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BRANCHING OF SOLUTIONS OF NONLINEAR EQUATIONS[1]

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CONTENTS

1. **Introduction.** Many problems in analysis and applied mathematics involving a parameter η require that, for (z, η) near $(0, 0)$, one determine the number of nontrivial solutions $w = w(z, \eta)$ near $w = 0$ of an equation of the form

$$(\ddagger) \qquad \mathfrak{L}\, w + \mathcal{M}(w, z) = \eta w, \qquad w, z \in \mathfrak{B},$$

where $\mathfrak{B}$ is a Banach space, $\mathfrak{L}$ is a linear operator, and $\mathcal{M}$ is a nonlinear operator satisfying $\mathcal{M}(0, 0) = 0$. This basic problem of determining the number of such nontrivial solutions near $w = 0$ has been considered by many authors since the early papers of Lyapunov [**45**] and Schmidt [**61**] on nonlinear integral equations, and has led to an entire theory of branching of solutions of nonlinear equations. For a historical and mathematical introduction to the theory of branching of solutions of nonlinear equations as well as an overall view of recent developments in the subject, the reader is referred to the books of Krasnosel'skiĭ [**40**], Vainberg [**68**], Pimbley [**52**], and Keller and Antman [**35**], and the survey articles of Krasnosel'skiĭ [**39**], Vainberg

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and Trenogin [73], Prodi [54], Vainberg and Aizengendler [69], and Stakgold [63].

The present paper may serve as a supplement to the above-mentioned books and articles in so far as it is devoted mainly to the study of equations of the form (‡) where the null space of $\mathfrak{L}$ has dimension $n \geqq 2$, and $\mathfrak{L}$ and $\mathcal{M}$ are, in general, unbounded noncompact operators. In particular, as indicated in the above listed contents, the major portion of the present article is concerned with the solution of the associated "branching equation" in the finite-dimensional spaces $\boldsymbol{C}^n$ (n-dimensional unitary space) or $\boldsymbol{R}^n$ (n-dimensional Euclidean space); the emphasis here is on constructive ways of solving the "branching equation" so that the indicated algebraic or topological degree methods are used mostly to complement more constructive analytic methods.

2. **Some results on branching of solutions.** In this section we consider some classes of nonlinear equations in a (real or complex) Hilbert space $\mathcal{H}$ and show how the basic problem of determining nontrivial solutions of such equations in $\mathcal{H}$ can be reduced to the problem of determining sufficiently small solutions of certain "branching equations" in the spaces $\boldsymbol{C}^n$ or $\boldsymbol{R}^n$. In §A we consider some classes of nonlinear equations involving unbounded operators, and in §B we consider the same equations in the special situation of bounded operators.

A. *Unbounded operators.* The following approach is a generalization of the method of Lyapunov and Schmidt referred to above and is based upon a paper of Gustafson and Sather [30] which is in turn a generalization of some earlier work of Cesari [16], Locker [43], and Reeken [55]. For the sake of simplicity we take $\mathcal{H}$ to be a Hilbert space (see [30] for a treatment of nonlinear operators on a Banach space) and consider an equation of the form

$$(*) \qquad Lw + N(w, z) = \eta w, \qquad w, z \in \mathcal{H},$$

where η is a scalar. Our initial hypotheses on the operators L and N are as follows:

(LF) $L : \mathfrak{D}(L) \to \mathcal{H}$ is a linear (not necessarily bounded) Fredholm operator, i.e., L is a closed linear operator such that

(a) the domain $\mathfrak{D}(L)$ is dense in $\mathcal{H}$,

(b) the range $\mathcal{R}(L)$ is closed in $\mathcal{H}$,

(c) the null space $\mathfrak{N}(L)$ of L and the null space $\mathfrak{N}(L^*)$ of the adjoint operator L^* are finite-dimensional with $\dim \mathfrak{N}(L) = n$ and $\dim \mathfrak{N}(L^*) = m$;

(NC) there exists an open set $\mathcal{O} \subset \mathcal{H}$ such that

(a) $N : (\mathcal{D}(L) \cap \mathcal{O}) \times \mathcal{O} \to \mathcal{H}$ is a nonlinear operator with $N(0,0) = 0$,

(b) N satisfies a local Lipschitz condition of the following form: if $w_1, w_2 \in \mathcal{D}(L) \cap \mathcal{O}$ and $z \in \mathcal{O}$ then

$$\|N(w_1, z) - N(w_2, z)\| \leqq Q(|||w_1|||, |||w_2|||, \|z\|)\, |||w_1 - w_2|||$$

where $\|\cdot\|$ denotes the norm derived from the inner product $(\cdot, \cdot)$ in $\mathcal{H}$, $Q : \boldsymbol{R}^3 \to [0, \infty)$ satisfies $\lim_{|x|\to 0} Q(x_1, x_2, x_3) = 0$, and $|||w||| = \|Lw\| + \|w\|$, $w \in \mathcal{D}(L)$, denotes the L-norm on $\mathcal{D}(L)$,

(c) $\|N(w, z) - N(w, 0)\| \to 0$ as $\|z\| \to 0$, uniformly for $|||w||| \leqq a$ $(a > 0)$, $w \in \mathcal{D}(L) \cap \mathcal{O}$.

The particular form of equation (*) and the above hypotheses on L and N were chosen with certain applications in mind to nonlinear eigenvalue problems, and nonlinear problems involving perturbations of unbounded linear operators. For example, if $\eta = \lambda - \lambda_0$ and L is of the form $L = A - \lambda_0 I$, where λ_0 is an isolated real eigenvalue of A such that $\mathfrak{N}(A - \lambda_0 I)$ and $\mathfrak{N}(A^* - \lambda_0 I)$ are finite-dimensional, then finding solutions of equation (*) near $\eta = 0$ is equivalent to finding solutions of $Aw + N(w, z) = \lambda w$ for λ near the eigenvalue λ_0 of the linearized problem.

Let us also remark here that we are interested mainly in the case where $n \geqq 2$ and $m \geqq 1$ in part (c) of hypothesis (LF). The special cases of either $n \geqq 1$ and $m = 0$, or $n = 0$ and $m \geqq 1$ are considered for bounded operators in [73], whereas the special case when $n = m = 1$ has been considered in recent years by a great many authors (e.g., see [4], [39], [40], [52], [54], [63], [73]).

Let us now assume that L satisfies (LF). Let $\{u_1, \cdots, u_n\}$ be an orthonormal basis for $\mathfrak{N}(L)$ and let $\{u_1^*, \cdots, u_m^*\}$ be an orthonormal basis for $\mathfrak{N}(L^*)$. Let P be the orthogonal projection operator of $\mathcal{H}$ onto $\mathfrak{N}(L^*)^\perp$, where $\mathfrak{N}(L^*)^\perp$ denotes the orthogonal complement of $\mathfrak{N}(L^*)$ in $\mathcal{H}$, and let Q be the projection operator of $\mathcal{H}$ onto $\mathfrak{N}(L)$. Then

$$Qw = \sum_{j=1}^{n} (w, u_j)u_j \tag{2.1}$$

and, since $\mathcal{R}(L) = \mathfrak{N}(L^*)^\perp$ (e.g., see [26, p. 95]),

$$(I - P)w = \sum_{j=1}^{m} (w, u_j^*)u_j^* \tag{2.2}$$

is the projection operator of $\mathcal{H}$ onto $\mathfrak{N}(L^*)$, and $\mathcal{H}$ may be written as the orthogonal direct sum of $\mathfrak{N}(L^*)$ and $\mathcal{R}(L)$.

