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THE σ-REPRESENTATIONS OF AMENABLE GROUPOIDS

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ABSTRACT. Techniques of Zimmer are exploited to show that for an ergodic equivalence relation arising from a group action, injectivity of any σ-regular representation von Neumann algebra implies injectivity of all of the others; this is so in particular if the group acting is amenable. The σ-representations of more general groupoids also are discussed.

Ergodic action of a group G on a measure space (S, μ) defines an ergodic equivalence relation $\mathscr{E}_G$ which has a representation theory analogous in some ways to the representation theory for groups. Of special interest is the regular representation of $\mathscr{E}_G$, which generalizes the group-measure space construction and is primary in the sense that its commuting algebra is a factor. Just as for groups, to every 2-cocycle σ on $\mathscr{E}_G$ are associated σ-representations, in particular, a σ-regular representation, which also is primary. The flow of weights of the factors obtained from different cocycles always is the same [8], but Connes' example [1] shows that the factors themselves may be different.

Zimmer has introduced the concept of amenability for measure groupoids such as $\mathscr{E}_G$ [13] and proved the equivalence of amenability of $\mathscr{E}_G$ for discrete G to possession by the regular representation factor of property E [14, 15]. The discreteness assumption was removed in [4] using reductions. In this paper Zimmer's methods are extended to σ-representations using results in [6, 9]. We show in particular that if one σ-representation factor is injective, then all of them are, so (except in case they are of type III_1) they must coincide. This holds in particular for the measure-preserving actions of amenable groups.

If G is abelian, then the relation $\mathscr{E}_G$ is approximately finite (AF): there is an ascending sequence of smooth subrelations on (S, μ) whose measure-theoretic union in G [4]. It has been conjectured that the same is true if G is any amenable group; and indeed some progress in that direction has been made [3, 11]. For AF actions, the 2-cohomology is trivial and all representations are AF. Zimmer's papers and the present work in a sense bridge the gap left by the question of approximate finiteness of the relations by proving the representation theory to be injective directly from amenability.

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The paper is organized as follows: in § 1 we give the definitions eschewed in this introduction. § 2 is devoted to two main technical results generalizing to σ-representations of discrete relations the methods of Zimmer. In fact, very little change is necessary; and where we omit details, they may be found by consulting [14, 15]. The main results are derived in § 3 by reduction to the discrete case using the theory in [4] and a generalization of an unpublished theorem in [6]. Finally, in § 4 we consider more general measure groupoids and apply the techniques of § 2 to the groupoids arising in Zeller-Meier's crossed product [12] as an example.

1. **Preliminary Notions.** Let $(\mathcal{G}, C)$ be a measure groupoid. By this we mean that, first, $\mathcal{G}$ is an analytic set with the algebraic structure of category with inverses for which $\mathcal{G}^{(2)} = \{(x, y) \in G \times G : xy$ is defined$\}$ is Borel and the maps $x \mapsto x^{-1} : \mathcal{G} \to \mathcal{G}$ and $(x, y) \mapsto xy : \mathcal{G}^{(2)} \to G$ are Borel. Second, C is a measure class on $\mathcal{G}$ containing a probability measure λ symmetric under $x \mapsto x^{-1}$ and with a disintegration $\lambda = \int \lambda^u d\tilde{\lambda}(u)$ with respect to $r = (x \mapsto xx^{-1})$ satisfying the following condition: for some $\tilde{\lambda}$-conull Borel subset $U_0 \subset U_{\mathcal{G}} = r(\mathcal{G})$, $r(x) \in U_0$ and $d(x) = x^{-1}x \in U_0$ imply that $E \mapsto \int 1_E(xy)\, d\lambda^{d(x)}(y)$ and $\gamma^{r(x)}$ are equaivalent measures. It is possible to find σ-finite measures (called *Haar measures* [7] for which the above equivalences are equality: $\int 1_E(xy)\, d\nu^{d(x)}(y) = \int 1_E(y)\, d\nu^{r(x)}(y)$. The isotropy groups $\mathcal{G}_u = \{x \in G : r(x) = d(x) = u\}$ possess a locally compact topology.

