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MULTI-PARAMETER SPECTRAL MEASURES, GENERALIZED RESOLVENTS, AND FUNCTIONS OF POSITIVE TYPE

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1. Introduction.

1.1. In this paper we extend the development of spectral triples as introduced in McKelvey [12] to the case of several parameters. Our central theme is the study of the interplay of certain classes of functions $\{E(t), Q(\lambda), V(s)\}$ whose values are bounded operators on a complex Hilbert space. In the proto-type for the general situation these functions arise from a sequence, $\mathbf{A}^j$, $j = 1, \cdots, m+n$, of $m+n$ selfadjoint operators in the space H the last n of which are, in addition, positive. Corresponding to each operator $\mathbf{A}^j$ there is defined its resolution of the identity $\mathbf{E}^j_{t^j}$, $-\infty < t_j < \infty$,

$$\mathbf{A}^j = \int t^j \, d\mathbf{E}^j_{t^j},$$

its resolvent function $\mathbf{Q}^j_{\lambda^j}$,

$$\mathbf{Q}^j_{\lambda^j} = \lambda^j(I - \lambda^j\mathbf{A}^j)^{-1}, \quad \operatorname{Im} \lambda^j \neq 0 \text{ unless } \lambda^j = 0,$$

and the unitary group $\mathbf{U}^j_{s^j}$,

$$\mathbf{U}^j_{s^j} = e^{-is^j\mathbf{A}^j}, \quad -\infty < s^j < \infty.$$

In case $\mathbf{A}^j$ is positive we prefer to work with the semi-group $\mathbf{V}^j_{s^j}$,

$$\mathbf{V}^j_{s^j} = e^{-s^j\mathbf{A}^j}, \quad s^j \geqq 0.$$

Furthermore in this case the resolution of the identity vanishes on the half-axis $t^j \leqq 0$ and the resolvent is defined on the negative half-axis $\lambda^j < 0$.

We assume that the operators $\mathbf{A}^j$ commute pairwise, that is the resolutions of the identity $\mathbf{E}^j$ commute pairwise. Then all the operator families Q^j, U^j and V^j commute pairwise and we may define the multiparameter operator functions $\mathbf{E}(t)$, $\mathbf{Q}(\lambda)$, and $\mathbf{V}(s)$ according to the equations

$$\mathbf{E}(t^1, \cdots, t^{m+n}) = \prod_{j=1}^{m+n} \mathbf{E}^j_{t^j}, \quad (t \in \Gamma_+)$$

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$$Q(\lambda^1, \cdots, \lambda^{m+n}) = \prod_{j=1}^{m+n} Q^j_{\lambda^j}, (\lambda \in \Omega)$$

and

$$V(s^1, \cdots, s^{m+n}) = \prod_{j=1}^{m} U^j_{s^j} \prod_{j=m+1}^{m+n} V^j_{s^j}, \ (s \in \Gamma_+),$$

where $\Gamma_+ = (-\infty, \ \infty)^m \times [0, \ \infty)^n$ and $\Omega = [(C - R) \cup \{0\}]^m \times [C - (0, \infty)]^n$.

These functions $E(t)$, $Q(\lambda)$ and $V(s)$ are called, respectively, an (m, n) – *parameter resolution of the identity, resolvent and continuous semi-group.* We denote the class of all such functions by $\mathcal{E}_0^{(m,n)}$ (H), $\mathcal{Q}_0^{(m,n)}$ (H), and $\mathcal{V}_0^{(m,n)}$ (H), respectively. When understood from the context, we omit the H and the (m, n). Collectively these three classes are called the *total spectral triples,* while individually such an operator family is referred to as a *total triple function.* Finally a triple E, Q, V from these classes each corresponding to the same sequence of self-adjoint operators A^j is called a *total matched triple.*

1.2. Now a relation among functions $\{E(t), Q(\lambda), V(s)\}$ persists, and is of significance, for broader classes than those described above, at several levels of generality. This is true of compressions (cf. Halmos [10]) or projections (cf. Sz. Nagy [20]) of total triple functions which we now describe. Let the Hilbert space H be a subspace of the Hilbert space H and let P be the orthogonal projection of H onto H. In general, if T is an operator acting in H, then the operator T defined on dom T $\cap$ H by

$$T = \mathrm{PT}|_H \equiv \mathrm{pr}\ \mathrm{T}$$

is called the *projection of T* onto H and is denoted by the third member of this equation. Alternatively we say that T is a *dilation* of T to H which is called a *dilation space* of H. In either case it is a simple consequence of this definition that

$$(1) \qquad (Tx, y) = (\mathrm{T}\,x, y), (x, y \in H).$$

In particular, we call the operator family $E(t)$, $Q(\lambda)$, or $V(s)$ an (m, n) – *parameter spectral function,* a *generalized* (m, n) —*parameter resolvent,* or an (m, n) —*parameter Bochner function,* respectively, if it is the projection of some (m, n) — parameter resolution of the identity, resolvent, or continuous semi-group respectively. We denote the class of all such functions by $\mathcal{E}^{(m,n)}(H)$, $\mathcal{Q}^{(m,n)}(H)$, and $\mathcal{V}^{(m,n)}(H)$ respectively. Collectively these three classes are called the *generalized spectral triples,* while individually such an operator family is referred to as a *generalized triple function.*

In § 4 we give intrinsic characterizations of the various triple functions and thereby allow each to be studied in its own right.

It is easy to see that these projected operator families are in fact generalizations of the total triple functions, that is $\mathcal{E}_0(H) \subset \mathcal{E}(H)$, $\mathcal{Q}_0(H) \subset \mathcal{Q}(H)$, and $\mathcal{V}_0(H) \subset \mathcal{V}(H)$. For if $\mathbf{H} = H$, then each generalized triple function is equal to its corresponding total triple function.

1.3. An autonomous relation among the generalized triples stems from the multi-parameter functional calculus based upon an (m, n)-parameter spectral function $E(t)$. This calculus is defined in the weak operator topology and enjoys all of the usual properties except multiplicativity. However multiplicativity is restored under certain conditions, for example in the event that $E \in \mathcal{E}_0$.