Let us next note that the restriction of L to $\mathcal{D}(L) \cap \mathfrak{N}(L)^\perp$ is a one-one mapping of $\mathcal{D}(L) \cap \mathfrak{N}(L)^{\perp}$ onto $\mathcal{R}(L)$ so that $K = (L|_{\mathcal{D}(L)\cap\mathfrak{N}(L)^{\perp}})^{-1}$ is well defined, and $\mathcal{D}(K) = \mathcal{R}(L)$ and $\mathcal{R}(K) = \mathcal{D}(L) \cap \mathfrak{N}(L)^{\perp}$. Moreover, since K is a closed operator from (the Hilbert space) $\mathcal{R}(L)$ into $\mathfrak{N}(L)^{\perp}$, it follows from the closed graph theorem that K is continuous (e.g., see [**26**, p. 94]). Hence KP is a continuous linear operator defined on all of $\mathcal{H}$. The following lemma summarizes some useful properties of K (see also [**16**], [**43**, p. 405] and [**50**, p. 72]).

LEMMA 2.1. *The linear operators K, P and Q satisfy*
(a) $KLw = (I - Q)w$ *for all* $w \in \mathcal{D}(L)$,
(b) $LKPw = Pw$ *for all* $w \in \mathcal{H}$.

The proof of property (a) follows by direct calculation, i.e., if $w \in \mathcal{D}(L)$ then $(I - Q)w \in \mathcal{D}(L) \cap \mathfrak{N}(L)^{\perp}$ and

$$KLw = KL[(I - Q)w + Qw] = KL(I - Q)w = (I - Q)w,$$

and property (b) is just the statement that K is a right inverse for L on $\mathcal{R}(L)$.

Let us assume for the moment that $w \in \mathcal{D}(L)$ is a solution of equation (*). Then, since $\mathcal{R}(L) = \mathfrak{N}(L^*)^{\perp}$, an application of property (a) yields the relationship

$$v + KP[-\eta(u + v) + N(u + v, z)] = 0 \tag{1}$$

where we have set $v = (I - Q)w$ and $u = Qw$. This relationship suggests the following natural question: For which u, z and η does a solution $\hat{v} = v(u, z, \eta)$ in $\mathfrak{N}(L)^{\perp}$ of equation (1) yield a solution $w = u + \hat{v}$ of equation (*)? A complete answer to this question is furnished by

LEMMA 2.2. *Suppose that η is a scalar, $z \in \mathcal{O}$ and $u \in \mathfrak{N}(L)$, and $\hat{v} = v(u, z, \eta)$ in $\mathfrak{N}(L)^{\perp}$ is a solution of equation* (1). *Then $w = u + \hat{v}$ is a solution of equation* (*) *if and only if u, z and η satisfy the equation*

$$(I - P)[-\eta(u + \hat{v}) + N(u + \hat{v}, z)] = 0. \tag{2}$$

It is clear that if $\hat{v} = v(u, z, \eta)$ is a solution of (1) then $\hat{v} \in \mathcal{D}(L) \cap \mathfrak{N}(L)^{\perp}$. Thus, by property (b) of Lemma 2.1,

$$\begin{aligned} 0 &= L\hat{v} + LKP[-\eta(u + \hat{v}) + N(u + \hat{v}, z)] \\ &= L(u + \hat{v}) - \eta(u + \hat{v}) + N(u + \hat{v}, z) \\ &\quad - (I - P)[-\eta(u + \hat{v}) + N(u + \hat{v}, z)] \end{aligned} \tag{2.3}$$

which implies Lemma 2.2.

REMARK 2.1. An equation such as (2) is usually referred to as a "branching equation". Let us observe that if we set $u = \sum_{j=1}^{n} \xi_j u_j$, where $\{u_1, \cdots, u_n\}$ is an orthonormal basis for $\mathfrak{N}(L)$, then equation (2) is equivalent to the system

$$(2)^* \qquad \left(-\eta \left(\sum_{j=1}^{n} \xi_j u_j + \hat{v} \right) + N \left(\sum_{j=1}^{n} \xi_j u_j + \hat{v}, z \right), u_i^* \right) = 0$$

$$(i = 1, \cdots, m)$$

so that the "branching equation" is actually a system of m nonlinear equations in the $n + 2$ variables $\xi_1, \xi_2, \cdots, \xi_n, \eta$ and z.

The following theorem provides a solution of equation (1) in the space $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$.

THEOREM 2.1. *Suppose that L and N satisfy hypotheses* (LF) *and* (NC). *Then there exists a positive constant* $b \leqq a/2$ *and positive constants c and d such that, for each* $u \in \mathfrak{N}(L)$, $z \in \mathcal{O}$ *and each* η *satisfying* $\|u\| \leqq b$, $\|z\| \leqq c$ *and* $|\eta| \leqq d$, *equation* (1) *has a unique solution* $\hat{v} = v(u, z, \eta)$ *belonging to* $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$. *Moreover, for* $\|u\| \leqq b$, $\|z\| \leqq c$ *and* $|\eta| \leqq d$, *the solution* $\hat{v}$ *satisfies* $|||\hat{v}||| \leqq \|u\| + 2(1 + \|K\|)\|N(0, z)\|$, *and* $\hat{v}$ *depends continuously in the L-norm on* u, z *and* η.

PROOF. Let us note first of all that, since L is a closed operator, the linear space $\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ is complete in the L-norm, so that we may use the contraction mapping principle in the Banach space $(\mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp, |||\ \ |||)$.

Let $v \in \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ and $u \in \mathfrak{N}(L)$, and define the mapping

$$(2.4) \qquad \Phi(v, u, z, \eta) = KP[\eta(u + v) - N(u + v, z)].$$

Then, by the definition of K and property (b) in Lemma 2.1, for each fixed $u \in \mathfrak{N}(L)$ and $z \in \mathcal{O}$, and each fixed η, we have $\Phi : \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp \to \mathfrak{D}(L) \cap \mathfrak{N}(L)^\perp$ and

$$(2.5) \qquad \begin{aligned} &|||\Phi(v_1, u, z, \eta) - \Phi(v_2, u, z, \eta)||| \\ &= \|P[\eta(v_1 - v_2) + N(u + v_2, z) - N(u + v_1, z)]\| \\ &\quad + \|KP[\eta(v_1 - v_2) + N(u + v_2, z) - N(u + v_1, z)]\| \\ &\leqq (1 + \|K\|)[|\eta| + Q(|||u + v_1|||, |||u + v_2|||, \|z\|)]\ |||v_1 - v_2|||, \end{aligned}$$

where $\|K\|$ denotes the norm of K. Let $d = 1/8(1 + \|K\|)$ and let $\tilde{c} > 0$, $b > 0$ be such that $Q(|||u + v|||, 0, \|z\|) \leqq 1/8(1 + \|K\|)$ and

$Q(|||u + v_1|||, |||u + v_2|||, \|z\|) \leqq 1/4(1 + \|K\|)$ whenever $\|u\| \equiv |||u||| \leqq b$, $|||v||| \leqq b$, $|||v_i||| \leqq b$, and $\|z\| \leqq \tilde{c}$. Let $c \leqq \tilde{c}$ be such that $\|N(0, z)\| \leqq b/2(1 + \|K\|)$. Then, for $\|u\| \leqq b$, $\|z\| \leqq c$ and $|\eta| \leqq d$, Φ maps the ball $B = \{|||v||| \leqq b\}$ into itself, and Φ satisfies

$$(2.6) \quad |||\Phi(v_1, u, z, \eta) - \Phi(v_2, u, z, \eta)||| \leqq \tfrac{1}{2}\, |||v_1 - v_2|||, \qquad v_1, v_2 \in B.$$

Therefore, by the contraction mapping principle, there is a unique fixed point $\hat{v} = v(u, z, \eta)$ of Φ such that $\hat{v}$ satisfies equation (1) and $|||\hat{v}||| \leqq \|u\| + 2(1 + \|K\|)\|N(0, z)\|$ whenever $\|u\| \leqq b$, $\|z\| \leqq c$ and $|\eta| \leqq d$. The desired continuity of the function $v(u, z, \eta)$ in u, z and η for $\|u\| \leqq b$, $\|z\| \leqq c$, and $|\eta| \leqq d$ is now immediate; namely, if $v = v(u, z, \eta)$ and $v^* = v(u^*, z^*, \eta^*)$ then

$$(2.7) \quad \begin{aligned} |||v - v^*||| &= |||KP[\eta(u + v) - \eta^*(u^* + v^*) \\ &\qquad + N(u^* + v^*, z^*) - N(u + v, z)]\,||| \\ &\leqq 2b(1 + \|K\|)|\eta - \eta^*| + \tfrac{1}{2}\|u - u^*\| + \tfrac{1}{2}\,|||v - v^*||| \\ &\qquad + (1 + \|K\|)\|N(u + v, z^*) - N(u + v, z)\|. \end{aligned}$$

This completes the proof of the theorem.

