ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 9, Number 3, Summer 1979

ALMOST QUASI-PURE INJECTIVE ABELIAN GROUPS

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1. **Introduction and preliminaries.** The quasi-pure projective, quasi-pure injective and strongly homogeneous torsion free abelian groups of finite rank have been classified in [1], [2] and [3]. This note investigates the torsion free abelian groups quasi-isomorphic to groups in each of the above three classes, and some generalizations.

In what follows, all groups will be torsion free abelian of finite rank. A group G is called *almost quasi-pure projective* (aqpp) if there is non-zero integer n such that the index of the image of $\mathrm{Hom}(G, G)$ in $\mathrm{Hom}(G, G/A)$ is bounded by n for every pure subgroup, A, of G. If n can be taken to be 1, G is called *quasi-pure projective* (qpp). In [9] it is shown that the class of *aqpp* groups is closed under quasi-isomorphism and that a group G is *aqpp* if and only f it is *qpp*. Hence, the class of *qpp* groups is closed under quasi-isomorphism. This result can be obtained by dualizing some results of § 3 and § 4.

The situation is more complicated in the quasi-pure injective case. A group G is called *almost quasi-pure injective* (aqpi) if there is a non-zero integer n such that the index of the image of $\mathrm{Hom}(G, G)$ in $\mathrm{Hom}(A, G)$ is bounded by n for every pure subgroup, A, of G. If n can be taken to be 1, G is called *quasi-pure injective* (qpi). Three classes of groups arise naturally: C_1, the class of *qpi* groups; C_2, the class of groups quasi-isomorphic to groups in C_1; C_3, the class of *aqpi* groups. We show that $C_1 \subsetneqq C_2 \subsetneqq C_3$ (Example 2 and the remarks following it), and characterize the groups in C_3.

Finally, G is called *almost strongly homogeneous* (ash) if there is a non-zero integer n such that given any two pure rank one subgroups A and B of G, there is a monomorphism $f: G \to G$ with $nB \subseteq f(A) \subseteq B$. If f can be taken to be an automorphism (and hence $n = 1$), G is called *strongly homogeneous* (*sh*). *The ash* groups are classified in Theorem 2.2, and an example is given of an *ash* group which is not quasi-isomorphic to any *sh* group.

The standard notions of height ($h(x)$) and type ($\tau(x)$) of an element x in a group G will be used. At times $\mathscr{E}(G)$ will be used to denote $\mathrm{Hom}_Z(G, G)$. Tensor products are all taken over the integers. Finally, G quasi-isomorphic to H will be written $G \sim H$, and if S is any subset of a group G, $\langle S \rangle_*$ will denote the pure subgroup generated by S.

The proofs of some of the theorems, which closely parallel earlier ar-

Received by the editors on May 17, 1976, and in revised form on September 6, 1977.

guments, will be omitted.

2. **Ash groups.** We begin with a lemma which helps justify the "almost" definitions.

LEMMA 2.1. *Let G and H be groups with $G \sim H$.*
(a) *If G is aqpi then H is aqpi.*
(b) *If G is ash then H is ash.*

PROOF. We may assume $mH \subseteq G \subseteq H$ for some positive integer m. Thus if A is a pure subgroup of H and $f \in \mathrm{Hom}(A, H)$, then $A \cap G$ is pure in G and $mf \in \mathrm{Hom}(A \cap G, G)$.

(a) Let A be pure in H and $f: A \to H$. Since G is *aqpi*, there is a non-zero integer n and $g: G \to G$ such that $(g - nmf)(A \cap G) = 0$. But $mg: H \to H$ and $(mg - nm^2 f)A = 0$. Thus H is *aqpi* with associated integer nm^2.

(b) Let A and B be pure rank one subgroups of H. Since G is *ash* there is a non-zero integer n and monomorphism $g: G \to G$ with $n(B \cap G) \subseteq g(A \cap G) \subseteq B \cap G$. Then $mg: H \to H$ is a monomorphism and $nm^2 B \subseteq nm(B \cap G) \subseteq mg(A \cap G) \subseteq mg(A) \subseteq g(A \cap G) \subseteq B$. That is, $nm^2 B \subseteq mg(A) \subseteq B$ and H is *ash*.