A *2-cocycle* from $\mathcal{G}$ into the circle $\mathbf{T}$ is a function $\sigma : \mathcal{G}^{(2)} \to \mathbf{T}$ satisfying $\sigma(x, y)\sigma(xy, z) = \sigma(x, yz)\sigma(y, z)$ and $\sigma(r(x), x) = \sigma(x, d(x)) = 1$ for x, y, and z belonging to an *inessential reduction* (*i.r.*) $\mathcal{G} \mid U_0 = \{x \in \mathcal{G} : r(x) \in U_0$ and $d(x) \in U_0\}$, $U_0 \subset U_{\mathcal{G}}$ conull and Borel. A weakly Borel function W of $\mathcal{G}$ into the unitary group of separable Hilbert space H *is a σ-repesentation* if $W(xy) = \sigma(x, y)W(x)W(y)$ for x, y in an i.r. The *commuting algebra* W' of W consists of all decomposable operators $T = \int T(u)\, d\tilde{\lambda}(u)$ on the direct integral space $\int H d\tilde{\lambda}(u) = L^2(U_{\mathcal{G}}, \tilde{\lambda}, H)$ such that $T(r(x))W(x) = W(x)T(d(x))$ a.e.; W' is a von Neumann algebra. For those groupoids with a.a. $L^2(G, \lambda^u)$ of the same dimension, there is a σ-regular representation W^σ defined by $W_\sigma(x)f(y) = \sigma(y^{-1}, x)^{-1}f(x^{-1}y)$, $W_\sigma(x) : L^2(G, \nu^{d(x)}) \to L^2(G, \nu^{r(x)})$ and then $W^\sigma(x) = V(r(x))W_\sigma(x)V(d(x))^{-1}$, where $V = \int V(u)\, d\tilde{\lambda}(u) : \int L^2(G, \nu^u)\, d\tilde{\lambda}(u) \xrightarrow{\sim} \int H d\tilde{\lambda}(u)$ is an isomorphism.

If G is a locally compact second countable group acting on an analytic probability space (S, μ) so that $(s, g) \mapsto sg : S \times G \to S$ is Borel and the measure class $[\mu]$ is invariant, $(S \times G, [\mu] \times \text{Haar})$ is a measure groupoid with multiplication $(s, g)(sg, g') = (s, gg')$. The map

$x \mapsto (r(x), d(x))$ projects any groupoid G onto another, $(r, d)(\mathscr{G})$, which is *principal*: the isotropy groups are trivial. $(r, d)(S \times G)$ is denoted $\mathscr{E}_G$, the measured equivalence relation furnished by the action of G on (S, μ). A necessary and sufficient condition for a principal $\mathscr{G}$ to be similar [9] to an $\mathscr{E}_G$ is that there be a Borel set $E \subset U_{\mathscr{G}}$ such that $[E] = r(d^{-1}(E))$ is conull and $[u] \cap E$ is countable for a.a. u [4]. Such principal groupoids are called *concrete*.

Zimmer's concept of amenability for measure groupoids is as follows: Let $u \mapsto K_u$ be a family of non-empty weakly compact convex subsets of the dual E^* of a separable Banach space E. Let $\gamma : \mathscr{G} \to \operatorname{Aut} E$ be a homomorphism ($\gamma(xy) = \gamma(x)\gamma(y)$ on an i.r.), Borel for the strong operator topology; and let $\gamma^*(x) = \gamma(x^{-1})^*$. If $\{(u, \phi) \in U_{\mathscr{G}} \times E^* : \phi \in K_u\}$ is Borel and $\gamma^*(x)K_{d(x)} = K_{r(x)}$ a.e., $u \mapsto K_u$ is called invariant. $\mathscr{G}$ is amenable if for every γ and invariant $u \mapsto K_u$, there is a Borel $u \mapsto \phi_u \in K_u$ with $\gamma^*(x)\phi_{d(x)} = \phi_{r(x)}$ a.e. $u \mapsto \phi_u$ is called an invariant section.

Finally, from the theory of von Neumann algebras we recall the following property equivalent to injectivity ([2], Proposition 6.2): a von Neumann subalgebra M of the algebra $\mathscr{B}(H)$ of all bounded operators on the separable Hilbert space H is said to have property E if there is a norm one projection P of $\mathscr{B}(H)$ onto M. $P(I) = I$ and $S_1P(T)S_2 \neq P(S_1TS_2)$ for all S_1, $S_2 \in M$ and $T \in \mathscr{B}(H)$.