As a consequence of Lemma 2.2 and Theorem 2.1, finding solutions of equation $(*)$ in $\mathcal{H}$ is now reduced to finding sufficiently small solutions (u, z, η) of a finite-dimensional problem; it is convenient to state this relationship as

THEOREM 2.2. *Let the operators L and N satisfy the hypotheses of Theorem* 2.1 *and, for* $\|u\| \leqq b$, $\|z\| \leqq c$ *and* $|\eta| \leqq d$, *let* $\hat{v}$ *be the unique solution of equation* (1) *as determined in Theorem* 2.1. *If, in addition, there exists a solution* $\hat{u} = u(z, \eta)$ *of the branching equation* (2) *then* $\hat{w} = \hat{u} + \hat{v}$ *is a solution of equation* $(*)$.

Before turning to the study of the branching equation (2), let us remark that our particular hypotheses on the operators L and N were chosen for the sake of convenience and with certain applications in mind to problems involving nonlinear perturbations of unbounded linear operators. Other related approaches are to be found in the work of Cesari [14], [15], [16], Hale [31], [32], [3], Trenogin [67], Vainberg and Trenogin [72], Locker [43], [44], Reeken [55], and Gustafson and Sather [30], and the references cited therein. A completely different approach may be found in some recent papers of Gustafson and Sather [28], [29] wherein the problems considered are such that certain results in monotone operator theory may be used rather than the contraction mapping principle.

Let us now consider the branching equation (2). It is, of course, well known that finding solutions of equation (2) may be a very difficult problem because the explicit form of $\hat{v}$ may not be known. In fact, in the case of a real Hilbert space $\mathcal{H}$ the complete solution of (2) is usually quite difficult even when $m = n$.

In order to simplify somewhat the study of the branching equation (2) we introduce the following stronger hypotheses on N:

(NS) N is a smooth operator of the form

$$N(w, z) = F_1 z + \sum_{r+s=2}^{q} F_{rs}(w, z) + G(w, z),$$

where $F_1 : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, $F_{rs} : \mathcal{D}(L) \times \mathcal{H} \to \mathcal{H}$ is a homogeneous polynomial of degree r in w and degree s in z which satisfies a local Lipschitz condition of the form stated in hypothesis (NC) for some Q_{rs}, and $G : (\mathcal{D}(L) \cap \mathcal{O}) \times \mathcal{O} \to \mathcal{H}$ is a "higher order" nonlinear operator such that $\|G(w, z)\| = o(|||w|||^q + \|z\|^q)$ as $|||w||| + \|z\| \to 0$, G satisfies also a local Lipschitz condition of the form stated in (NC), and G is continuous in z as in (NC).

It is clear that if L and N satisfy (LF) and (NS) then the existence of a unique solution $\hat{v} = v(u, z, \eta)$ of equation (1) is a consequence of Theorem 2.1. Moreover, if N satisfies (NS) then several of the lower order terms of $\hat{v}$ may be calculated as in Bartle [4, p. 373] and Pimbley [52, pp. 24–28] by successively substituting

$$KP[\eta(u + \hat{v}) - N(u + \hat{v}, z)]$$

for $\hat{v}$. In this way one obtains more detailed information about the branching equation (2) which allows one in some cases to carry out the solution of (2) near $\|u\| = \|z\| = \eta = 0$; such an approach is considered in some detail in §2B for the special case when $\mathcal{D}(L) = \mathcal{H}$ and $m = n$ in (LF), and $\eta = 0$ in equation $(*)$.

On the other hand, since the calculations in the method of successive substitutions are very involved, one would like instead an approach which does not require detailed information about $\hat{v}$. Such an approach may, in fact, be developed in the case where, in addition to (LF), L is an operator having ascent $\alpha(L) = 1$ (e.g., see [**64**]). For the sake of simplicity, let us consider in the present paper only the special case where L is a selfadjoint operator; the reader is referred to Gustafson and Sather [**30**] for a more general branching analysis of equation $(*)$ for unbounded operators L with $\alpha(L) = 1$, and to Pimbley [52, p. 120] and Stakgold [**63**] for the case of bounded operators.

The basic equations (1) and (2) above assume an especially convenient form when L satisfies the following stronger hypothesis:

(LSA) L is of the form $L = A - \lambda_0 I$, where A is a (not necessarily bounded) selfadjoint operator and λ_0 is an isolated eigenvalue of A of finite multiplicity.

Then, since $\mathfrak{N}(L) = \mathfrak{N}(L^*)$ implies $\mathcal{R}(L) = \mathfrak{N}(L)^\perp$, $\mathfrak{N}(L)$ and $\mathcal{R}(L)$ are complementary orthogonal subspaces of $\mathcal{H}$ so that the projection operators P and Q in (2.1) and (2.2) satisfy $Q = I - P$, and the equations (1) and (2) reduce to

$$(1)_{sa} \qquad v + K[-\eta v + PN(u + v, z)] = 0,$$

$$(2)_{sa} \qquad -\eta u + QN(u + v, z) = 0.$$

It will be seen in §4 that this reduction is especially useful when $\mathcal{H}$ is a real Hilbert space and N is a smooth operator, in that one can determine solutions $u = u(z, \eta)$ of the branching equation $(2)_{sa}$ without detailed information about the function $\hat{v} = v(u, z, \eta)$ arising from the solution of equation $(1)_{sa}$.

Let us now consider some applications of the above method to nonlinear partial differential equations.

Elliptic equations. Let Ω be an open bounded set in R^n and let $\partial\Omega$ denote its boundary. Let $x = (x_1, x_2, \cdots, x_n)$ denote a point of Ω and let $\|w\|_p = (\int_\Omega |w(x)|^p\, dx)^{1/p}$ where dx denotes n-dimensional Lebesgue measure. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is any n-tuple of nonnegative integers, we set $|\alpha| = \sum_{j=1}^n \alpha_j$, $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$, and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $D_j = (1/i)(\partial/\partial x_j)$ $(j = 1, 2, \cdots, n)$. Let $\tilde{E}$ be the partial differential operator of order $2m$ defined by $\tilde{E} = \sum_{|\alpha| \leqq 2m} a_\alpha(x) D^\alpha$, where the $a_\beta(x)$ $(|\beta| \leqq 2m - 1)$ are complex-valued functions on Ω and (for the sake of simplicity) the a_α with $|\alpha| = 2m$ are real-valued functions on Ω. The operator $\tilde{E}$ is elliptic on Ω if its characteristic polynomial $P(x, \xi) = \sum_{|\alpha| = 2m} a_\alpha(x)\xi^\alpha$ does not vanish in Ω for any real n-vector $\xi \neq 0$. It is uniformly elliptic on Ω if there exists a constant $c > 0$ such that $|P(x, \xi)| \geqq c|\xi|^{2m}$ for every point x in Ω and for any real n-vector ξ.

