)}(a) = 0$, $i = 0, \cdots, k - 1$,

$$u(x) = \int_a^x K(x, \tau) \left[-D \frac{W_{k+1}(\tau)}{W_k(\tau)}\right] d\tau$$

where $K(x, \tau)$ is the Cauchy function for $\ell_k[y] = 0$ (so $K^{(i)}(\tau, \tau) = \delta_{i,k-1}$, $i = 0, \cdots, k - 1$). Now $K(x, \tau)$ is a linear combination of $u_0(x, b), \cdots, u_{k-1}(x, b)$ implies that $K^{(j)}(b, \tau) = 0$, $j = k, \cdots, n - 1$. It follows that

$K^{(i)}(x, \tau) > 0$ for $a \leqq \tau < x \leqq b$, $i = 0, \cdots, k - 1$. From

$$u^{(i)}(x) = \int_a^x K^{(i)}(x, \tau) \left[-D \frac{W_{k+1}(\tau)}{W_{k+1}(\tau)} \right] d\tau, \; i = 0, \cdots, x\, 1,$$

we get that

$$(-1)^{n-k} u^{(i)}(x) > 0 \text{ for } a < x \leqq s, \; i = 0, \cdots, k-1.$$

We now set out to prove that

$$\lambda v^{(k)}(x) > 0$$

for $s \leqq x < b$. From (6),

$$v(x) = u(x) + D u_{n-1}(x, s).$$

Hence, $v^{(k)}(x)$ is a linear combination of the $n - k + 1$ functions $u_k^{(k)}(x, a), \cdots, u_{n-1}^{(k)}(x, a), u_{n-1}^{(k)}(x, s)$. Set

$$\omega_{n-k+1}(x) = W[u_k^{(k)}(x, a), \cdots, u_{n-1}^{(k)}(x, a), u_{n-1}^{(k)}(x, s)], \tag{8}$$

the Wronskian of $u_k^{(k)}(x, a), \cdots, u_{n-1}^{(k)}(x, a), u_{n-1}^{(k)}(x, s)$.

Since $u_k(x, a), \cdots, u_{n-1}(x, a), u_{n-1}(x, s)$ are solutions of (1) we get that

$$\omega_{n-k+1}(x) = \begin{vmatrix} u_k^{(k)}(x, a) & \cdots u_{n-1}^{(k)}(x, a) & u_{n-1}^{(k)}(x, s) \\ \cdots & \cdots & \cdots \\ u_k^{(n-1)}(x, a) & \cdots u_{n-1}^{(n-1)}(x, a) & u_{n-1}^{(n-1)}(x, s) \\ \lambda p(x) u_k(x, a) & \cdots \lambda p(x) u_{n-1}(x, a) & \lambda p(x) u_{n-1}(x, a) \end{vmatrix}$$

$$= \lambda p(x) \begin{vmatrix} z_{n-k-1}^{(n-k-1)}(a, x) \cdots & (-1)^{n-1+k} z_{n-k-1}(a, x) & (-1)^{n-1+k} z_{n-k-1}(s, x) \\ \cdots & \cdots & \cdots \\ (-1)^{n+k-1} z_0^{(n-k-1)}(a, x) \cdots & z_0(a, x) & z_0(s, x) \\ (-1)^k z_{n-1}^{(n-k-1)}(a, x) \cdots & (-1)^{n-1} z_{n-1}(a, x) & (-1)^{n-1} z_{n-1}(s, x) \end{vmatrix}$$

$$= (-1)^{n-1} \lambda p(x) \begin{vmatrix} z_{n-k-1}^{(n-k-1)}(a, x) \cdots & z_{n-k-1}(a, x) & z_{n-k-1}(s, x) \\ \cdots & \cdots & \cdots \\ z_0^{(n-k-1)}(a, x) \cdots & z_0(a, x) & z_0(s, x) \\ z_{n-1}^{(n-k-1)}(a, x) \cdots & z_{n-1}(a, x) & z_{n-1}(s, x) \end{vmatrix} .$$

It follows that $\omega_{n-k+1}(x_0) = 0$ for $x_0 \in (s, b]$ iff there is a solution $z(x)$

of (4) such that $z(s) = 0$, $z^{(i)}(a) = 0$, $i = 0, \cdots, n-k-1$, and $z^{(j)}(x_0) = 0$, $j = n-k, \cdots, n-2$. But then by Rolle's theorem there is a point $c \in (a, x_0)$ such that $z^{(n-1)}(c) = 0$. This contradicts the fact that (4) is disfocal on $[a, b]$. Hence, $\omega_{n-k+1}(x) \neq 0$ for $x \in (s, b]$.