This functional calculus yields the following representations.

$$Q(\lambda) = \int_{\Gamma_+} \prod_{j=1}^{m+n} \frac{\lambda^j}{1 - \lambda^j t^j} dE(t), (\lambda \in \Omega) \tag{2}$$

$$V(s) = \int_{\Gamma_+} e^{-[i]s \cdot t} dE(t), (s \in \Gamma_+) \tag{3}$$

where $[i]s = (is^1, \cdots, is^m, s^{m+1}, \cdots, s^{m+n})$, and

$$Q(\lambda) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=1}^{m+n} \pm ie^{\pm isj/\lambda j} \right] \cdot V(\mp s^1, \cdots, \mp s^m, \mp is^{m+1}, \cdots, \mp is^{m+n})\, ds \tag{4}$$

for $\mp \operatorname{Im} \lambda^j > 0$.

Moreover these three equations may be inverted to obtain $E(t)$ in terms of $Q(\lambda)$ or $V(s)$ and $V(s)$ in terms of $Q(\lambda)$ respectively. This is achieved for formulas (2) and (3) by repeated application of the inversion of Stieltjes and Laplace transforms respectively. We carry out this process in § 3. Naturally, formula (3) combined with the inversion of (2) furnishes $V(s)$ in terms of $Q(\lambda)$. Thus these equations establish a one-to-one-to-one correspondence among the spectral triples and are referred to as the *matching formulas*.

These considerations lead to our first fundamental proposition which we prove in § 3.

THEOREM 1. *Each function* $E(t) \in \mathcal{E}^{(m,n)}(H)$, $Q(\lambda) \in \mathcal{Q}^{(m,n)}(H)$, *or* $V(s) \in \mathcal{V}^{(m,n)}(H)$ *belongs to a unique triple of functions from these classes, such that the members of the triple are interrelated by the matching formulas.*

A triple of functions, $\{E(t), Q(\lambda), V(s)\} \in \mathcal{E}, \mathcal{Q}, \mathcal{V}$, whose members are related as in Theorem 1, will be called a *matched* triple. Similarly, a triples of classes $\{\mathcal{E}', \mathcal{Q}', \mathcal{V}'\}$ will be called *matched* if

(i) $\mathcal{E}' \subset \mathcal{E}, \mathcal{Q}' \subset \mathcal{Q}, \mathcal{V}' \subset \mathcal{V}$ and

(ii) when $\{E(t), Q(\lambda), V(s)\}$ is a matched triple, then any one of the relations $E(t) \in \mathcal{E}', Q(\lambda) \in \mathcal{Q}', V(s) \in \mathcal{V}'$ implies all three.

In this terminology we have the following.

THEOREM 2. *The triple of classes* $\{\mathcal{E}_0, \mathcal{Q}_0, \mathcal{V}_0\}$ *is matched.*

Not only are the generalized spectral triples matched, but also a matched triple possesses a matched dilation.

THEOREM 3. *Let* E, Q, V *be a matched triple of functions belonging to the classes* $\mathcal{E}^{(m,n)}(H)$, $\mathcal{Q}^{(m,n)}(H)$, *and* $\mathcal{V}^{(m,n)}(H)$ *respectively. There exists a Hilbert space* $\mathbf{H} \supset H$ *and a matched triple* $\mathbf{E}$, $\mathbf{Q}$, $\mathbf{V}$ *belonging to the classes* $\mathcal{E}_0^{(m,n)}(\mathbf{H})$, $\mathcal{Q}_0^{(m,n)}(\mathbf{H})$, $\mathcal{V}_0^{(m,n)}(\mathbf{H})$ *respectively such that*

$$E(t) = \mathrm{pr}\, \mathbf{E}(t), Q(\lambda) = \mathrm{pr}\, \mathbf{Q}(\lambda), V(s) = \mathrm{pr}\, \mathbf{V}(s).$$

Moreover $\mathbf{H}$ *may be chosen to be minimal in the sense that, up to isomorphism,*

$$\begin{aligned} \mathbf{H} &\cong \mathrm{span}\, \{\mathbf{E}(t)x : x \in H, t \in \Gamma_+\} \\ &\cong \mathrm{span}\, \{\mathbf{Q}(\lambda)x : x \in H, \lambda \in \Omega\} \\ &\cong \mathrm{span}\, \{\mathbf{V}(s)x : x \in H, S \in \Gamma_+\}. \end{aligned}$$

1.4. It is possible to define $m + n$ one-parameter triple functions given a single multi-parameter function. If $E \in \mathcal{E}^{(m,n)}$, $Q \in Q^{(m,n)}$, and $V \in \mathcal{V}^{(m,n)}$, then the one-parameter operator families

$$E^j_{t^j} = E(\infty, \cdots, \infty, t^j, \infty, \cdots, \infty), \tag{5}$$

$$Q^j_{\lambda^j} = \lim_{\lambda/\lambda^j \to 0} \frac{\lambda^j Q(\lambda)}{\Pi\lambda}, \tag{6}$$

and

$$V^j_{s^j} = V(0, \cdots, 0, s^j, 0, \cdots, 0) \tag{7}$$

are their various *marginals*. The notation $\Pi\lambda$ means $\lambda^1\lambda^2 \cdots \lambda^{m+n}$ while $\lim_{\lambda/\lambda^j \to 0}$ denotes the strong limit as all the parameters $\lambda^1, \cdots, \lambda^{m+n}$ tend to zero except λ^j which remains fixed.

It is immediate that a total triple function is decomposed into the product of its marginals. However this is not always the case for a generalized triple function. Nevertheless, a certain marginal matching does hold.

THEOREM 4. *Let E, Q, V be a generalized matched triple of (m, n)-parameter functions on H. Then the* jth *triple of marginals E^j, Q^j, V^j is a generalized matched triple of one-parameter functions on H. Moreover, there exists a common dilation space for all the marginals, namely the minimal dilation space of the matched triple given in Theorem* 3.

1.5. One situation in which the generalized spectral triples do decompose into the product of marginals occurs when the Bochner function is, in fact, a semi-group. More precisely, by the *quadrant*, Θ, of $\mathbf{R}^m$ containing t, $t^j \neq 0, j = 1, 2, \cdots, m$, we mean the set

$$\Theta = \{s \in \mathbf{R}^m : \operatorname{sign}(s^j) = \operatorname{sign}(t^j) \text{ or } s^j = 0, j = 1, \cdots, m\}.$$

Then an operator-valued function $V(s)$ defined on some quadrant Θ is a *continuous semi-group of contractions* if

(i) $\|V(s)\| \leqq 1$,
(ii) $V(s)$ is continuous as a function of s and $V(0) = I$,
(iii) $V(s_1 + s_2) = V(s_1)V(s_2)$.