Let $W^{l,2}(\Omega)$ be the set of functions u in $\mathfrak{L}^2(\Omega)$ which have generalized derivatives $D^\alpha u$ belonging to $\mathfrak{L}^2(\Omega)$ for $|\alpha| \leqq l$. Let $\mathfrak{D}$ be defined by $\mathfrak{D} = \{u : u \in W^{2m,2}(\Omega),\ D^\alpha u = 0 \text{ on } \partial\Omega \text{ for } |\alpha| < m\}$ and let E (the Dirichlet operator with zero boundary conditions on $\partial\Omega$ of order m) be defined by $\mathfrak{D}(E) = \mathfrak{D}$ and $Eu = \tilde{E}u$ for $u \in \mathfrak{D}(E)$. Then $E : \mathfrak{D}(E) \to \mathfrak{L}^2(\Omega)$ is linear with $\mathfrak{D}(E)$ dense in $\mathfrak{L}^2(\Omega)$. Under reasonable assumptions on the coefficients a_α and the open set Ω, it can then be shown (e.g., see Browder [12, p. 46 and p. 62] and Agmon, Douglis and Nirenberg [1, §12]) that if $\tilde{E}$ is uniformly elliptic on Ω

then E is a Fredholm operator; thus, by Theorem 2.2, the existence of solutions of an equation of the form $Eu + N(u, z) = \eta u$ (with zero Dirichlet boundary conditions on $\partial \Omega$ of order m) can be reduced to a finite-dimensional problem provided that N satisfies (NC). However the problem of determining classes of such N can be resolved in several ways by means of the Sobolev inequalities and some "a priori" bounds for elliptic operators.

For example, if Ω is sufficiently regular, one can show (e.g., see [12, p. 44] and [1, p. 694]) that there exists a constant K_1 such that if $u \in \mathfrak{D}(E)$ then

$$\sum_{|\alpha| \leqq 2m} \|D^\alpha u\|_2 \leqq K_1(\|u\|_2 + \|Eu\|_2). \tag{2.8}$$

Moreover, there exists a constant K_2 (e.g., see Sobolev [62, p. 56]) such that if $u \in \mathfrak{D}(E)$ and $|l|$ and n satisfy $n + 2|l| < 4m$, then, for $|\beta| \leqq |l|$,

$$\|D^\beta u\|_\infty \leqq K_2 \sum_{|\alpha| \leqq 2m} \|D^\alpha u\|_2, \tag{2.9}$$

where $\|w\|_\infty$ denotes the $\mathfrak{L}^\infty$-norm on Ω of w. Thus, combining (2.8) and (2.9), we see that if $n + 2|l| < 4m$ and N is of the form $N(u) = F(u, Du, \cdots, D^l u)$, where F is a polynomial satisfying $F(0, 0, \cdots, 0) = 0$, then N is defined on $\mathfrak{D}(E)$ and N satisfies a local Lipschitz condition of the type given in (NC).

As some indication of another possible use of the Sobolev inequalities in determining suitable nonlinear operators N, let us consider the simple boundary value problem

$$\begin{aligned} -\Delta w - \mu w + \sigma \alpha + \beta w^2 &= \eta w \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial \Omega, \end{aligned} \tag{2.10}$$

where α and β are real-valued functions that are continuous on the closure of the square $\Omega = \{(x_1, x_2) : 0 < x_1 < \pi \text{ and } 0 < x_2 < \pi\}$, μ and σ are real parameters, and Δ is the Laplacian in R^2. Let $C_0^\infty(\Omega)$ denote the set of infinitely differentiable functions with compact support in Ω. Let the operator $\tilde{A}$ be defined by $\mathfrak{D}(\tilde{A}) = C_0{}^\infty(\Omega)$ and $\tilde{A}u = -\Delta u$ for $u \in \mathfrak{D}(\tilde{A})$. In particular then $\mathfrak{D}(\tilde{A})$ is dense in the real Hilbert space $\mathfrak{L}^2(\Omega)$. Let A be the selfadjoint Friedrichs extension of $\tilde{A}$ (e.g., see [33]). Then $\mathfrak{D}(A) \subset \mathring{W}^{1,2}(\Omega)$ (the closure of $C_0{}^\infty(\Omega)$ in the norm of the real Hilbert space $W^{1,2}(\Omega)$), and A has eigenvalues $\mu_{kl} = k^2 + l^2$ and corresponding eigenfunctions $u_{kl} = (2/\pi) \sin kx_1 \sin lx_2$ $(k, l = 1, 2, \cdots)$. Thus, for example, the eigenvalue $\mu = 5$ has multiplicity two and the eigenvalue $\mu = 65$ has multiplicity four. Clearly, equation (2.10) can be written in the form of equation (*) with $L = A - \mu_{kl}I$, and $N(w, \sigma) = \sigma\alpha + \beta w^2$. Let us now show that N satisfies hypothesis (NC). If $w \in \mathfrak{D}(A)$ then $w \in \mathring{W}^{1,2}(\Omega)$ and $(Aw, w) = \|\nabla w\|_2^2$ $\nabla w \equiv \operatorname{grad} w$, so that one of the basic Sobolev inequalities (e.g., see [62, p. 57]) implies $w \in \mathfrak{L}^4(\Omega)$ and

$$\begin{aligned} \|w\|_4^2 &\leqq K_1\{\|w\|_2^2 + \|\nabla w\|_2^2\} = K_1\{\|w\|_2^2 + (Aw, w)\} \\ &\leqq \tfrac{3}{2} K_1\{\|w\|_2^2 + \|Aw\|_2^2\}, \end{aligned} \tag{2.11}$$

where K_1 is independent of w. Hence, if $u, v \in \mathfrak{D}(A)$ then (2.11), together with Schwarz's inequality, yields

$$\begin{aligned} \|N(u,\sigma) - N(v,\sigma)\|^2 &\leqq K_2(\|u\|_4^2 + \|v\|_4^2)\|u - v\|_4^2 \\ &\leqq K_3(|||u||| + |||v|||)^2 |||u - v|||^2 \end{aligned} \tag{2.12}$$

where $|||w||| = \|w\| + \|(A - \mu_{kl}I)w\|$. Thus, the existence of (generalized) solutions of problem (2.10) reduces to solving the appropriate branching equation $(2)_{\mathrm{sa}}$ in $\boldsymbol{R}^n$; this latter problem will be considered in §4A and §4B.

The Hartree equation for the Helium atom. As a second application of the above method let us consider a branching analysis of the Hartree equation for the Helium atom. This particular nonlinear equation, namely

$$\text{(He)} \quad -\frac{1}{2}\Delta w - \frac{2}{|x|} w + w \int \frac{w^2(y)}{|x - y|} dy = \lambda w, \qquad w(x) \in \mathfrak{L}(\boldsymbol{R}^3),$$

where Δ is the Laplacian in $\boldsymbol{R}^3$, arises naturally in the self-consistent field method of quantum mechanics. In this method approximate solutions w_0, $\int w_0{}^2(x)\,dx = 1$, and λ_0 of equation (He) provide approximations to the first eigenvalue and corresponding eigenfunction of the Schrödinger equation for the Helium atom.

Recently, the existence of an actual solution pair $(\hat{w}, \hat{\lambda})$ of equation (He) has been investigated by Reeken [55] who extended the analysis of Bazley and Zwahlen in [6] so as to apply to equation (He). The basic idea in [6], [55] is to first determine a branch of small solutions $w = w(\lambda)$ of equation (He) near $\lambda = -2$ (the first eigenvalue of the operator $-\frac{1}{2}\Delta - 2/|x|$) and then continue this local branch to a solution $\hat{w} = w(\hat{\lambda})$ of equation (He) satisfying $\int (\hat{w}(x))^2\,dx = 1$.

The branching analysis presented below is related to that of Reeken

[55]. On the other hand, an entirely different and independent approach to the problem of determining a local branch of solutions of equation (He) has also been given by Gustafson and Sather [28] who based their analysis instead upon certain results in monotone operator theory.

Let us note first of all (e.g., see [33, pp. 301–303]) that the operator $A = -\frac{1}{2}\Delta + q$, with $q = -2/|x|$ and $\mathfrak{D}(A) = W^{2,2}$ consisting of functions in $\mathfrak{L}^2(\boldsymbol{R}^3)$ which have generalized derivatives belonging to $\mathfrak{L}^2(\boldsymbol{R}^3)$ up to the second order, is selfadjoint and the lower part of its spectrum consists of isolated eigenvalues $\lambda_m = -2/m^2$ of multiplicity m^2 $(m = 1, 2, \cdots)$. Thus, if for fixed λ_m one sets $L = A - \lambda_m I$, then L satisfies hypothesis (LSA) with $\mathcal{H} = \mathfrak{L}^2(\boldsymbol{R}^3)$ and $\dim \mathfrak{N}(L) = m^2$.