The first theorem characterizes certain subrings of algebraic number fields which appear in subsequent results. Some well-known facts are used repeatedly: If R is a subring containing 1 of an algebraic number field K, and J is the ring of algebraic integers in K, then the integral closure of R is $JR = \overline{R}$ which is quasi-isomorphic to R. Furthermore $\overline{R}$ is a Dedekind domain, so that any ideal may be uniquely expressed as a product of prime ideals. Finally, the symbol R_P is used to denote the usual localization of the ring R at the prime ideal P.

THEOREM 2.2. *Let R be an integrally closed integral domain such that the quotient field, K, of R is an algebraic number field, and let J be the ring of algebraic integers in K. Then the following are equivalent:*

(a) *There exists $0 \neq n \in Z$ such that every $0 \neq r \in R$ can be written $r = ks$, where $k \in Z$, $s \in R$ and $nR \subseteq sR$.*

(b) *If p is a rational prime then $pR = P^{e_p}$ for some maximal ideal P of R and $0 \leqq e_p \in Z$, such that $e_p \leqq 1$ for all but a finite number of p.*

(c) *$R = \bigcap_{P \in S} J_P$ where S is a collection of maximal ideals of J such that if p is a rational prime with $pJ = P_1^{e_1} \cdots P_k^{e_k}$ a product of powers of distinct maximal ideals of J, then at most one P_i is in S and for all but a finite number of p, the corresponding $e_i = 1$.*

PROOF. (a) $\Rightarrow$ (b). Let p be a prime such that $(p, n) = 1$. If $pR \neq R$, choose $r \in R \setminus pR$ and write $r = ks$ with $k \in Z$, $nR \subseteq sR \subseteq R$. Since

$r \notin pR$, then $(p, k) = 1$. Hence $R \supseteq pR + Rr = pR + Rks \supseteq pR + knR = R$. Therefore, pR is maximal in R.

Now suppose $n = p^t n_1$ with $t \geqq 1$, $(p, n_1) = 1$. Let $pR = P_1{}^{e_1} \cdots P_k{}^{e_k}$ be the product of powers of distinct maximal ideals in R and suppose $k \geqq 2$. Then $p^tR = P_1{}^{te_1} \cdots P_k{}^{te_k}$. Since $P_1{}^{te_1+1} \subseteq pR$ would imply $P_1 \subseteq P_2$, we can choose $r \in P_1{}^{te_1+1} \setminus pR$. Write $r = \ell s$ with $\ell \in Z$, $nR \subseteq sR \subseteq R$. Since $r \notin pR$, $(p, \ell) = 1$. But $\ell nR \subseteq \ell sR = rR \subseteq P_1{}^{te_1+1}$, and $\ell nR = p^t n_1 \ell R = (\ell n_1 R)(P_1{}^{te_1} \cdots P_k{}^{te_k})$. Therefore $\ell n_1 \in P_1$, and $R = \ell n_1 R + pR \subseteq P_1$ a contradiction. Thus $pR = P_1{}^{e_1}$.

(b)$\Rightarrow$(c). Let S be the set of all prime ideals P of J such that $P = J \cap M$ for M a maximal ideal in R. Then

$$R \subseteq \bigcap_{P \in S} J_P \subseteq \bigcap_{M \text{ maximal in } R} R_M = R.$$

Furthermore, for all rational primes p such that $R \neq pR$ is maximal, $pR \cap J = P$, for some $P \in S$. If $P = J \cap pR$, then $R_{pR} = J_P$, $PJ_P = pJ_P$ and $R/pR \cong J_P/pJ_P$ is a field. Hence, if $pJ = P_1{}^{e_1} \cdots P_k{}^{e_k}$, then $P = P_i$ for exactly one i, and $e_i = 1$.