Such a function $V(s)$ can be extended in a natural way to the opposing quadrant of $\mathbf{R}^m$ by setting

(iv) $V(-s) = V^*(s)$.

In this case property (iii) holds whenever s_1 and s_2 belong to the same quadrant. By an *extended continuous semi-group of contractions* we mean an operator-valued function $V(s)$, $s \in \mathbf{R}^m$, defined on all of $\mathbf{R}^m$ which satisfies (i), (ii) and (iv), and (iii) whenever s_1 and s_2 belong to the same quadrant. The class of all extended continuous semi-groups of contractions will be denoted by $\mathcal{V}_\mu{}^m$.

THEOREM 5. *Every extended continuous semi-group of contractions is also an $(m, 0)$-parameter Bochner function.*

In view of this fact, each function $V \in \mathcal{V}_\mu{}^m$ is part of a matched triple E, Q, V of functions interrelated by the matching formulas. Of course spectral functions and generalized resolvents corresponding to an extended continuous semi-group of contractions have certain special properties. Put $\Omega' = ((\mathbf{C} - \mathbf{R}) \cup \{0\})$. By the *halfplane* Φ containing $\mu \in \Omega'$, $\mu^j \neq 0$, all j, we mean the set

$$\Phi = \{\lambda \in \Omega' : \operatorname{sign}(\operatorname{Im} \lambda^j) = \operatorname{sign}(\operatorname{Im} \mu^j), j = 1, \cdots, m\}$$

Let a halfplane Φ and a spectral function $E \in \mathcal{E}^{(m,0)}$ be given. We define the class $B_\Phi(dE)$ to be all those functions $f : \mathbf{R}^m \to \mathbf{C}$ such that f can be decomposed

$$f(t) = \prod_{j=1}^{m} f^j(t^j)$$

into the product of m functions f^j each of which belongs to McKelvey's class $B_\pm(dE^j)$ where E^j is the jth marginal of E. The choice of $+$ or $-$ depends on whether the imaginary part of the jth coordinate of points in Φ are greater than or less than zero respectively. (A complex-valued function $f^0(t^0)$ of the real variable t^0 belongs to the class $B_\pm(dE^j)$ if it is Borel measurable and bounded on the support of the operator measure dE^j and if there exists a function $f^0(z^0)$ of the complex-variable z^0, which is bounded and holomorphic on the halfplane $\pm$ Im $z^0 < 0$ and is such that

$$\lim_{\delta \downarrow 0} f^0(t^0 \pm i\delta) = f^0(t^0), (dE^j)).$$

Now a function $E \in \mathcal{E}^{(m,0)}$ is *semi-multiplicative* if the equation

$$\int_{\mathrm{R}^m} f_1(t) f_2(t)\, dE(t) = \int_{\mathrm{R}^m} f_1(t)\, dE(t) \int_{\mathrm{R}^m} f_2(t)\, dE(t)$$

holds in the weak operator topology whenever f_1 and f_2 belong to the class B (dE) for the same halfplane. We denote the class of all semi-multiplicative spectral functions by $\mathcal{E}_\mu{}^m$.

On the other hand, the operator-valued function $Q(\lambda)$, $\lambda \in \Omega'$, is said to be a *μ-resolvent* if its restriction $S(u) = Q(iu)$, $u \in \mathrm{R}^m$, satisfies conditions (i) and (ii) below and also (iii) when u_1 and u_2 belong to the same quadrant.

(i) $S(u^*) = S^*(u)$.

(ii) for each subset $\{k_1, \cdots, k_p\} \subset \{1, \cdots, m\}$, the joint limit

$$\lim_{u^{k_1}, \cdots, u^{k_p} \to 0} \frac{S(u)}{i^p \pi_j u^{k_j}} \tag{8}$$

exists in the srong operator topology and

$$\lim_{u^1, \cdots, u^m \to 0} \frac{S(u)}{i^m \Pi u} = I.$$

(9) (iii) $\Pi(u_2 - u_1) S(u_2) S(u_1) = i^m \Pi u_2 \Pi u_1\, \Delta_{u_1}^{u_2} S$.

Here we have used the notation

$$\Delta_{u_1}^{u_2} S = \sum (-1)^{\Sigma k_j} S(u_{k_1}^1, \cdots, u_{km}^m),$$

where the summation extends over all $k_1, \cdots, k_{m+n} = 1$ or 2. The class of all μ-resolvents will be denoted by $\mathcal{Q}_\mu{}^m$.

THEOREM 6. *There is a one-to-one-to-one correspondence between the classes $\mathcal{E}_\mu{}^m$, $\mathcal{Q}_\mu{}^m$, $\mathcal{V}_\mu{}^m$ given by the matching formulas. Given a matched triple E, Q, V from these classes, there exists a decomposition of each triple function into the product of its marginals.*

$$E = \Pi E^j, Q = \Pi Q^j, V = \Pi V^j$$

where the triple E^j, Q^j, V^j is a one-parameter matched triple of functions with $E^j \in \mathcal{E}_\mu{}^1$, $Q^j \in \mathcal{Q}_\mu{}^1$, and $V^j \in \mathcal{V}_\mu{}^1$. Finally, the various marginals commute pairwise.

1.6. Let us summarize our notational conventions. The topological Cartesian products Γ_+ and Ω have been given in section 1.1. Put

$$\Gamma_+ = (-\infty, \infty)^m \times (-\infty, 0]^n.$$

An $m + n$-tuple $(u^1, \cdots, u^{m+n})$ will be denoted by u. The sum $u + v$ and the dot product $u \cdot v$ are defined in the usual way. By Πu we mean the product $u^1 \cdots u^{m+n}$ and by u^* we mean $(-u^1, \cdots, -u^m, u^{m+1}, \cdots, u^{m+n})$. The symbol $[i]_j$ will stand for $\sqrt{-1}$ if $1 \leqq j \leqq m$ and for 1 if $m + 1 \leqq j \leqq m + n$, while $[i]u$ is to signify $(iu^1, \cdots, iu^m, u^{m+1}, \cdots, u^{m+n})$. We denote by $\lim_{u/u^j \to 0}$, the joint limit as all the variables $u^1, \cdots, u^{m+n}$ tend to zero (in their respective spaces) except u^j which remains fixed. Finally, the various partial derivatives of an operator-valued function $S(u)$ will be denoted by subscripts in the usual way, e.g., $\partial^2 S/\partial u^j \partial u^k = S_{kj}(u)$.