2. **Fundamental Technical Results.** We are prepared now to show how Zimmer's techniques may be adapted to treat σ-representations of discrete principal groupoids. The proof of the present Proposition A is essentially that of Theorem 2.1 of [15]; for Proposition B, the proof of the theorem of [14] is modified. The discreteness assumption will be removed later.

PROPOSITION A. *The commuting algebra of any σ-representation of an amenable standard countable equivalence relation $\mathscr{E}$ has property E.*

PROOF (sketch). Let G be a discrete group acting on (S, μ) to furnish $\mathscr{E}$. $\sigma \sim \sigma'$, where σ' has the property $\sigma((s, t), (t, s) = 1$. By Lemma 4.10 of [8], we may assume for σ this property. Let M_r be the isomorphism of $L^\infty(S, \mu)$ with the diagonalizable operators in $\mathscr{B}(\int H\, d\mu(s))$, $\mathscr{D}$ the decomposable operators. Let ρ be defined by $\int f(sg)\, d\mu(s) = \int f(s)\rho(s, g)\, d\mu(s)$. $U_g\phi(s) = \rho(s, g)^{1/2}W(s, sg)\phi(sg)$ defines a unitary operator on $\int H\, d\mu(s)$ and if $T = \int T(s)\, d\mu(s) \in \mathscr{D}$, $U_gTU_g^{-1} \in D$ and

$$\Phi_gT(s) = U_gTU_g^{-1}(s) = W(s, sg)T(sg)W(s, sg)^{-1} \tag{1}$$

$U_g{}^{-1} = U_{g^{-1}}$ $g \mapsto U_g$ is not necessarily a homomorphism of G, but $g \mapsto \Phi_g$ is, and

$$\gamma(s,\, t)Q = W(s,\, t)QW(t,\, s) \tag{2}$$

defines a homomorphism of $\mathscr{E}$ into the automorphisms of trace class (H). γ^* on (tr class$(H))^* = B(H)$ is defined by the same formula.

Let $T \in \mathscr{D}$. Apply Lemma 2.2 of [15] to the maps $s \mapsto \sum_{i=1}^n M_r(f_i)\Phi_{g_i}T(s)$ $(\sum_{i=1}^n f_i \equiv 1)$ to obtain a Borel family $s \mapsto C_s(T)$ of compact convex subsets of $\mathscr{B}(H)$. Since

$$\gamma^*(s,\, sg)\Phi_{g_1}T(sg) = \Phi_{gg_1}T(s), \tag{3}$$

this is an invariant family, a cross-section for which belongs to $C_S(T) \cap W'$, where $C_S(T)$ is the closed convex hull of the family of operators $\sum_{i=1}^n M_r(f_i)U_{g_i}TU_{g_i}^{-1}$ in either the weak or the $\sigma(L^\infty(S,\, \mu,\, \mathscr{B}(H),$ $L^1(S,\, \mu,$ tr class$(H)))$ topology.

Now let F consist of all $\sum_{i=1}^n M_r(f_i)\Phi_{g_i}$ as above and let $\overline{F}$ be the closed convex hull in the $\boldsymbol{\sigma} = \sigma(B(\mathscr{D}),\, \mathscr{D} \otimes_{\max} \mathscr{D}*)$ topology on the algebra of bounded operators on $\mathscr{D}$. Before proceeding further, we need a lemma.

LEMMA. *Let $\Phi_\gamma \in F$ converge in $\mathscr{B}(\mathscr{D})$ to Φ_0 and let $\Phi \in F$. Then $\Phi\Phi_\gamma \longrightarrow \Phi\Phi_0 \in \overline{F}$.*