(c)$\Rightarrow$(a). Let $0 \neq r \in R$ and write $r = ks$ where $k \in Z$, and s has minimal (idempotent) height in R. This is possible since any element in R has the same type as 1. Then $s \notin pR$ if p is a rational prime and $pR \neq R$. Write $sR = Q_1{}^{f_1} \cdots Q_m{}^{f_m}$ as a product of maximal ideals. Now observe that if $M \notin S$ is a maximal ideal of J, then $MR = R$ since $MJ_P = J_P$ for all $P \in S$. Hence if $\{p_j\}_{j=1}^{\ell}$ are the rational primes not maximal in R, then for each j, either $p_jR = R$ or $p_jR = P_j{}^{e_j}R$ for some $P_j \in S$, $e_j > 1$. Furthermore, each P_jR is maximal in R and the set of Q_i's is a subset of the set of P_jR's. Thus if $n = p_1p_2 \cdots p_\ell$, then $nR \subseteq sR$ since the multiplicity of P_jR in the factorization of sR must be less than e_j (otherwise p divides s).

Any integrally closed ring satisfying one, hence all of the conditions of Theorem 2.2 will be said to satisfy *condition* (†). Such rings appear immediately in the following characterization of *ash* groups, the proof of which parallels that of Theorem 1 in [2].

THEOREM 2.3. *A group G is almost strongly homogeneous if and only if $G \sim R \otimes_Z H$, where $H = \oplus \sum_{\text{finite}} A$ for some rank one group and R is a subring of an algebraic number field satisfying* (†).

PROOF. ($\Leftarrow$). First, R, $+$ is *ash*. For let X, Y be pure rank one subgroups of R. Choose $x \in X$, $y \in Y$ of minimal (idempotent) height. Then $nR \subseteq yR$ and $n = yy'$ for some $y' \in R$. Therefore left multiplication by xy' is a monic endomorphism of R such that $nX \subseteq xy'Y \subseteq X$.

Second, $R \otimes_Z A$ is *ash*. Choose $0 \neq a \in A$ and define $i: R \to R \otimes A$ by $i(r) = r \otimes a$. Now let X' and Y' be pure rank one subgroups of $R \otimes A$. Let X and Y be the pure subgroups of R such that $i(X) = X' \cap i(R)$, $i(Y) = Y' \cap i(R)$. Following the first paragraph, $nX \subseteq xy'Y \subseteq X$. It is then easy to show $nX' \subseteq xy'Y' \subseteq X'$.

Finally, $G = R \otimes H$ is *ash*. Let X be a pure rank one subgroup of G, and let $B = \langle RX \rangle_*$. Then B is an R-pure submodule of G, for using property (a) of Theorem 2.2, $rg \in B$ implies $mg \in B$ for some $0 \neq m \in Z$, and hence $g \in B$. Following the methods in Fuchs ([5], p. 115, Lemma 86.8) we will show that B is a quasi-summand of G.

Choose $0 \neq x \in X$ and since $G = \oplus \sum_{i=1}^{k} R \otimes A_i$, $A_i = A$, write $x = \sum_{i=1}^{k} r_i \otimes a_i$ where, for all i, $a_i \in A_i$ satisfies $h_p{}^A(a_i) = h_p{}^G(x)$ for all p such that $h_p{}^G(x) < \infty$. Using (†), write $r_i = k_i s_i$ with $nR \subseteq s_i R$ such that each s_i has idempotent height in R. Without loss of generality the set $\{k_i\}$ may be taken to be relatively prime. Hence, as in Fuchs, there is a basis $x, b_2, \cdots, b_k$ for the group $G' = \oplus \sum_{i=1}^{k} \langle s_i \otimes a_i \rangle_*$. Furthermore, by the second paragraph above, for each i there is a $u_i \in R$ such that $n\langle 1 \otimes A_i \rangle_* \subseteq u_i \langle s_i \otimes A_i \rangle_* \subseteq \langle 1 \otimes A_i \rangle_*$. This implies $nG \subseteq \langle Rx \rangle_* \oplus \sum_{i=2}^{k} \langle Rb_i \rangle_* \subseteq G$, the sum being direct since each summand is R-pure, and $\dim_F F \otimes_R G = k$, where $F =$ quotient field of R.