Let $W^{1,2}$ denote the set of functions in $\mathfrak{L}^2(\boldsymbol{R}^3)$ having generalized first order derivatives belonging to $\mathfrak{L}^2(\boldsymbol{R}^3)$. Then the fundamental inequality (e.g., see [42, p. 16])

$$\int \frac{w^2(y)}{|x-y|^2} dy \leqq 4\|\nabla w\|^2 \qquad (\nabla w = \operatorname{grad} w), \tag{2.13}$$

together with Schwarz's inequality, implies

$$\int w^2(x) \left(\int \frac{w^2(y)}{|x-y|} \, dy \right)^2 dx \leqq 4\|w\|^4 \, \|\nabla w\|^2. \tag{2.14}$$

Thus, if we set $N(w) = w \int (w^2(y)/|x-y|) \, dy$ then N is defined on $W^{1,2} \supset \mathfrak{D}(A)$ and equation (He) may be written in the form of equation (*) with $L = A - \lambda_m I$ and $\eta = \lambda - \lambda_m$.

Let us now show that the nonlinear operator N satisfies hypothesis (NC). First of all, by applying (2.13) and Schwarz's inequality, one obtains

$$\begin{aligned} &\|N(w_1) - N(w_2)\| \\ &\quad \leqq 2\{\|w_1\| \, \|\nabla w_1\| + \|w_2\| \, \|\nabla w_2\| \\ &\qquad + \|w_1\| \, \|\nabla w_2\| + \|w_2\| \, \|\nabla w_1\|\} \|w_1 - w_2\|. \end{aligned} \tag{2.15}$$

Thus, in order to show that N satisfies the local Lipschitz condition in (NC), it suffices to show that, for some constant C,

$$\|\nabla w\| \leqq C(\|Lw\| + \|w\|), \qquad w \in \mathfrak{D}(L). \tag{2.16}$$

However, since q is relatively bounded with respect to $-\Delta$ (e.g., see [33, p. 303]) in the sense that

$$\|qw\| \leqq a\|w\| + b\|-\Delta w\|, \qquad w \in \mathfrak{D}(A), \tag{2.17}$$

where a and b are constants and b may be chosen arbitrarily small, one easily sees for $b = \frac{1}{4}$ that

$$(2.18) \qquad \|qw\| \leqq 2a\|w\| + \|Aw\|, \qquad w \in \mathfrak{D}(A).$$

Moreover, since (e.g., see [13])

$$(2.19) \qquad \|\nabla w\|^2 = \|(-\Delta)^{1/2}w\|^2 = (-\Delta w, w), \qquad w \in \mathfrak{D}(A),$$

it follows for $w \in \mathfrak{D}(L) = \mathfrak{D}(A)$ that

$$(2.20) \qquad \|\nabla w\|^2 = 2(Aw, w) - 2(qw, w) \leqq \|Aw\|^2 + \|qw\|^2 + 2\|w\|^2$$

which together with (2.18) and the definition of L implies the desired inequality (2.16).

Thus, if we now consider instead of equation (He) the equivalent system $(1)_{sa}$ and $(2)_{sa}$ then, by Theorem 2.1, equation $(1)_{sa}$ can be solved for a unique $v = v(u, \eta)$ whenever $\|u\| \leqq b$, $u \in \mathfrak{N}(L)$, and $|\eta| \leqq d$, so that by Theorem 2.2 finding nontrivial solutions of equation (He) in $\mathfrak{L}^2(\boldsymbol{R}^3)$ reduces to finding nontrivial solutions $u = u(\lambda)$ of the corresponding branching equation $(2)_{sa}$. This latter problem is solved, for example, in [**28**, §4] for $\lambda_m = \lambda_1$ and $\dim \mathfrak{N}(L) = 1$; in particular, one can show that there exists a positive constant δ such that for $\lambda_1 < \lambda < \lambda_1 + \delta$ equation (He) has a continuous nontrivial solution branch $w = u(\lambda) + \hat{v}(u(\lambda), \lambda)$ satisfying $\lim_{\lambda \to \lambda_1^+} w(\lambda) = 0$. Some techniques which may apply in the general case of $\dim \mathfrak{N}(L) = m^2$ to determine nontrivial solutions of equation $(2)_{sa}$ are developed in §4B.

B. *Bounded operators.* In the special case when $\mathfrak{D}(L) = \mathcal{H}$ so that L is a bounded operator, some results closely related to Theorems 2.1 and 2.2 are derived in the articles of Friedrichs [**23**], [**24**], Cronin [**18**], [**19**], Bartle [**4**], Graves [**27**], Trenogin [**65**], [**66**], Nirenberg [**50**, p. 83], and Vainberg and Trenogin [**73**], all of which are generalizations of the methods used by Lyapunov [**45**] and Schmidt [**61**] in their work on nonlinear integral equations. Most of the approaches just mentioned employ projection operators similar to the operators P and Q introduced above. However, under additional assumptions, some of these approaches lead to other types of associated branching equations (e.g., see [**19**] and [**50**]) as well as more detailed descriptions of the associated finite-dimensional problem.

In order to obtain a more detailed description of the branching equation in the case of bounded everywhere defined operators, we introduce the following hypotheses on L and N:

(LFB) L satisfies (LF) with $\mathfrak{D}(L) = \mathcal{H}$ and $m = n$;

(NSB) N is a smooth operator of the form

$$N(w, z) = F_1 z + \sum_{r+s=2}^{q} F_{rs}(w, z) + G(w, z)$$

where $F_1 : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, $F_{rs} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is a continuous homogeneous polynomial of degree r in w and degree s in z, and $G: \mathcal{O} \times \mathcal{O} \to \mathcal{H}$ is a continuous operator which is higher order in the sense that $\|G(w, z)\| = o(\|w\|^q + \|z\|^q)$ as $\|w\| + \|z\| \to 0$, and satisfies a Lipschitz condition of the form stated in hypothesis (NC) with $\mathcal{D}(L) = \mathcal{H}$ and the L-norm replaced by the norm in $\mathcal{H}$.

It can then be shown (e.g., see [4, p. 373]) that if N satisfies (NSB) then N satisfies (NC) with $\mathcal{D}(L) = \mathcal{H}$ and the L-norm replaced by the norm in $\mathcal{H}$. Thus, if L and N satisfy (LFB) and (NSB) then the existence of a unique solution $\hat{v} = v(u, z, \eta)$ of equation (1) is a consequence of Theorem 2.1.

Let us consider in some detail the special situation where $\eta = 0$ in equation (*), and L and N satisfy (LFB) and (NSB). Then equation (*) reduces to

$$(**) \qquad Lw + N(w, z) = 0, \qquad w, z \in \mathcal{H},$$

and equations (1) and (2) reduce to

$$(1^\circ) \qquad v + KPN(u + v, z) = 0$$

and

$$(2^\circ) \qquad (I - P)N(u + v, z) = 0.$$

It is convenient to study the case when z is of the form $z = \sigma z_0$ where σ is a scalar and z_0 is a fixed unit vector. If we now set $u = \sum_{j=1}^{n} \xi_j u_j$ as in Remark 2.1 then equation (2°) is equivalent to the system

$$(2^\circ)^* \quad \left(N \left(\sum_{j=1}^{n} \xi_j u_j + v(\xi, \sigma), \sigma z_0 \right), u_i^* \right) = 0 \qquad (i = 1, \cdots, n),$$

where $v = v(\xi, \sigma)$ denotes the unique continuous solution of (1°). Moreover, by using the method of successive substitutions as in [4], [52, p. 24], the system $(2^\circ)^*$ may be written in the special form