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2. Functional Calculus and Matching Formulas.

2.1. A resolution of the identity $E(t) = \Pi E^j_{t^j} \in \mathcal{E}_0{}^{(m+n)}$ defines an operator measure $dE(B)$ on the Borel subsets B of Γ_+ (cf. § 4.1 and Billingsley [4], p. 226). This can be achieved by extending the projection-valued function $dE(\cdot)$ defined on $m + n$ rectangles $[t_1, t_2) = [t_1, t_2{}^1) \times \cdots \times [t_1{}^{m+n}, t_2{}^{m+n})$ according to the equation

$$\begin{aligned} dE([t_1, t_2)) &= \prod_j (E^j_{t_2{}^j} - E^j_{t_1{}^j}) \\ &= \sum (-1)^{\Sigma k_j} E(t^1_{k_1}, \cdots, t^{m+n}_{k_{m+n}}) \end{aligned} \tag{10}$$

where the summation extends over all choices $k_1, \cdots, k_{m+n} = 1, 2$. The resulting extension satisfies the conditions:

$$0 = dE(\phi) \leqq d\mathbf{E}(B) \leqq d\mathbf{E}(\Gamma_+) = I, \quad (B \subset \Gamma_+),$$

$$d\mathbf{E}(B_1 \cup B_2) = d\mathbf{E}(B_1) + d\mathbf{E}(B_2), \quad \text{for disjoint } B_1, B_2),$$

$$d\mathbf{E}(B_1 \cap B_2) = d\mathbf{E}(B_1)\, d\mathbf{E}(B_2), \quad (\text{for arbitrary } B_1, B_2).$$

In fact this measure is precisely the product measure

$$d\mathbf{E} = \prod d\mathbf{E}^j$$

of the marginal measures. In view of equation (1) for projections, each spectral function $E \in \mathcal{E}^{(m,n)}$ likewise induces a weak operator measure $(dE(B)x, y)$, $(x, y \in H)$. Let $B(dE)$ denote the class of all scalar-valued functions $f(t)$, $t \in \Gamma_+$, which are measurable, finite, and defined almost everywhere with respect to all the measures $(dE(\,\cdot\,)x, x)$, $(x \in H)$. Then, by a theorem due to Riesz, the integral

$$(Fx, y) = \int_{\Gamma_+} f(t)\,(dE(t)x, y)$$

determines an operator F in H with domain consisting of all x for which

$$\int |f(t)|^2\,(dE(t)x, x) < \infty\,.$$

This functional correspondence, which we indicate by writing $F \sim f(t)$, enjoys most of the properties of the usual operator calculus following exactly as in that case. In general though, it is not multiplicative. However multiplicativity is restored whenever $E \in \mathcal{E}_0$ is a resolution of the identity.

THEOREM. *With notation as above, let $F \sim f$, $G \sim g$, $F_n \sim f_n$, and $G_n \sim g_n$. Then the functional correspondence:*

(i) *is linear: $c_1F + c_2G \sim c_1f + c_2g$, ($c_1, c_2$ scalars),*

(ii) *preserves conjugates: $F^* \sim \bar{f}$,*

(iii) *is positive: $f(t) \geqq 0$ on* supp *dE implies $F \geqq 0$,*

(iv) *is norm decreasing: $\|F\| \leqq \operatorname{ess\,sup}|f(t)|$ on* $\operatorname{supp}(dE)$ *if the latter exists,*

(v) *preserves limits: when $f_n(t) \to f(t)$ boundedly a.e. (dE), then $F_n \to F$ strongly,*

and satisfies

(vi) $\|Fx\|^2 \leqq \int_{\Gamma_+} |f(t)|^2\,(dE(t)x, x)$

(vii) *when $f_n(t) \to f(t)$, as $n \to \infty$, uniformly on* supp dE, *then* $\|F_n - F\| \to 0$,

(viii) $I \sim 1$.

If in addition $E \in \mathcal{E}_0$, then the correspondence is multiplicative,

$FG \sim fg$, and the inequality in (vi) *may be replaced by equality.*

2.2. One important consequence of this theorem in the total setting is the simple relationship between a scalar function $f(t) \in B(d\mathbf{E})$ and its corresponding operator F. If $f(t)$ is a multi-variate polynomial in the variables $t^1, \cdots, t^{m+n}$, then it follows from linearity and multiplicativity that F is the multi-variate operator obtained from $f(t)$ by replacing each occurrence of t^j by the self-adjoint operator A^j. Moreover, in view of the limit properties of the operational calculus, the same can be said for arbitrary $f(t) \in B(d\mathbf{E})$. Hence we may regard the operator $F \sim f(t)$ as the multi-variate function $f(A)$ of the "operator" $A = (A^1, \cdots, A^{m+n})$.

THEOREM. *For each function $f(t) \in B(d\mathbf{E})$ the operator $f(A^1, \cdots, A^{m+n})$ can be represented according to the integral*

$$f(A) = \int_{\Gamma_+} f(t)\, d\mathbf{E}(t),$$

convergent in the strong operator topology.

As a corollary to this theorem we obtain the matching formulas (2)–(4) interrelating a matched triple.

PROOF OF THE MATCHING FORMULAS. Equations (2) and (3) are immediate from the functional calculus. Equation (4) is a consequence of the one-parameter scalar equation,

$$\frac{\lambda}{1 - \lambda t} = \pm i \int_0^\infty e^{\pm is/\lambda} e^{\pm ist}\, ds, \ (\pm \ \mathrm{Im}\, \lambda > 0).$$

Of course the domain of $V(s)$ must be extended to the imaginary axis in its last n coordinates. But this is always possible since the same is true for a dilation of V to a semi-group.