Now let Y be another pure rank one subgroup of G. By the same argument $nG \subseteq \langle RY \rangle_* \oplus G' \subseteq G$ for some $G' = \oplus \sum \langle Rb_i' \rangle_*$. Furthermore, G is homogeneous so that $X \simeq Y$, and this implies $RX \simeq RY$. Now $n\langle RX \rangle_* \subseteq RX \subseteq \langle RX \rangle_*$, for suppose $y \in \langle RX \rangle_*$. Then $ty = rx$ with $0 \neq t \in Z$, $r \in R$, $x \in X$. Using (†), $r = ks$ with $nR \subseteq sR$. Write $n = ss'$. Then $ts'y = knx$, implying $s'y \in X$ and hence $ny = ss'y \in RX$. The composition of homomorphisms

$$G \xrightarrow{n^2} n^2 G \subseteq n\langle RY \rangle_* \oplus \sum_{i=2}^{k} n\langle Rb_i' \rangle_*$$

$$\subseteq RY \oplus \sum_{i=2}^{k} R\langle b_i' \rangle_* \simeq RX \oplus \sum_{i=2}^{k} R\langle b_i \rangle_* \subseteq G,$$

is a monic endomorphism f of G such that $n^2 X \subseteq f(Y) \subseteq X$.

($\Rightarrow$). First G is irreducible (has no proper pure fully invariant subgroups) since any pure rank one subgroup can be mapped to any other pure rank one subgroup. Hence by Reid [8], $G \sim \oplus \sum_{i=1}^{m} G_0$ where G_0 is strongly indecomposable and irreducible and $Q \otimes \mathscr{E}(G_0)$ is a division algebra with Q-dimension equal to the rank of G_0. It is immediate that G_0 is also *ash*.

Let $R = \mathscr{E}(G_0)$ and A be a pure rank one subgroup of G_0. Then the map $f\colon\ R \otimes A \to G_0$ given by $f(r \otimes a) = ra$ is a quasi-isomorphism since G_0 is *ash* and rank $R =$ rank G_0.

Now let $\overline{R}$ be the integral closure of R. Then $\overline{R} \sim R$ so that $\overline{G}_0 = \overline{R} \otimes_R G_0 \sim G_0$. Thus $\overline{G}_0$ is *ash* by Lemma 2.1. Furthermore $\overline{R} \subseteq \mathrm{End}(\overline{G}_0)$ in a natural way, and since $\overline{R} \sim \mathrm{End}(\overline{G}_0)$, there is an integer $t > 0$ such that $t\,\mathrm{End}(\overline{G}_0) \subseteq \overline{R} \subseteq \mathrm{End}(\overline{G}_0)$. We will show $\overline{R}$ satisfies (†). Let $0 \neq r \in \overline{R}$, X a pure rank one subgroup of $\overline{G}_0$, and $Y = \langle rX \rangle_*$. Since $\overline{G}$ is *ash*, let $n > 0$ be the associated integer, and pick $s \in \overline{R}$ such that $tnY \subseteq sX \subseteq Y$. Since $r, s \in \mathrm{Hom}(X, Y)$ which has rank one, there are relatively prime integers a, b such that $ar = bs$. Write $\ell a + mb = 1$. Then $s = \ell as + mbs = a(\ell s + mr) = as'$, where $s' = \ell s + mr$ satisfies $tnY \subseteq s'X \subseteq Y$ and $r = bs'$. Now choose $u \in \overline{R}$ such that $tnX \subseteq uY \subseteq X$. Then $(tn)^2X \subseteq tnuY \subseteq us'X \subseteq uY \subseteq X$. It follows that $(tn)^2\overline{R} \subseteq us'\overline{R}$: consider $(tn)^2x$ for some $0 \neq x \in X$. By the above, $(tn)^2x = us'\,c/d\,x$ for some relatively prime integers c and d such that $c/d\,x \in X$. It is sufficient to consider the case where none of the primes dividing d are units in $\overline{R}$. Choose $x_1 \in X$ such that $p \mid d \Rightarrow h_p(x_1) = 0$. Then $(tn)^2x_1 = us'\,c_1/d_1\,x_1$ where $(c_1, d_1) = 1$ and $c_1/d_1\,x_1 \in X$. Clearly $(d_1, d) = 1$. But since $Q \otimes \overline{R}$ is a division ring, $(tn)^2 = us'\,c_1/d_1 = us'\,c/d$. It follows that $d_1 = d = 1$ and $c_1 = c$. Thus $(tn)^2\overline{R}$. We now have $r = bs'$ with $(tn)^2\overline{R} \subseteq s'\overline{R}$ Therefore $\overline{\mathrm{R}}$ satisfies (†).