$$(2.21) \qquad \begin{aligned} \varphi^i(\xi, \sigma) \equiv a^{i0}\sigma + \sum_{r+s=2}^{q} \sigma^s \sum_{|l|=r} a^{is}_{l_1 \cdots l_n} \xi_1^{l_1} \cdots \xi_n^{l_n} \\ + \rho^i(\xi, \sigma) = 0 \qquad (i = 1, \cdots, n) \end{aligned}$$

where the second sum is over all nonnegative integers l_i such that $\sum_{i=1}^{n} l_i = r$, and the coefficients $a^{is}_{l_1 \cdots l_n}$ are calculable constants. Also the ρ^i $(i = 1, \cdots, n)$ are continuous near $|\xi| = \sigma = 0$ and satisfy the conditions

$$\rho^i(\xi, \sigma) = o(|\xi|^q + |\sigma|^q), \tag{2.22}$$

$$|\rho^i(\xi, \sigma) - \rho^i(\xi^*, \sigma)| \leqq \omega(|\xi|, |\xi^*|, |\sigma|)|\xi - \xi^*|, \tag{2.23}$$

where $\omega : \boldsymbol{R}^3 \to [0, \infty)$ satisfies $\lim_{|x| \to 0} \omega(x_1, x_2, x_3) = 0$. Thus, in the case when L and N satisfy (LFB) and (NSB), the problem of finding solutions of equation (**) in $\mathcal{H}$ reduces to the finite-dimensional problem of solving a system of the form (2.21). A fairly complete solution of this finite-dimensional problem in the case where $\mathcal{H}$ is a *complex* Hilbert space will be given in §3. The more difficult problem when $\mathcal{H}$ is a *real* Hilbert space will be considered in §4.

Let us consider also the case when N is an analytic operator and determine the special form of the branching equation (2°). Suppose that L satisfies (LFB) and that N satisfies the following analyticity assumption (the case when $\mathfrak{D}(N)$ is only dense in $\mathcal{H}$ is considered in Trenogin [**67**] and Vainberg and Trenogin [**72**]):

(NAB) N has a Taylor series representation, convergent in an open set $\mathcal{A} \subset \mathcal{H} \times \mathcal{H}$,

$$N(w, z) = F_1 z + \sum_{r+s \geqq 2} F_{rs}(w, z), \qquad (w, z) \in \mathcal{A},$$

where $F_1 : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator, and $F_{rs} : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is a continuous homogeneous polynomial of degree r in w and degree s in z.

Then, under the above assumptions on L and N, Vainberg and Trenogin [**73**, §4] (see also [**69**, p. 41]) show that if $z = \sigma z_0$ then the unique solution $v = v(\xi, \sigma)$ of equation (1°) is given by a convergent series of the form

$$v(\xi, \sigma) = A_1 \sigma + \sum_{|l| + p \geqq 2} A^p_{l_1 \cdots l_n} \xi_1^{l_1} \cdots \xi_n^{l_n} \sigma^p, \tag{2.24}$$

where the summation is over all nonnegative integers l_i and p such that $\sum_{i=1}^{n} l_i + p \geqq 2$, and the coefficients $A^p_{l_1 \cdots l_n}$ belong to $\mathcal{H}$ and are uniquely determined by certain recurrence formulas. Therefore, upon substituting the above series for v into (2°)*, one obtains, at least theoretically, a system of the form

$$\begin{aligned}\Phi^i(\xi,\sigma) \equiv & \sum_{|l|\geqq 2} b^i_{l_1\cdots l_n}\xi_1^{\ l_1}\cdots\xi_n^{\ l_n} \\ & + \sum_{|l|\geqq 0}\xi_1^{\ l_1}\cdots\xi_n^{\ l_n}\sum_{p\geqq 1} c^{ip}_{l_1\cdots l_n}\sigma^p = 0 \ \ (i=1,\cdots,n)\end{aligned} \tag{2.25}$$

where the coefficients $b^i_{l_1\dots l_n}$ and $c^{ip}_{l_1\cdots l_n}$ are calculable scalars and the Φ^i are analytic in a neighborhood of $|\xi| = \sigma = 0$. Thus, in the case when L and N satisfy (LFB) and (NAB), the basic problem of finding solutions of equation (**) in $\mathcal{H}$ reduces to a finite-dimensional problem involving analytic functions of $n + 1$ variables. This finite-dimensional problem will be studied in considerable detail in §§3A and 4A.

REMARK 2.2. In connection with our study of the branching equation in §§3 and 4, it is convenient to point out here that if $w = 0$ is an isolated solution of the equation $Lw + N(w, 0) = 0$ then, under the appropriate smoothness assumptions on N, $\xi = 0$ is an isolated solution of either the equation $\varphi(\xi, 0) = 0$ resulting from (2.21), or the equation $\Phi(\xi, 0) = 0$ resulting from (2.25).

3. **Solution of the branching equation in C^n.** The results of this section together with Theorem 2.2 provide a fairly complete solution to the basic problem of the existence of branches of equation (**) in a (complex) Hilbert space $\mathcal{H}$. Some of these results on analytic functions of several complex variables were obtained in the early part of this century whereas others were obtained only in the last several years.

The points of C^m will be denoted by m-tuples such as $\xi = (\xi_1, \cdots, \xi_m)$ and the norm in C^m will be given by $|\xi|^2 = |\xi_1|^2 + \cdots + |\xi_m|^2$. In addition, if $\xi = (x_1 + iy_1, \cdots, x_m + iy_m)$ then its conjugate $\bar{\xi}$ is given by $\bar{\xi} = (x_1 - iy_1, \cdots, x_m - iy_m)$.

A. *Algebraic methods and Newton's polygon for several variables.* Let us recall first of all that if the operators L and N in equation

$$Lw + N(w, z) = 0 \tag{**}$$

satisfy (LFB) and (NAB) in a complex Hilbert space $\mathcal{H}$, then the branching equation is equivalent to a system of the form

$$\begin{aligned}\Phi^i(\xi,\sigma) = & \sum_{|l|\geqq 2} b^i_{l_1\cdots l_n}\xi_1^{\ l_1}\cdots\xi_n^{\ l_n} \\ & + \sum_{|l|\geqq 0}\xi_1^{\ l_1}\cdots\xi_n^{\ l_n}\sum_{p\geqq 1} c^{ip}_{l_1\cdots l_n}\sigma^p = 0 \qquad (i=1,\cdots,n)\end{aligned} \tag{3.1}$$

where the Φ^i are analytic in a ball $|\xi|^2 + \sigma^2 < a^2$ in C^{n+1}, and $z = \sigma z_0$, $\|z_0\| = 1$. Thus, it is natural for our purposes to consider the problem of the existence of solutions $\xi = \xi(\sigma)$ in C^n of such systems or, more generally, to study the problem of the existence of solutions $x = x(\mu)$ of systems of the form

$$\Psi^i(x, \mu) = 0, \qquad x \in C^p, \mu \in C^q \; (i = 1, \cdots, p), \tag{3.2}$$

where the Ψ^i are analytic in a ball $|x|^2 + |\mu|^2 < a^2$ in C^{p+q}.

Let us suppose throughout the first part of this section that $x = 0$ is an isolated zero of the system $\Psi^i(x, 0) = 0$ $(i = 1, \cdots, p)$. Since the functions Ψ^i do not vanish identically near $x = 0$, let us denote by $\psi^i_{k_i}(x)$ the homogeneous polynomial in $\Psi^i(x, 0)$ of lowest degree k_i which does not vanish identically, and let $R(\psi)$ denote the resultant of the $\psi^i_{k_i}$ $(i = 1, \cdots, p)$. Then, by employing the Weierstrass preparation theorem, one can establish the following result due to MacMillan [46] and Bliss [11].