3. Generalized Spectral Triples.

3.1. **Proof of Theorem 2.** Let E, Q, and V be a matched triple and suppose $Q \in \mathcal{Q}_0^{(m,n)}$. (The argument for $E \in \mathcal{E}_0^{(m,n)}$ or $V \in \mathcal{V}_0^{(m,n)}$ is similar). Then $Q(\lambda) = \Pi Q^j_{\lambda_j}$ and each Q^j is a one-parameter resolvent for some self-adjoint operator A^j. Letting E^j and V^j denote the resolution of the identity and the unitary group (or semi-group depending upon j) for A^j, then $E' = \Pi E^j \in \mathcal{E}_0$, $V' = \Pi V^j \in \mathcal{V}_0$, and E', Q, V' is a matched triple. To complete the proof it remains to show that $E' = E$ and $V' = V$. But this is a consequence of the one-to-one nature of the correspondence given by the matching formulas. A fact which is demonstrated in 3.2 below.

3.2. **Proof of Theorem 1.** Let $Q \in \mathcal{Q}^{(m,n)}(H)$ be a generalized resolvent. It has a dilation to a multi-parameter resolvent $\mathcal{Q}$ which, as was shown above, is part of a matched triple **E**, **Q**, **V**. Since the projections $E = pr\mathbf{E}$, $Q = pr\mathbf{Q}$, and $V = pr\mathbf{V}$ satisfy the matching formulas, therefore we have proved that each member of $\mathcal{Q}$ belongs to at least one matched triple.

Since a similar argument implies the same conclusion when applied to $\mathcal{E}$ or $\mathcal{V}$ it remains only to show that the matching is unique. Clearly, given E, then Q and V are uniquely determined by the first two matching formulas. The same reasoning applied to the third matching formula shows that $Q(\lambda)$ is uniquely determined by $V(s)$. To complete the proof it remains to see that $E(t)$ is uniquely determined by $Q(\lambda)$. For this we proceed by induction on $m + n$.

For $m + n = 1$ it is a familiar fact that the scalar integral

$$(Q(\lambda^1)x, x) = \int_{\Gamma_+} \frac{\lambda^1}{1 - \lambda^1 t^1}(dE(t^1)x, x) = \int_{\Gamma_+} \frac{(dE(t^1)x, x)}{1/\lambda^1 - t^1}$$

uniquely determines the integrator $(E(t^1)x, x)$ under a normalization such as our condition (ii) of Theorem 3 (cf. Widder [22], p. 336 or Stone [19], p. 163). Here Γ_+ is the set of non-negative real numbers if $n = 1$ and the entire set of real numbers if $m = 1$. Since this holds for all $x \in H$, it follows from the polarization formula that $(E(t^1)x, y)$ is uniquely determined for all $x, y \in H$, and hence the conclusion holds for $m + n = 1$ by an application of a well-known theorem of Riesz.

Next assume the uniqueness holds for all integers up to $m + n - 1$ and let

$$(11)\quad dp(t^1) = \int_{-\infty}^{\infty} \cdots \int_0^{\infty} \prod_{k=2}^{m+n} \frac{\lambda^k}{1 - \lambda^k t^k}(dE(t)x, x)$$

where the integration is taken over the last $m + n - 1$ variables $t^2, \cdots, t^{m+n}$. Naturally $p(t^1)$ depends upon $\lambda^2, \cdots, \lambda^{m+n}$ as well as t^1; we omit this dependence in our notation for convenience. It is easy to see, that as a function of t^1, p is of bounded variation, continuous from below and vanishes at $t^1 = -\infty$. All these conclusions follow more or less directly from the same properties of the integrator and from the bound

$$\left| \prod_{k=2}^{m+n} \frac{\lambda^k}{1 - \lambda^k t^k} \right| \leqq \prod_{k=2}^{m+n} \frac{|\lambda^k|}{|\operatorname{Im} \lambda^k|}$$

for the integrand. Therefore by the induction assumption $p(t^1)$ is uniquely determined by the equation

$$(Q(\lambda)x, x) = \int \frac{\lambda^1}{1 - \lambda^1 t^1} dp(t^1).$$

But then a second application of the assumption applied to (11) shows that $(E(t)x, x)$ is uniquely determined for each fixed t^1 from which the desired conclusion about $E(t)$ now follows for the integer $m + n$ and induction is complete.

3.3. **Proof of Theorem 3.** The first part of our argument in § 3.2 above shows the existence of a simultaneous dilation for a matched triple. It remains to be shown that the minimal dilation space may be computed up to isomorphism by any of the three formulas. Let $\mathbf{H}_E = \operatorname{span}\{\mathbf{E}(t)x : x \in H, \ t \in \Gamma_+\}$ and similarly define $\mathbf{H}_Q$ and $\mathbf{H}_V$ using Q and V respectively in place of E. Let E and V′ be matched in $\mathbf{H}_E$ and let E′ and V be matched in $\mathbf{H}_V$. Then it follows from the functional calculus that

$$\mathbf{H}_V = \operatorname{span}\{\mathbf{E}'(t)x : x \in H, T \in \Gamma_+\}.$$

Therefore both pairs $(\mathbf{H}_E, \mathbf{E})$ and $(\mathbf{H}_V, \mathbf{E}')$ are minimal dilations of (H, E) and so are isomorphic (cf. Nagy [20]). A similar argument applied to $\mathbf{H}_0$ and $\mathbf{H}_E$ shows they are isomorphic and the proof is complete.

4. **Intrinsic Characterizations.**

4.0. It is clear that any attempt to study spectral theory as a multiparameter phenomenon will require an intrinsic characterization of the various operator families. This has been done for spectral functions by Naimark [13]. For Bochner functions and generalized resolvents the integral representations of the functional calculus are used to reduce these problems to analogous questions for scalar functions. As applied to Bochner functions, the scalar integrals have been studied by Bochner [5], Akhiezer [2], and Devinatz [7]. As applied to generalized resolvents, scalar characterizations have been found by Nevanlinna [15], Korânyi [11], and Shonkwiler [16].

The process mentioned above for obtaining operator characterizations based upon scalar ones has been completed for Bochner functions, and generalized resolvents in Shonkwiler [17] and [18]. For the sake of completeness, characterizations of all the triple functions will be stated here.