As in Arnold [2], the fact $R = \mathrm{Hom}(G_0, G_0)$, $R \otimes A \sim G_0$ and $Q \otimes R$ a division algebra imply $R \sim \mathrm{Hom}_Z(R, R)$. By a result of Reid [8] this implies R, hence $\overline{R}$ is a full subring of an algebraic number field.

Example 1. Let ξ be a primitive root of $x^{p^r} - 1 = 0$, for some prime p and $r \geqq 1$. Let J be the ring of integers in $Q(\xi)$. Then in J, $(p) = P^{\phi(p^r)}$ where ϕ is the Euler ϕ-function ([6], p. 74). Thus in the localization, J_P, of J at P, all primes except p are units, and $pJ_P = (PJ_P)^{\phi(p^r)}$, so J_P satisfies the condition (b) of Theorem 2.2. Furthermore, J_P is not quasi-isomorphic to any ring in which (p) is maximal and the other primes are units, hence by Theorem 2.3 and Proposition 5 of [2], is an example of a group which is *ash*, but not quasi-isomorphic to a strongly homogeneous group.

3. **Strongly indecomposable *aqpi* groups.**

THEOREM 3.1. *The following are equivalent for a group G*:

(a) *G is strongly indecomposable, homogeneous, and aqpi*;

(b) *G is ash and every pure subgroup is strongly indecomposable*;

(c) $G \sim R \otimes_Z A$ *where*

(i) *R is a subring of an algebraic number field and satisfies* (†),

(ii) *A is a rank one group with type* $(\mathscr{E}(A))$ = type (R),

(iii) *For all* $0 \neq r \in R$ *there is a rational prime p, with* $pR \neq R$, *and* $a \in R \cap Q$ *such that* $r - a \in pR$.

PROOF. (a)$\Rightarrow$(b). As in [1], Theorem B.

(b)$\Rightarrow$(c). Since G is strongly indecomposable, by Theorem 2.3 and its proof, we need only show R satisfies (iii). Let $r \in R \setminus R \cap Q$. Then by (b), $B = \langle r \rangle * \oplus R \cap Q$ is not pure in R. Thus there is a prime p with $pR \neq R$ and $x \in R \setminus B$ such that $px \in B$. That is $px = c/d\, r + a$ for some relatively prime integers c and d, and $a \in R \cap Q$. Since $x \notin B$, it follows that $(p, c) = 1$ and $r \in R \cap Q + pR$.

(c)$\Rightarrow$(a). As in [1] it can be shown that for any pure subgroup H of R, + and $f: H \rightarrow R$, then nf is just left multiplication by an element of R. Therefore R, hence $R \otimes A$, hence G, is strongly indecomposable, homogeneous and *aqpi*.

The final result of this section covers the non-homogeneous case.

THEOREM 3.2. *group G is strongly indecomposable and aqpi if and only if G is a torsion free R module such that*

(i) *R is quasi-isomorphic to a ring with* (†), *and satisfies* (iii) *of Theorem* 3.1 *above,*

(ii) $\mathrm{Hom}_R(X, Y) = 0$ *for every pair X, Y of distinct Z-pure R submodules of G of R-rank one,*

(iii) *For all pure rank one subgroups A of G, type* $\mathscr{E}(A)$ = *type R.*

PROOF. As in [1], Theorem D.

EXAMPLE 2. If R is the ring J_P of Example 1 and $G = (R, +)$ then by the above theorem G is *aqpi* and by Theorem B of [1] G is not quasi-isomorphic to any *qpi* group. We remark that by Theorem A of [1] it follows that there are groups quasi-isomorphic to *qpi* groups which are not *qpi*.