THEOREM 3.1. *Suppose that the resultant $R(\psi)$ of the homogeneous polynomials $\psi^i_{k_i}$ does not vanish. Then there exist positive constants b and c such that, for $|\xi| < b$ and $|\mu| < c$, the system* (3.2) *has exactly $M = \prod_{i=1}^p k_i$ solutions $x^l(\mu)$ (counting multiplicities) such that the $x^l(\mu)$ are continuous functions with $x^l(0) = 0$ $(l = 1, 2, \cdots, M)$.*

Since the $\psi^i_{k_i}$ are homogeneous polynomials, it is well known [73, p. 15] that the system $\psi^i_{k_i}(x) = 0$ $(i = 1, \cdots, p)$ has nontrivial solutions if and only if $R(\psi) = 0$. Thus, one may also state Theorem 3.1 in terms of the condition that the vector field $\{\psi^1_{k_1}, \cdots, \psi^p_{k_p}\}$ vanishes only at $x = 0$.

A somewhat related result, which may apply when $R(\psi) = 0$ but ($p - 1$ of the $\psi^i_{k_i}$ are linear and) the rank of the Jacobian matrix $(\partial\Psi/\partial x)$ at $(0, 0)$ is $p - 1$, is the following theorem due to Clements [17].

THEOREM 3.2. *Suppose that the Jacobian $J_1 = |\partial\Psi/\partial x| = 0$ at $(0, 0)$, $J_l = 0$ at $(0, 0)$ $(l = 2, 3, \cdots, k - 1)$, and $J_k \neq 0$ at $(0, 0)$, where*

$$J_l = \frac{\partial(J_{l-1}, \Psi_2, \cdots, \Psi_p)}{\partial(x_1, x_2, \cdots, x_p)} \quad (l = 2, 3, \cdots, k). \tag{3.3}$$

Then there exist positive constants b and c such that for $|\xi| < b$ and $|\mu| < c$ the system (3.2) *has exactly k solutions $x^l(\mu)$ such that the $x^l(\mu)$ are continuous functions with $x^l(0) = 0$ $(l = 1, 2, \cdots, k)$.*

Let us remark here that in the simple case when $|\partial\Psi/\partial x| \neq 0$ at $(0,0)$, either of the above theorems yield the existence of a unique solution $x(\mu)$ of the system (3.2) which is, in addition, analytic near $\mu = 0$.

The following theorem due to Osgood [51, p. 194] requires only that $x = 0$ is an isolated zero of the system $\Psi^i(x,0) = 0$ $(i = 1, \cdots, p)$, and is apparently one interpretation of a result stated by Poincaré [53, p. 14].

THEOREM 3.3. *Suppose that $x = 0$ is an isolated zero of the system $\Psi^i(x,0) = 0$ $(i = 1, \cdots, p)$. Then there exists a neighborhood $\mathcal{U}$ of $\mu = 0$, a set $E \subset \mathbf{C}^q$ of q-measure zero, and an integer M such that to each $\mu \in (\mathcal{U} - E)$ there correspond M points $x^l(\mu)$ such that the pairs $(x^l(\mu), \mu)$ are solutions of the system* (3.2).

As pointed out by Osgood, this last theorem is somewhat deficient in the sense that certain values of μ near $\mu = 0$ are excluded and, in general, the integer M is unknown; both of these deficiencies can, of course, be overcome by placing more restrictive assumptions on the Ψ^i as in Theorems 3.1 and 3.2. Some related but less definitive results for smooth vector fields will be obtained in §3B by means of topological degree theory.

A stronger result (see MacMillan [47]) than that given in Theorem 3.1 can be obtained in the special case when $q = 1$; namely, in the case of a single parameter μ, the solutions $x(\mu)$ obtained in Theorem 3.1 can be expanded in a power series of integral or fractional powers of μ. The knowledge of such an expansion, or even the lowest term, is of course of great importance in certain applications. On the other hand, the restriction to a single parameter μ, or several parameters $\mu_1, \mu_2, \cdots, \mu_q$ related to a single parameter μ by relations of the form $\mu_l = \mu\,\theta_l$ $(l = 1, \cdots, q)$ is necessary since, as pointed out by MacMillan, the equation $x^2 - 2\mu_1 x + \mu_2{}^2 = 0$ has solutions $x = \mu_1 \pm \sqrt{\mu_1{}^2 - \mu_2{}^2}$ which do not have power series expansions in terms of μ_1 and μ_2.

Let us consider therefore a system of the form (3.1). Let $\varphi^i_{k_i}(\xi)$ denote the homogeneous polynomial in $\Phi^i(\xi, 0)$ of lowest degree k_i which does not vanish identically, and let $R(\varphi)$ denote the resultant of the $\varphi^i_{k_i}$ $(i = 1, \cdots, n)$. Then, under the assumption $R(\varphi) \neq 0$, there are exactly $M = \prod_{i=1}^n k_i$ continuous solutions $\xi(\sigma)$ of the system (3.1) which satisfy $\xi(0) = 0$. The problem of expanding such a solution $\xi(\sigma)$ in a power series of integral or fractional powers of σ was considered by MacMillan in a second paper [47] and, in doing so, he was apparently the first to systematically make use of a Newton polygon method for several variables.

As some indication of the method developed by MacMillan, let us consider the simple example [47, p. 181]

$$(3.4)\qquad \begin{aligned} \Phi^1(\xi,\sigma) &\equiv (\xi_1{}^6 + \xi_2{}^6) + (\xi_1{}^4 + \xi_2{}^4)\sigma + (3\xi_1 - \xi_2)\sigma^2 + \sigma^4 = 0, \\ \Phi^2(\xi,\sigma) &\equiv \xi_1{}^8 + (\xi_1 - \xi_2)\sigma^2 + \sigma^3 = 0. \end{aligned}$$

First of all, since the vector field $\{\varphi_6{}^1, \varphi_8{}^2\} \equiv \{\xi_1{}^6 + \xi_2{}^6, \xi_1{}^8\}$ vanishes only at $\xi_1 = \xi_2 = 0$, it follows from Theorem 3.1 that there are exactly $M = 48$ continuous solutions of (3.4) near $\sigma = 0$. In order to determine expansions for these solutions, it is useful to rewrite the system (3.4) as

$$(3.5)\qquad \begin{aligned} \Phi^1 &= \varphi_6{}^1 + \varphi_4{}^1\sigma + \varphi_1{}^1\sigma^2 + \varphi_0{}^1\sigma^4, \\ \Phi^2 &= \varphi_8{}^2 + \varphi_1{}^2\sigma^2 + \varphi_0{}^2\sigma^3, \end{aligned}$$

where each $\varphi_j{}^i$ denotes the appropriate homogeneous polynomial of degree j. Let us now construct a Newton polygon (e.g., see [73, §5]) for *each* of the equations $\Phi^i = 0$ $(i = 1, 2)$ such that each term $\varphi_j{}^k\sigma^i$, which does not vanish identically, corresponds to a point on the polygon for $\Phi^k = 0$ having Cartesian coordinates (i, j); for example, consider the equation $\Phi^1 = 0$ and construct a polygonal line joining the point $(0, 6)$ with the point $(4, 0)$ such that the (negative) slopes of the line segments are increasing, and the polygonal line forms a boundary between the region in which points (i, j) exist and that in which they do not. Let l_1, l_2, and l_3 denote the line segments having slopes $-7/2$, $-5/2$, and -1, respectively. Corresponding to l_1 one now makes the substitution $\xi_i = \sigma^{2/7}y_i$ $(i = 1, 2)$ in (3.4) and, after dividing the first equation by $\sigma^{12/7}$, the second by $\sigma^{16/7}$, and letting $\sigma \to 0$, one obtains the equations

$$(3.6)\qquad y_1{}^6 + y_2{}^6 = 0,$$

$$(3.7)\qquad y_1{}^8 + y_1 - y_2 = 0.$$

By eliminating y_2 one obtains the equation

$$(3.8)\qquad y_1{}^6[(y_1{}^7 + 1)^6 + 1] = 0$$

which has 48 solutions, 6 of which are $y_1 = 0$ and 42 of which are nontrivial and distinct from one another. Therefore, by the usual implicit function theorem for a system of analytic equations, each of the 42 nontrivial solutions of (3.8) generates a solution ξ_1, ξ_2 of (3.4) which can be expanded as a power series in the fractional power $\sigma^{1/7}$. Similarly, corresponding to the lines l_2 and l_3 one makes the substitutions $\xi_i = \sigma^{2/5}y_i$ and $\xi_i = \sigma y_i$, respectively, and proceeds as above to determine five solutions of (3.4) each of which can be expanded as a power series in the fractional power $\sigma^{1/5}$, and one solution of (3.4) which can be expanded as a power series in σ. Thus, the method of MacMillan [47] yields a complete solution of the system (3.4) when σ is sufficiently small.