PROOF. $\boldsymbol{\sigma}$-convegence implies pointwise weak convergence on $\mathscr{D}$. Let $T \in \mathscr{D}$, $\phi, \psi \in \int h\, d\mu(s)$. Let $\Phi = \sum_{i=1}^n M_r(f_i)\Phi_{g_i}$. By (1),$\langle \Phi(\Phi_\gamma(T))\, (\phi),$ $\psi\rangle = \sum_{i=1}^n \langle U_{g_i}(\Phi_\gamma(T))\, (U_{g_i}^{-1}(\phi)),\, M_r(f_i)\psi\rangle \rightarrow \sum_{i=1}^n \langle U_{g_i}(\Phi_0(T))\ \ (U_{g_i}^{-1}(\phi)),$ $M_r(f_i\psi\rangle = \langle \Phi(\Phi_0(T))(\phi),\, \psi\rangle$. On bounded subsets of $\mathscr{B}(\mathscr{D})$, pointwise weak convergence implies σ-convergence, so $\sigma - \lim \Phi\Phi_\gamma = \Phi\Phi_0$. The multiplicativity $\Phi_{g_1}\Phi_{g_2} = \Phi_{g_1g_2}$ of $\mapsto\Gamma_g$ implies that each $\Phi\Phi_\gamma$ belongs to F, so $\Phi\Phi_0 \in \overline{F}$.

Returning to the proof of Proposition A, we partially order $\overline{F}$ by $\phi_1 \geqq \phi_2$ if $C_S(\Phi_1(T)) \subset C_S(\Phi_2(T))$ for all $T \in \mathscr{D}$. As in Proposition 4.4.15 of [10], one sees that Zorn's Lemma applies to give a maximal element Φ_0 (the key point is that $T_1 \in C_S(T)$ implies $C_S(T_1) \subset C_S(T)$, which holds by an argument as in the lemma). Let $T_1 \in C_S(\Phi_0(T)) \cap W'$. Let $\Phi_\gamma \in F$ be a net such that $\Phi_\gamma(\Phi_0(T)) \longrightarrow T_1$. We may assume that $\Phi_\gamma\Phi_0$ has a limit Φ_1, so that $\Phi_1(T) = T_1$. $C_S(\Phi_1(T)) = C_S(T_1) \subset C_S(\Phi_0(T))$ and since Φ_0 is an accumulation point of F, $\Phi_1 \in \overline{F}$ by the lemma. By maximality of Φ_0, $\{T_1\} = C_S(\Phi_0(T)) =$ $[\Phi_0(T)\}$. Thus Φ_0 is the map required for property E.

PROPOSITION B. *Let W^σ be the left σ-regular representation of the standard countable equivalence relation $\mathscr{E}$. If $(W^\sigma)'$ has property E, then $\mathscr{E}$ is amenable.*

PROOF (sketch). Let G, S, μ, and ρ be as in the proof of Proposition A. Again we may assume $\sigma((s, t), (t, s)) = 1$. Let β_s be a counting measure on $sG \subset S$, so that $\nu = \int \delta_s \times \beta_s \, d\mu(s)$ defines a Haar measure (ν, μ). $\Delta = (d\nu^{-1}/d\nu)^{-1}$ satisfies $\rho(s, g) = \Delta(sg^{-1}, s)$ a.e. and may be taken to be a homomorphism. $(W^\sigma)''$ is spatially isomorphic to the von Neumann algebra L_σ generated by the operators T_f defined by Equation 4.2 of [8].

LEMMA. *$U_g j(s, t) = \sigma((t, s), (s, sg))^{-1}\Delta(sg, s)^{1/2} j(sg, t)$ defines a unitary $U_g \in L_\sigma$. $V_g j(s, t) = \sigma((s,), (t, tg)) j(s, tg)$ defines a unitary $V_g \in R_\sigma = L_\sigma'$. $V_g^{-1} j(s, t) = \sigma((s, tg^{-1}), (tg^{-1}, t))^{-1} j(s, tg^{-1})$a.e.*

PROOF. Let D be the characteristic function of $\{(s, t) \in \mathscr{E} : s = t\}$. Letting $f(s, t) = D(sg, t)\Delta(sg, s)^{1/2}$ in Equation 4.2 of [8], we obtain U_g after perhaps a limit argument involving Δ-boundedness. $V_g = J_\sigma U_g J_\sigma$ by equation (4.6) of [8]. Thus $U_g \in L_\sigma$ and $V_g \in R_\sigma$. The remaining statements involve only computation.