4.1. THEOREM 7. *An operator-valued function $E(t)$, $t \in \Gamma_+$, is an (m, n)-parameter resolution of the identity if and only if*

(i) *$E(t)$ is selfadjoint, non-decreasing in each variable, and for each $m + n$-dimensional rectangle* $[t_1, t_2) = [t_1, t_2{}^1) \times \cdots \times [t_1{}^{m+n}, t_2{}^{m+n})$

the operator $dE([t_1, t_2))$ *given by the last member of* (10) *is n*
negative selfadjoint,

(ii) $E(t)$ *is continuous from below as a function of* t, $E(t) = 0$ *if a*
one of the last n coordinates of t vanishes, $E(t) \to 0$ *as any one of*
first m coordinates of t tend to $-\infty$, *and* $E(t) \to I$ *as all coordinates*
t tend to ∞ *(strong limits) and*

(iii) $E(t_1)\, E(t_2) = E(u)$ *where* $u^k = \min\{t_1{}^k, t_2{}^k\}$.

Proof. One easily sees that the marginal defined by equation (5
a one-parameter resolution of the identity. Finally property (iii) assu
that $E = \Pi E^j$.

This theorem admits a restatement in terms of measures.

Theorem 8. *The operator-valued measure dE defined on the Bo*
subsets of Γ_+ *determines an (m, n)-parameter spectral function if a*
if and only if it is non-negative, finite with $dE(\Gamma_+) = I$, *and for ar*
trary Borel subsets B_1 *and* B_2 *of* Γ_+,

$$dE(B_1 \cap B_2) = dE(B_1)dE(B_2).$$

The operator-valued function $E(t)$ may be recovered from its m
sure according to the formula

$$E(t) = dE([(-\infty, \cdots, -\infty, 0, \cdots, 0), t)). \tag{12}$$

4.2. The following characterization is due to Naimark.

Theorem 9. *An operator-valued measure dE defined on the Bo*
subsets of Γ_+ *determines an (m, n)-parameter spectral function if a*
only if it is non-negative and finite with $dE(\Gamma_+) = I$. *Moreover, th*
is a minimal dilation $d\mathbf{E}$ *acting on* $\mathbf{H}$, *unique up to isomorphi*
given by

$$\mathbf{H} = \operatorname{span}\{\mathbf{E}(t)x : t \in \Gamma_+, x \in H\}$$

where $\mathbf{E}$ *is defined by equation* (12) *of* § 4.1. *In this case* $dE(B) =$
if $d\mathbf{E}(B) = 0$ *where B is a Borel subset of* Γ_+.

Reformulated in terms of spectral functions this theorem reads
follows.

Theorem 10. *An operator-valued function* $E(t)$, $t \in \Gamma_+$, is
(m, n)-parameter spectral function if and only if (i) *and* (ii) *of Theor*
7 § 4.1 *hold.*

4.3. While defined on the entire topological space Ω, the mu
parameter resolvent $Q(\lambda)$ is in fact completely determined by its val
on a somewhat smaller space. Thus we denote by $S(u)$ the restrict

of $Q(\lambda)$ to the imaginary axis in its first m coordinates and to the non-positive real axis in its last n coordinates,

$$S(u) = Q([i]u), u \in \Gamma_-.$$

The intrinsic characterization below is proved in Shonkwiler [17].

THEOREM 11. *The operator-valued function* $S(u)$, $u \in \Gamma_-$, *is the restriction of an* (m, n)*-parameter resolvent* $Q(\lambda)$ *if and only if*

(i) $S(u^*) = S^*(u)$,

(ii) *for each subset* $\{k_1, \cdots, k_p\} \subset \{1, \cdots, m+n\}$ *the joint limit*

$$\lim_{u^{k_1}, \cdots, u^{k_p} \to 0} \frac{S(u)}{\prod_{j=1}^{p} [i]_{k_j} u^{k_j}}$$

exists in the strong operator topology and

$$\lim_{u \to 0} \frac{S(u)}{i^m \Pi u} = I,$$

and (iii) *for* $u_1, u_2 \in \Gamma_-$

$$\Pi(u_2 - u_1) S(u_2) S(u_1) = i^m \Pi u_2 \Pi u_1 \, \Delta_{u_1}^{u_2} S$$

where

$$\Delta_{u_1}^{u_2} S = \sum_{k_1, \cdots, k_{m+n} = 1,2} (-1)^{\Sigma k_j} S(u_{k_1}^1, \cdots, u_{k_{m+n}}^{m+n}). \tag{13}$$

The equation in (iii) *is the multi-parameter analog of the familiar resolvent equation.*

REMARK. The resolvent $Q(\lambda)$ can be reconstructed from $S(u)$ as above through the marginals

$$S_{u^j} = \lim_{u/u^j \to 0} \frac{[i]_j u^j S(u)}{\Pi[i]u}, \; j = 1, \cdots, m+n.$$

In fact the selfadjoint operator

$$A^j = \frac{1}{[i]_j u^j}(I - [i]_j u^j S_{u^j}^{-1})$$

does not depend upon the choice of u^j and is the jth operator in the defining sequence for Q. Thus $Q(\lambda)$ is the (m, n)-parameter resolvent of the operator sequence $A^1, \cdots, A^{m+n}$.

4.4. Just as in the case of a resolvent, the generalized resolvent $Q(\lambda)$ may be characterized in terms of its restriction $S(u) = Q([i]u)$, $u \in \Gamma_-$. The projection of such a restriction may easily be seen to satisfy conditions (i) and (ii) of the Theorem of § 4.3, but not necessarily the resolvent equation (iii). Nevertheless, there is a suitable "projection invariant" alternative to (iii) through the use of a certain kernel $K(\cdot, \cdot)$ defined on $\Gamma_- \times \Gamma_-$ as follows. With the notation as in equation (13) of § 4.3 we put

$$K(u_2, u_1) = \frac{\Delta_{u_1}^{u_2} {}^*S}{\pi[i](u_2{}^* - u_1)} \tag{14}$$

if the denominator does not vanish. Otherwise the kernel is defined by the joint limit of (14) as the appropriate factors in the denominator tend to zero or equivalently as the multiple difference of the resulting mixed partial derivative.

We may now state our next theorem. For further details and the proof the reader is referred to Shonkwiler [17].