Letting $x = s$ in (8) we see that $(-1)^{n-k}\lambda\omega_{n-k+1}(s) > 0$, hence

$$(-1)^{n-k}\lambda\omega_{n-k+1}(x) > 0$$

for $s \leqq x \leqq b$.

Define the $(n-k)$-th order operator M_{n-k} by

$$M_{n-k}[y] = \frac{W[u_k^{(k)}(x, a), \cdots, u_{n-1}^{(k)}(x, a), y]}{\omega_{n-k}(x)}$$

where

$$\omega_{n-k}(x) \equiv W[u_k^{(k)}(x, a), \cdots, u_{n-1}^{(k)}(x, a)] > 0$$

on $[a, b]$, by the $(0, \cdots, k-1; k, \cdots, n-1)$-disfocalness of (1) and $\omega_{n-k}(a) = 1$.

Since $v^{(k)}(x) = u^{(k)}(x) + Du_{n-1}^{(k)}(x, s)$,

$$M_{n-k}[v^{(k)}(x)] = \frac{D\omega_{n-k+1}(x)}{\omega_{n-k}(x)} .$$

Since, further, $v^{(k)}(b) = \cdots = v^{(n-1)}(b) = 0$, we have that

$$v^{(k)}(x) = \int_b^x C(x, \tau) \frac{D\omega_{n-k+1}(\tau)}{\omega_{n-k}(\tau)} d\tau$$

where $C(x, \tau)$ is the Cauchy function for $M_{n-k}[y] = 0$ (so $C^{(i)}(\tau, \tau) = \delta_{i,n-k-1}$, $i = 0, \cdots, n-k-1$). Since $W[u_k^{(k)}(x, a), \cdots, u_i^{(k)}(x, a)] > 0$ on $[a, b]$ for $i = k, \cdots, n-1$, $M_{n-k}[y] = 0$ is disconjugate on $[a, b]$. Since $C(x, \tau)$ has a zero of order $n-k-1$ at τ with $C^{(n-k-1)}(\tau, \tau) = 1$, we have that

$$(-1)^{n-k-1}C(x, \tau) > 0 \quad \text{for} \quad x < \tau < b.$$

Since

$$\lambda v^{(k)}(x) = \int_x^b [(-1)^{n-k-1}C(x, \tau)] \frac{D[(-1)^{n-k}\lambda\omega_{n-k+1}(x)]}{\omega_{n-k}(\tau)} d\tau$$

we have that

$$\lambda v^{(k)}(x) > 0 \qquad \text{on } [s, b).$$

We will now use this last inequality to show that $(-1)^{n-k}v^{(i)}(x) > 0$ on $[s, b]$, $i = 0, \cdots, k-1$.

For $0 \leqq i \leqq k-1$, consider

$$v^{(i)}(b) = \begin{vmatrix} u_{n-1}^{(i)}(b, s) & u_k{}^{(i)}(b, a) & \cdots u_{n-1}^{(i)}(b, a) \\ u_{n-1}^{(k)}(b, s) & u_k{}^{(k)}(b, a) & \cdots u_{n-1}^{(k)}(b, a) \\ \cdots & & \cdots \\ u_{n-1}^{(n-1)}(b, s) & u_k{}^{(n-1)}(b, a) & \cdots u_{n-1}^{(n-1)}(b, a) \end{vmatrix} .$$

By use of (5) and properties of determinants we get that

$$v^{(i)}(b) = (-1)^{n+i-1} \begin{vmatrix} z_{n-i-1}(s, b) & z_{n-i-1}^{(n-k-1)}(a, b) \cdots z_{n-i-1}(a, b) \\ z_{n-k-1}(s, b) & z_{n-k-1}^{(n-k-1)}(a, b) \cdots z_{n-k-1}(a, b) \\ \cdots & \cdots \\ z_0(s, b) & z_0{}^{(n-k-1)}(a, b) \cdots z_0(a, b) \end{vmatrix}$$

Hence, $v^{(i)}(b)]_{s_0} = 0$ iff there is a nontrivial solution $z(x)$ of (4) such that $z^{(i)}(a) = 0$, $i = 0, \cdots, n-k-1$ $z(s_0) = 0$, $z^{(j)}(b) = 0$, $j = n-k, \cdots, n-1$ but $j \neq n-i-1$. But then by Rolle's theorem there is a point $c \in (a, b)$ such that $z^{(n-i-1)}(c) = 0$. This contradicts the fact that (4) is disfocal on $[a, b]$. Hence, $v^{(i)}(b) \neq 0$ for all $s \in (a, b)$. Further

$$v^{(i)}(b)]_{s=b} = (-1)^{n-k} \begin{vmatrix} u_k{}^{(i)}(b, a) & \cdots u_{n-1}^{(i)}(b, a) \\ u_k{}^{(k)}(b, a) & \cdots u_{n-1}^{(k)}(b, a) \\ \cdots & \cdots \\ u_k{}^{(n-2)}(b, a) & \cdots u_{n-1}^{(n-2)}(b, a) \end{vmatrix} .$$

Therefore

$$(-1)^{n-k} v^{(i)}(b) > 0$$

for $a < s \leqq b$, $i = 0, \cdots, k-1$.