Returning now to the proof of Proposition B, for $f \in L^\infty(\mathscr{E}, \nu)$ define $f^g(s, t) = f(s, tg)$. Let M be the representation of $L^\infty(\mathscr{E}, \nu)$ on $L^2(\mathscr{E}, \nu)$ by multiplication. $M(f^g) = V_g M(f) V_g^{-1}$. Let P be the projection of $\mathscr{B}(L^2(\mathscr{E}, \nu))$ onto R_σ guaranteed by property E and let $R(f) = P \circ M(f)$. $V_g R(f) V_g^{-1} = P(V_g M(f) V_g^{-1} = R(f^g)$. Moreover, as $R(f) \in R_\sigma \subset M_r(L^\infty(S, \mu))'$ (Theorem 4.1 of [8]), $R(f) = \int R(f)(s)\, d\mu(s)$ is decomposable and $R(f)(s) = W_\sigma(s, sg) R(f)\,(sg) W_\sigma(sg, s)$ for μ-a.a. s.

Let $\tau(f)(s) = \int R(f)(s) D(s, \cdot) D(s, \cdot)\, d\beta_s$. $\tau : L^\infty(\mathscr{E}, \nu) \longrightarrow L^\infty(S, \mu)$ is a norm one unital positive projection.

$$
\begin{aligned}
\tau(f^g)(s) &= \int R(f)(s)(V_g^{-1} D)(s, \cdot) V_g^{-1} D(s, \cdot)\, d\beta_s \\
&= \int R(f)(sg) W(sg, s)(V_g^{-1} D)(sg, \cdot) W(sg, s)(V_g^{-1} D)(sg, \cdot)\, d\beta_{sg} \\
&= \tau(f)(sg) \text{ a.e.}
\end{aligned}
$$

because $W(sg, s)(V_g^{-1}D)(sg, t) = \sigma((t, sg), (sg, s))^{-1} V_g^{-1} D(s, t) = \sigma((t, sg), (sg\, s))^{-1} \sigma((s, tg^{-1}), (tg^{-1}, t))^{-1} D(s, tg^{-1}) = D(sg, t)$. Another calculation shows that a.e.

$$
\tau(f 1_E \circ d)(s) = \tau(f)(s) 1_E(s). \tag{4}
$$

Now we use τ to produce an invariant section for a cocycle γ^* on $\mathscr{E}$ with $s \mapsto K_s \subset E^*$ an invariant Borel family of non-empty compact convex subsets. Choose $s \mapsto b(s) \in K_s$ a Borel function and define $F(t, s) = \gamma^*(s, t)b(t) \in K_s$. Choose $a(s)$ so that $\tau((u, v) \mapsto \langle \theta, F(u, v)\rangle)(s) = \langle \theta, a(s)\rangle$ a.e. By extension and (4), $\tau((t, v) \mapsto \langle \theta(v), F(u, v)\rangle)(s) = \langle \theta(s), a(s)\rangle$ a.e. For $g \in G$,

$$\begin{aligned}\langle \theta, a(sg)\rangle &= \tau((u, v) \mapsto \langle \theta, F(u, v)\rangle)(sg) \text{ a.e.}\\ &= \tau(((u, v) \mapsto \langle \theta, F(u, v)\rangle)^g)(s)\\ &= \tau((u, v) \mapsto \langle \theta, F(u, vg)\rangle)(s)\\ &= \tau((u, v) \mapsto \langle \theta, \gamma^*(vg, v)\gamma^*(v, u)b(u)\rangle)(s)\\ &= \tau((u, v) \mapsto \langle \gamma(v, vg)\theta, F(u, v)b(u)\rangle)(s)\\ &= \langle \gamma(s, sg)\theta, a(s)\rangle \text{ a.e.}\\ &= \langle \theta, \gamma^*(sg, s)a(s)\rangle\end{aligned}$$

so that by separability of E, $s \mapsto a(s)$ is invariant.

To show $a(s) \in K_s$ a.e. it suffices to prove that if $S_q = \{s \in S : \langle \theta, \alpha\rangle \geqq q \text{ for all } \alpha \in A_s\}$ is non-null, then $\langle \theta, a(s)\rangle \geqq q$ a.e. on S_q. But by (4), $\tau((u, v) \mapsto \langle \theta, F(u, v)\rangle 1_{S_q}(v))(s) \geqq \tau((u, v) \mapsto q1_{S_q}(v))(s) = q1_{S_q}(s)$ a.e. Thus $\langle 1_{S_q}(s)\theta, a(s)\rangle \geqq q1_{S_q}(s)$ a.e., so $\langle \theta, a(s)\rangle \geqq q$ for a.a. $s \in S_q$.