THEOREM 12. *The operator-valued function* $S(u)$, $u \in \Gamma_-$, *is the restriction of a generalized* (m, n)*-parameter resolvent* $Q(\lambda)$, $S(u) = Q([i]u)$, *if and only if condition* (ii) *of Theroem* 11 *holds and* $S(u)$ *has weakly continuous partial derivatives of up to* $(m + n)$th *order and the kernel* $K(u_2, u_1)$ *defined by* (14) *is of positive type.*

Moreover, the dilation $Q(\lambda)$ *acting on the dilation space* H *can be chosen to be minimal in the sense that*

$$\mathrm{H} = \operatorname{span}\{Q(\lambda)x : x \in H, \lambda \in \Omega\}.$$

4.5. The following criteria for multi-parameter semi-groups is completely analogous to that of the familiar one-parameter case.

THEOREM 13. *The operator-valued function* $V(s)$, $s \in \Gamma_+$, *is a continuous* (m, n)*-parameter semi-group if and only if*

(i) $V(s^*) = V^*(s)$ *and* $\|V(s)\|$ *is uniformly bounded,*
(ii) $V(0) = I$ *and* $V(0, \cdots, s^j, \cdots, 0) \to V(0)$ *weakly as* $s^j \to 0$, *for* $j = 1, \cdots, m + n$, *and*
(iii) V *has the semi-group property*

$$V(s_1 + s_2) = V(s_1)V(s_2).$$

PROOF. One easily verifies that the marginals defined by equation (7) are the one-parameter component factors of a function V satisfying (i)–(iii). The necessity of the conditions is immediate.

4.6. In connection with the multi-parameter Bochner function we repeat here a characterization similar to one which can be found in Akhiezer [2], p. 231. As above the conditions are also couched in terms of a kernel of positive type.

THEOREM 14. *The operator-valued function* $V(s)$, $s \in \Gamma_+$, *is an* (m, n)*-parameter Bochner function if and only if*

(i) $\|V(s)\| \leqq 1, s \in \Gamma_+$;

(ii) $V(0) = I$ *and the following weak limits obtain as* $s^k \to 0$, $V(0, \cdots, s^k, \cdots, 0, t^{m+1}, \cdots, t^{m+n}) \to V(0, \cdots, 0, t^{m+1}, \cdots, t^{m+n})$ *if* $1 \leqq k \leqq m$ *while* $V(0, \cdots, 0, t^{m+1}, \cdots, s^{m+1}, \cdots, t^{m+n}) \to V(0, \cdots, 0, t^{m+1}, \cdots, o, \cdots, t^{m+n})$ *if* $m + 1 \leqq m + j = k \leqq m + n$; *and*

(iii) *the kernel*

$$K(s_2, s_1) = V(s_1 + s_2{}^*) \tag{15}$$

is of positive type.

Moreover, the dilation $\mathbf{V}(s)$ *acting on the dilation space* $\mathbf{H}$ *can be chosen to be minimal in the sense that*

$$\mathbf{H} = \operatorname{span}\{\mathbf{V}(s)x : x \in H, s \in \Gamma_+\}.$$

The proof may be found in Shonkwiler [18]. Our result is an improvement since condition (ii) is weaker than the weak continuity of $V(s)$ required by Akhiezer.

REMARK. The well-known theorem of Bochner is contained in the above (in view of Stone's Theorem on the representation of continuous groups). In fact when $m = 1$ and $n = 0$ condition (i) is redundant and the other two are known to imply Bochner's conditions.

5. Marginals.

5.1. PROOF OF THEOREM 4. Let H be the minimal dilation space of the matched triple E, Q, V; let $\mathbf{E}$, $\mathbf{Q}$, $\mathbf{V}$ be the matched dilation; and let $\mathbf{E}^j$, $\mathbf{Q}^j$, $\mathbf{V}^j$ be their triple of jth marginals. If $\mathbf{P}$ denotes the orthogonal projection of $\mathbf{H}$ onto H as usual, then the generalized marginals E^j, Q^j, V^j are given by the equations

$$\begin{aligned} E^j_{t^j} &= E(\infty, \cdots, t^j, \cdots, \infty) = \mathbf{P}\,\mathbf{E}(\infty, \cdots, t^j, \cdots, \infty) \\ &= \operatorname{pr} \mathbf{E}^j_{t^j} \end{aligned}$$

$$\begin{aligned} Q^j_{\lambda^j} &= \lim_{\lambda/\lambda^j \to 0} \frac{\lambda^j Q(\lambda)}{\pi\lambda} = \lim_{\lambda/\lambda^j \to 0} \frac{\lambda^j \mathbf{P}\mathbf{Q}(\lambda)}{\Pi\lambda} \\ &= \mathbf{P} \lim_{\lambda/\lambda^j \to 0} \frac{\lambda^j \mathbf{Q}(\lambda)}{\pi\lambda} = \operatorname{pr} \mathbf{Q}^j_{\lambda^j}, \end{aligned}$$

$$V_s = V(0, \cdots, s^j, \cdots 0) = \mathbf{P}\,\mathbf{V}(0, \cdots, s_j, \cdots, 0) = \operatorname{pr} \mathbf{V}^j_{s^j}.$$

The conclusions of the theorem now follow immediately.

5.2. Unlike the situation in the case of total triple functions, the product of the $m + n$ marginals of a generalized triple function will not in general be that function. Indeed these marginals may not even commute as can be seen by a simple example.

REMARK. While a generalized triple function may fail to be the product of its marginals, it is nevertheless possible to recover the multi-parameter function from them. Associated with these marginals is a single dilation space and an orthogonal projection P. The calculation in the proof of our last theorem shows that the projection P of the product, in any order, of the dilations of the marginals is the required generalized triple function.

6. **Semi-Multiplicative Case.**

6.1. PROOF OF THEOREMS 5 AND 6. The former is a more-or-less immediate corollary of the latter since each extended continuous semi-group of contractions is matched with a semi-multiplicative spectral function. Evidently the same argument can be applied to a μ-resolvent.