Since $(-1)^{n-k} v^{(k-1)}(s) > 0$, $(-1)^{n-k} v^{(k-1)}(b) > 0$, and the derivative of $(-1)^{n-k} v^{(k-1)}(x)$ is either strictly positive or strictly negative on $[s, b)$, it follows that

$$(-1)^{n-k} v^{(k-1)}(x) > 0 \qquad \text{for } s \leqq x \leqq b.$$

By this same argument and finite mathematical induction we get that

$$(-1)^{n-k} v^{(i)}(x) > 0$$

for $s \leqq x \leqq b$ and $i = 0, \cdots, k-1$.

We then get the following corollary.

COROLLARY 5. *If the strictly positive assumption on $p(x)$ in Theorem 4 is replaced by $p(x) \geqq 0$ on $[a, b]$, then the conclusion becomes*

$$(-1)^{n-k} G_k^{(i)}(x, s) > 0 \qquad \text{for } a < x \leqq s < b$$

$$(-1)^{n-k} G_k^{(i)}(x, s) \geqq 0 \qquad \text{for } s < x < b.$$

PROOF. The proof of the first inequality in this corollary is the same as in Theorem 4. To prove the second inequality, consider the differential equation

$$y^{(n)} = \lambda p_\epsilon(x) y \tag{9}$$

where $p_\epsilon(x) = p(x) + \epsilon$ where $\epsilon > 0$. It is easy to argue that for $\epsilon > 0$, sufficiently small, equation (9) satisfies the hypothesis of Theorem 4. Using this and a limiting argument we get the desired result.

The differential equation $y^{(n)} = 0$ shows that we cannot get the conclusion in Theorem 4 with the assumptions of Corollary 5.

In [4], Nehari gives a concise formula for $G_k(x, s)$ when $p(x) = 0$ (note that Nehari uses different notation and considers the operator $(-1)^{n-k} d^n/dx^n$ instead of the operator d^n/dx^n.

Theorem 4 can be generalized to the differential equation

$$\rho_{n+1}(x) \frac{d}{dx} \, \rho_n(x) \, \frac{d}{dx} \cdots \frac{d}{dx} \, \rho_1(x) y = \lambda p(x) y \tag{10}$$

where $\lambda = \pm 1$ and p, ρ_i, $1 \leqq i \leqq n+1$, are positive continuous functions on $[a, b]$. Define quasi derivatives D_i, $0 \leqq i \leqq n$ by

$$D_0 y = \rho_1(x) y$$

$$D_i y = \rho_{i+1}(x)(D_{i-1} y)', \qquad i = 1, \cdots, n.$$

We say a solution $y(x)$ of (10) has a zero of order k at a point x_0 provided $D_i y(x_0) = 0$, $i = 0, \cdots, k-1$. We say (10) is disfocal (see [3]) on $[a, b]$ provided there is not nontrivial solution $y(x)$ of (10) such that there are points x_i, $0 \leqq i \leqq n-1$, in $[a, b]$ with $D_i y(x_i) = 0$, $i = 0, \cdots, n-1$.

Assume (10) is disfocal on $[a, b]$ and let $g_k(x, s)$, $1 \leqq k \leqq n-1$, be the Green's function for the k-focal point problem

$$\rho_{n+1}(x) \frac{d}{dx} \, \rho_n(x) \, \frac{d}{dx} \cdots \frac{d}{dx} \, \rho_1(x) y - \lambda p(x) y = h(x)$$

$$D_i y(a) = 0 \qquad i = 0, \cdots, k-1$$

$$D_j y(b) = 0 \qquad j = k, \cdots, n-1.$$

In the following theorem we will use the following notation

$$D_0 g_k(x, s) = \rho_1(x) g_k(x, s)$$

$$D_i g_k(x, s) = \rho_{i+1}(x) \frac{\partial}{\partial x} D_{i-1} g_k(x, s).$$

THEOREM 6. *If* (10) *is disfocal on* $[a, b]$, *then*

$$(-1)^{n-k} D_i g_k(x, s) > 0$$

on $(a, b) \times (a, b)$ *for* $i = 0, \cdots, k-1$.

The proof of Theorem 6 is similar to the development and proof of Theorem 4 in this paper. In most places it is a matter of replacing derivatives by the corresponding quasi derivatives and making sure the results still go through with this change.

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