3. **Reduction to the Discrete Case.** Statements about concrete principal groupoids—those principal groupoids furnished by group actions—often can be reduced to the discrete case by the result in [4] already described. To use this idea in the present case, we need a result telling how the von Neumann algebras behave under such reduction.

THEOREM. *Let $[\phi] : \mathcal{G} \to \mathcal{H}$ and $[\psi] : \mathcal{H} \to \mathcal{G}$ be a similarity of groupoids and W a σ-representation of $\mathcal{H}$. Then ϕ may be chosen so that $W \circ \phi$ is a $\sigma \circ (\phi \times \phi)$-representation of $\mathcal{G}$ and then W' and $(W \circ \phi)'$ are isomorphic von Neumann algebras. If $(r, d)(\mathcal{G})$ or $(r, d)(\mathcal{H})$ is concrete and W is the σ-regular representation of $\mathcal{H}$, then $W \circ \phi$ is (equivalent to) the $\sigma \circ (\phi \times \phi)$-regular representation of $\mathcal{G}$.*

PROOF. The first statement follows from the proof of Theorem 5.19 of [6] with only minor adaptation. The second is a restatement of Theorem 8.3 of [4].

The foregoing render our main result easily accessible. In the case $\sigma = 1$, equivalence of injectivity of the regular representation and amenability was obtained similarly in Section 8 of [4].

THEOREM 2. *Let $\mathcal{G}$ be a concrete principal groupoid, σ a cocycle on $\mathcal{G}$. The following are equivalent:*

1. *$\mathcal{G}$ is amenable.*
2. *The commuting algebra of the σ-regular representation is injective.*
3. *For every σ-representation W of G, W' is injective.*

PROOF. The property of amenability is invariant under similarity of measure groupoids. Therefore, in view of Theorem 1, it suffices to prove the equivalence of 1, 2, and 3 for some similar groupoid. Thus, we may assume that $\mathcal{G}$ is $\mathcal{E}_G$ for some countable group G acting on (S, μ). Then $1 \Rightarrow 3$ by Proposition A, $3 \Rightarrow 2$ is obvious, and $2 \Rightarrow 1$ by Proposition B, because injectivity and property E are equivalent properties.

A groupoid $(\mathcal{G}, C)$ is *ergodic* if $\int |1_E \circ r - 1_E \circ d| \, d\lambda = 0$ for some Borel set E implies $\tilde{\lambda}(E)\tilde{\lambda}(U_{\mathcal{G}} - E) = 0$. $\mathcal{G}$ is ergodic if and only if $(r, d)(\mathcal{G})$ is ergodic.

COROLLARY. *If $(\mathcal{E}, C)$ is a concrete principal groupoid and $\omega \mapsto (\mathcal{E}_\omega, C_\omega)$ on (Ω, p) is an ergodic decomposition (Theorem 6.1 of [8]), then $\mathcal{E}$ is amenable if and only if a.a. $\mathcal{E}_\omega$ are amenable.*

PROOF. First assume that $\dim L^2(\mathcal{E}, \nu^u)$ is essentially constant. The regular representation commuting algebra of $\mathcal{E}$ then decomposes correspondingly as a direct integral. The result follows from the fact that $\int M_\omega \, dp(\omega)$ is injective if and only if a.a. M_ω are. The general case is verified using Lemma 3.10 of [8] and a simple argument about countable disjoint unions of amenable groupoids.

The *range closure* $\overline{\Delta}$ of the modular homomorphism Δ of the ergodic groupoid $\mathcal{G}$ is the $\mathbf{R}$-action defined as follows (see [9], Section 7): $E \subset U_{\mathcal{G}} \times \mathbf{R}$ is invariant if $1_E(r(x), s) = 1_E(d(x), s + \Delta(x), s + \Delta(x))$ a.e. $\mathbf{R}$ acts on the measure algebra of invariant sets by translation by $(-r)$ of the second coordinate. $\overline{\Delta}$ is the point realization of this action.