4. **Decompositions.** In this section we complete the study of *aqpi* groups by characterizing the decomposable ones. The first result deals with those homogeneous, strongly indecomposable *aqpi* groups H such that $\oplus\, \Sigma H$ is *aqpi*. These are characterized by

CONDITION DOP. *If x and y are independent elements of a group H then $\langle x, y\rangle*/\langle y\rangle* \simeq Q$.*

REMARK. It can be shown that if H is *ash* then H satisfies DOP if and only if $E(H)/E(H) \cap Q$ is divisible. The latter condition on the endomorphism ring, rather than the DOP condition of the group is used in [1].

THEOREM 4.1. *Let H be strongly indecomposable, homogeneous and aqpi and $G \sim \oplus \sum_{i=1}^{m} H$, $m \geqq 2$. Then G is aqpi if and only if H has DOP.*

PROOF. Suppose H does not satisfy DOP. It suffices to show $H \oplus H$ is not *aqpi*. Assume the converse and let n be the *aqpi* integer. Then choose a prime p, positive integer k, and independent elements $x, y \in H$ such that $p^k \nmid n$, height $x =$ height y, $p -$ height $x = 0$, and $1/p\, x + \alpha y \notin H$ for any $\alpha \in Q$. This implies $A = \langle(x, p^k x)\rangle* \oplus \langle(y, 0)\rangle*$ is pure in $H \oplus H$. But $f(x, p^k x) = (0, x)$, $f(y, 0) = (0, 0)$ defines a map $f: A \rightarrow H \oplus H$ such that nf cannot be lifted.

The proof in the converse direction goes through as in [1], Theorem C.

We are now ready to give the main decomposition theorem.

THEOREM 4.2. *Let G be a reduced group of finite rank. Then G is aqpi if and only if*

$$(*) \qquad G \sim H_1 \oplus H_2 \oplus \cdots \oplus H_n \oplus K_1 \oplus \cdots \oplus K_m$$

where: (1) each H_i is homogeneous, (2) $H_i = \oplus \sum^{n_i}_{j=1} A_{ij}$ with A_{ij} *aqpi*, strongly indecomposable and mutually quasi-isomorphic for fixed i, and with $n_i > 1$ only if A_{ij} satisfies DOP, (3) each K_j is *aqpi*, non-homogeneous and strongly indecomposable and (4) if X and Y are pure rank one subgroups of distinct summands of (*), then $\tau(X) \cup \tau(Y) = \tau(Q)$.

PROOF. Assume G is *aqpi*. Since G is of finite rank, $G \sim G_1 \oplus G_2 \oplus \cdots \oplus G_\ell$ where each G_i is strongly indecomposable. By grouping together the quasi-isomorphic summands and applying Theorem 4.1 and adapting Lemma 1.1 and Theorem A of [1], the result follows.

Conversely, assume $G = H_1 \oplus \cdots \oplus H_n \oplus K_1 \oplus \cdots \oplus K_m$ with H_i's and K_j's satisfying the conditions of the theorem and let $\Pi_i : G \rightarrow H_i$, $\Pi_j : G \rightarrow K_j$ be the natural projections. Then given a pure subgroup A of G and $f \in \mathrm{Hom}(A, G)$, it suffices to construct $h_i \in \mathrm{hom}(\langle \Pi_i A\rangle*, H_i)$, $g_j \in \mathrm{Hom}(\langle \Pi_j A\rangle*, K_j)$ such that $h_i \Pi_i(a) = \Pi_i f(a)$, $g_j \Pi_j(a) = \Pi_j f(a)$ for all $a \in A$, as these maps can be quasi-lifted to the corresponding summands and the sum will provide a quasi-lifting of f. The h_i and g_j are constructed in a straightforward manner, exactly as in the proof of Theorem A of [1].

REMARKS. It is shown in [9] that for groups of finite rank, *aqpp* is equivalent to *qpp*. This result can be obtained by applying the Warfield duality of [10] to the above results on *aqpi*. At the suggestion of the referee, details are omitted.

The author would like to thank Dave Arnold for many useful suggestions and ideas concerning this paper.

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