COROLLARY. *The kernel of each μ-resolvent as given in § 4.4 is of positive type.*

Now for the proof of Theorem 6. Consider $V \in \mathcal{V}_\mu{}^m$ first. It is easy to see that its marginals V^j satisfy the conditions of a natural extension of a strongly continuous one-parameter semi-group of contractions (cf. McKelvey [12]). Moreover

$$\begin{aligned} V(s) &= V((s^1, 0, \cdots, 0) + (0, s^2, \cdots, 0) + \cdots + (0, 0, \cdots, s^m)) \\ &= V(s^1, 0, \cdots, 0)\, V(0, s^2, \cdots, 0) \cdots V(0, 0, \cdots, s^m) \\ &= \pi V^j_{s^j} \end{aligned}$$

since each point $(0, \cdots, s^j, \cdots, 0)$ belongs to the same quadrant, namely that of s. Finally we observe that

$$\begin{aligned} V^1_{s^1} V^2_{s^2} &= V(s^1, 0, \cdots, 0)\, V(0, s^2, \cdots, 0) \\ &= V(s^1, s^2) \\ &= V(0, s^2, \cdots, 0)\, V(s^1, 0, \cdots, 0) \\ &= V^2_{s^2} V^1_{s^1}. \end{aligned}$$

By a similar calculation for other pairs it follows that the marginals commute pairwise.

Next let E^j and Q^j correspond to V^j through the matching formulas, $j = 1, \cdots, m$. It is known that the triple E^j, Q^j, V^j is matched and belongs to the triple of classes $\mathcal{E}_\mu{}^1$, $\mathcal{Q}_\mu{}^1$, $\mathcal{V}_\mu{}^1$ respectively. Put $E = \pi E^j$. Then $E \in \mathcal{E}^{(m,0)}$ and so has an associated functional calculus. But

$$V(s) = \pi V^j_{s^j} = \prod_{j=1}^{m} \int_{-\infty}^{\infty} e^{-is^jt^j} dE^j_{t^j}$$

$$= \int_{\mathbf{R}^m} \pi\, e^{-is^jt^j} dE(t)$$

$$= \int_{\mathbf{R}^m} e^{-is;t}\, dE(t),$$

so that E and V are matched.

To show that $E \in \mathcal{E}_\mu{}^m$, let f_1 and f_2 have the required properties; $f_k(t) = \prod_{j=1}^m f_k^j(t^j)$. Then

$$\int_{\mathbf{R}^m} f_1(t)f_2(t)\, dE(t) = \prod_{j=1}^{m} \int_{-\infty}^{\infty} f_1^j(t^j)f_2^j(t^j)\, dE^j_{t^j}$$

$$= \prod_{j=1}^{m} \int_{-\infty}^{\infty} f_1^j(t^j)\, dE^j_{t^j} \int_{-\infty}^{\infty} f_2^j(t_j)\, dE^j_{t^j}$$

$$= \int_{\mathbf{R}^m} f_1(t)\, dE(t) \int f_2(t)\, dE(t).$$

Let $Q(\lambda) = \prod Q^j$ where Q^j is given above. Then the matching formulas hold for $Q(\lambda)$; in particular

$$Q(\lambda) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=1}^{m} \pm ie^{\mp is^j\lambda^j} \right] V(\mp s^1, \cdots, \mp s^m)\, ds$$

$$\text{where } \mp \operatorname{Im} \lambda^j > 0 \text{ depending on } \Phi,$$

in each half-plane Φ. It is easily seen from this formula that the semigroup property for V implies the resolvent equation for Q in the corresponding half-plane Φ. Thus $Q \in \mathcal{Q}_\mu{}^m$.

Next let $E \in \mathcal{E}_\mu{}^m \subset \mathcal{E}^{(m,0)}$. The function V defined by

$$V(s) = \int_{\mathbf{R}^m} e^{-is\cdot t}\, dE(t)$$

is matched with E. If s_1 and s_2 are in the same quadrant of $\mathbf{R}^m$, then there exist functions of the required type showing that $e^{-is_1{}^jt^j}$ and $e^{-is_2{}^jt^j}$ belong to the same class $B \pm (dE^j)$. Therefore

$$V(s_1+s_2) = \int_{R^m} e^{-i(s_1+s_2)\cdot t}\, dE(t)$$

$$= \int_{R^m} \left(\prod_{j=1}^{m} e^{-is_1{}^j t^j}\right) \left(\prod_{j=1}^{m} e^{-is_2{}^j t^j}\right) dE(t)$$

$$= \int_{R^m} \prod_{j=1}^{m} e^{-is_1{}^j t^j}\, dE(t) \int_{R^m} \prod_{j=1}^{m} e^{-is_2{}^j t^j}\, dE(t)$$

$$= V(s_1)\, V(s_2).$$

Therefore, $V \in \mathcal{V}_\mu{}^m$. The marginals V^j of V belong to $\mathcal{V}_\mu{}^1$ and E^j is matched with V^j, then $E^j \in \mathcal{E}_\mu{}^1$. Moreover, $E(t) = \prod E$ In fact

$$V(s) = \prod V^j_{s^j} = \prod \int_{-\infty}^{\infty} e^{-is^j t^j} dE^j_{t^j}$$
$$= \int_{Rm} e^{-is\cdot t}\, dE^1_{t^1} \cdots dE^m_{t^m}.$$

But then by the uniqueness of the integral representation, we hav that the measure $dE(t)$ is equal to the product measure $dE^1_{t^1} \cdots dE^m_t$ Hence, $E = \prod E_j$. Finally, if Q is defined by the matching formula then the desired decomposition and matching follow here as it di above for a given $V \in \mathcal{V}_\mu{}^m$.

To complete the proof let $Q \in \mathcal{Q}_\mu{}^m$. By comparing the proof (Theorem 1 in Shonkwiler [17] it is seen that the basic constru(tions there may be applied to a given half-plane in the present settin to yield a decomposition.

$$Q(\lambda) = \prod Q^j_{\lambda^j}$$

of Q into the product of m pairwise commuting one-parameter μ resolvents in that half-plane. Letting E^j and V^j be the triple fun(tions corresponding to Q^j and $E = \prod E^j$, then $E \in \mathcal{E}^{(m,0)}$ and

$$Q(\lambda) = \prod Q^j_{\lambda^j} = \prod_{j=1}^{m} \int_{-\infty}^{\infty} \frac{\lambda^j}{1-\lambda^j t^j} dE^j{}_{t^j}$$

$$= \int_{R^m} \prod_{j=1}^{m} \frac{\lambda^j}{1-\lambda^j t^j} dE(t).$$

Hence, E, Q and V are matched. To show that $E \in \mathcal{E}_\mu{}^m$ we procee exactly as in an earlier part of this proof. Therefore, also $V \in \mathcal{V}_\mu$ and the proof is complete.

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