COROLLARY. *Let $(\mathcal{E}, C)$ be an amenable ergodic concrete equivalence relation such that $\overline{\Delta}$ is not translation by $\mathbf{R}$ on itself. Then for any 2-cocycle σ, $(W^\sigma)'$, $(W^\sigma)''$, and $(W^1)'$ are isomorphic factors.*

PROOF. The algebras are factors by Theorem 5.1 of [8]. For any groupoid $\mathcal{G}$, $K = f \mapsto \bar{f}$ is a conjugate linear isometry on $L^2(g, \nu)$ such that $KL_\sigma K = L_{\bar{\sigma}}$. Since $J_\sigma L_\sigma J_\sigma = R_\sigma \cong (W^\sigma)'$ and $L_\sigma \cong (W^\sigma)''$, we have $(W^1)' \cong (W^1)''$ and it suffices to prove for $\mathcal{E}$ the isomorphism of $(W^1)''$ and $(W^\sigma)''$ for every σ. If either $(W^1)''$ or $(W^\sigma)''$ is type I, then $\mathcal{E}$ is essentially transitive by Theorem 5.4 of [8], σ is trivial, and $(W^1)''$

and $(W^\sigma)''$ are isomorphic by Lemma 4.10 of [8]. If either $(W^1)''$ or $(W^\sigma)''$ is of type II_1, Theorem 5.6 and Proposition 8.1 of [4] show that $\mathscr{E}$ is a discrete relation, say on (S, μ), and existence of a trace permits us to choose μ invariant. This in turn allows construction of a trace using the conditional expectation (Proposition 2.9 of [4]), so that $(W^1)''$ and $(W^\sigma)''$ are the AF factor of type II_1. Finally, if $(W^1)''$ and $(W^\sigma)''$ are infinite and non-type I, they are the unique AF factor ([2], Part VII) with $\overline{\Delta}$ as their flow of weights ([4], Theorem 5.5).

The restriction placed on $\overline{\Delta}$ simply avoids the case in which the factors $(W^\sigma)'$ are all injective of type III_1. The ergodicity assumption can be removed by insistence that the type III_1 component in the ergodic decomposition be null.

Corollary. *If G is an amenable locally compact second countable group acting on (s, μ) and σ is a cocycle on the resultant equivalence relation $\mathscr{E}_G$, then the left σ-regular representation von Neumann algebra is injective.*

Proof. $S \times G$ is amenable by Theorem 2.1 of [13], so $\mathscr{E}_G = (r, d)(S \times G)$ is amenable.

4. **Non-Principal Groupoids.** For non-principal groupoids results are more difficult to obtain. We can treat at least the case of a groupoid intermediate between $S \times G$ and $\mathscr{E}_G$ for discrete G.

Theorem 3. *Let G be an amenable discrete group acting on (S, μ), $\mathscr{G}$ another groupoid with $(r, d)(\mathscr{G}) = \mathscr{E}_G$, Π a homomorphism of $S \times G$ onto $\mathscr{G}$. If σ is any* **T**-*valued 2-cocycle on $\mathscr{G}$, then the commuting algebra of every σ-representation of $\mathscr{G}$ is injective.*

Proof. Slight modification of the proof of Proposition A proves this result, too. $W(s, sg)$ is replaced by $W(\Pi(s, g))$ and (2) and (3) are rewritten

$$\gamma(\Pi(s, g))Q = W(\Pi(s, g))QW(\Pi(sg, g^{-1})) \tag{2'}$$

and

$$\gamma^*(\Pi(s, g))\Phi_{g_1}T(sg) = \Phi_{gg_1}T(s) \tag{3'}$$

As shown in Example 4.8 of [8], Zeller-Meier's cocycle twisted crossed product by the action of a discrete group on an abelian von Neumann algebra is a special case of a σ-regular representation of $S \times G$. Thus we obtain

COROLLARY. The Zeller-Meier twisted crossed products by the action of a discrete amenable group on an abelian von Neumann algebra are injective von Neumann algebras.

Calvin Moore has pointed out to us that his corollary is a consequence of Proposition 6.8 of [2].

Finally, we state without proof a theorem hinting at other results along these lines. The proof is an adaptation of Zimmer's argument for Theorem 2.1 of [13].

THEOREM 4. *Let $\mathscr{G}$ be a measure groupoid with $(r, d)(\mathscr{G})$ similar to $\mathscr{E}_G$, G amenable. If* a.a. *the isotropy groups $\mathscr{G}_u$ are amenable, then $\mathscr{G}$ is amenable.*

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