Let us remark here that the example (3.4) is a particularly simple one in that certain "intermediate resultants" do not vanish so that all of the solutions are obtained by means of substitutions derived from the corresponding Newton polygons. The solution of a problem in which this is not the case is illustrated by the example [47, p. 197]

$$\begin{aligned} \Phi^1 &= \xi_1{}^3 + (\xi_1{}^2 - \xi_2{}^2)\sigma + \sigma^4 = 0, \\ \Phi^2 &= \xi_2{}^3 + (\xi_1{}^2 - \xi_2{}^2)\sigma - \sigma^4 = 0. \end{aligned} \tag{3.9}$$

Clearly, there are nine continuous solutions of (3.9) near $\sigma = 0$. By constructing a Newton polygon for each equation, one easily sees as above that there are two substitutions $\xi_i = \sigma y_i$ and $\xi_i = \sigma^{3/2}y_i$ which are derived from the polygons. However, due to the vanishing of the resultant R of the coefficients of σ in (3.9), the substitution involving σ yields only two nontrivial solutions of (3.9) whereas the substitution involving $\sigma^{3/2}$ does not yield any solutions. Thus, there are seven solutions of (3.9) whose order in σ lies between σ and $\sigma^{3/2}$. By using additional substitutions related to the vanishing of R, one can, in fact, show that there are four solutions of (3.9) of order $\sigma^{5/4}$ and three solutions of (3.9) of order $\sigma^{4/3}$; the reader is referred to the paper of MacMillan for the details required to complete example (3.9) as well as for a general discussion of the problem of vanishing "intermediate resultants".

Let us remark here that a related Newton polygon method for functions of several variables has been formulated more recently by Graves [27]; this method and related results will be considered in §4A.

Clearly, as a consequence of Remark 2.2, all of the above results in this section have applications to the problem of solving the branching equation as given in (3.1) and, thus, together with Theorem 2.2 they provide a partial solution to the basic problem of the existence of branches of equation (**) in the case where $\mathcal{H}$ is a complex Hilbert space. A more complete theoretical solution of this basic problem has been given in some recent papers of Aizengendler [2] and Vainberg and Aizengendler [70] wherein an algebraic approach is developed which does not require $w = 0$ to be an isolated solution of the equation

$Lw + N(w, 0) = 0$; in fact, by employing the Weierstrass preparation theorem, the elimination method of Kronecker, the Newton polygon method for a single equation, and some results from the theory of commutative algebra, a method can be developed which theoretically determines all small solutions of a system of the form (3.1). For the sake of simplicity we outline the method for the case $n = 2$ as presented in [2], [**69**, p. 36]; the reader is referred to Vainberg and Aizengendler [**70**] for the general case $n \geqq 2$.

By means of a nonsingular linear transformation and the Weierstrass preparation theorem, the system (3.1) in C^2 is replaced with a system

$$G^i(z_1, z_2, \sigma) = b^i_{s_i} z_1{}^{s_i} + \sum_{j=1}^{s_i} z_1{}^{s_i - j} H^i_{s_i - j}(z_2, \sigma) = 0 \tag{3.10}$$

$$(i = 1, 2)$$

which is equivalent with respect to small solutions. Here s_i is the degree of the lowest order homogeneous polynomial in $\Phi^i(z_1, z_2, 0)$ which does not vanish identically, $b^i_{s_i} \neq 0$, $H^i_{s_i - j}(0, 0) = 0$, and the $H^i_{s_i - j}$ are analytic in a neighborhood of $(0, 0)$.

Let $R(\xi_2, \sigma) \equiv R(G^1, G^2)$ denote the resultant of the distinguished polynomials G^1 and G^2. Then the elimination of z_1 from the system (3.10) leads to the equation $R(z_2, \sigma) = 0$. If $R(z_2, \sigma) \not\equiv 0$ and (after possibly eliminating certain powers of σ) $R(0, 0) = 0$, then Newton's polygon method can be used to find all small solutions of $R(z_2, \sigma) = 0$ as convergent power series $z_2{}^\alpha(t)$ $(\alpha = 1, \cdots, p)$, where $t = \sigma^{1/\mu_\alpha}$ is some determined fractional power of σ. Substituting such a solution into (3.10) one obtains the system

$$g_\alpha{}^i(z_1, t) \equiv G^i(z_1, z_2{}^\alpha(t), t^{\mu_\alpha}) = 0 \qquad (i = 1, 2), \tag{3.11}$$

where the $g_\alpha{}^i$ are distinguished polynomials in z_1. Let $d_\alpha = (g_\alpha{}^1, g_\alpha{}^2)$ be the greatest common divisor of $g_\alpha{}^1$ and $g_\alpha{}^2$. Then d_α is also a distinguished polynomial and Newton's polygon method can again be used to find all small solutions of $d_\alpha(z_1, t) = 0$ as convergent power series $z_1{}^{\beta_\alpha}(\sigma)$ $(\beta_\alpha = 1, \cdots, q_\alpha)$ in fractional powers of σ. Each pair $(z_1{}^{\beta_\alpha}, z_2{}^\alpha)$ then determines a solution of the system (3.10) which is a power series in fractional powers of σ.

Therefore, (after possibly eliminating certain powers of σ) one has the following results [2]:

(a) If $R(0, 0) \neq 0$ then the system (3.1) has no small solutions.

(b) If $R(z_2, \sigma) \not\equiv 0$ and $R(0, 0) = 0$, then the system (3.1) has a finite number of small solutions each of which can be expanded in a power series of integral or fractional powers of σ.

On the other hand, if $R(z_2, \sigma) \equiv 0$ then the system (3.10) (and hence the system (3.1)) has, of course, an infinite number of solutions (see also [2], [69, p. 39]).

In addition to the above results, an important algorithm for determining the explicit form of the greatest common divisor d_α is also proposed in [2], [70]. Finally, for an application of the above method to nonlinear ordinary differential equations, the reader is referred to Vainberg and Aizengendler [71].

B. *Topological degree methods.* Let us suppose throughout the remainder of this section that the operators L and N are such that the branching equation (2°) is equivalent to a system of the form

$$\begin{aligned}\varphi^i(\xi, \sigma) \equiv a^i\sigma + \sum_{r+s=2}^{q} \sigma^s \sum_{|l|=r} a^{is}_{l_1 \cdots l_n} \xi_1^{l_1} \cdots \xi_n^{l_n} \\ + \rho^i(\xi, \sigma) = 0 \qquad (i = 1, \cdots, n),\end{aligned} \tag{3.12}$$

where the ρ^i are continuous in a neighborhood $\mathcal{U}$ of $|\xi| = \sigma = 0$ and "higher order" in the sense of (2.22). Moreover, as observed in Remark 2.2, if $w = 0$ is an isolated solution of the equation $Lw + N(w, 0) = 0$ then the vector field $\Phi_0(\xi) = \{\varphi^1(\xi, 0), \cdots, \varphi^n(\xi, 0)\}$ has an isolated zero at $\xi = 0$ so that (regarding Φ as a mapping of a subset of R^{2n} into R^{2n} which can be described in terms of functions of n complex variables) the topological index of Φ_0 at 0 is defined; i.e., for some open ball B centered at $\xi = 0$ of sufficiently small radius, $i(\Phi_0, 0) \equiv d(\Phi_0, B, 0)$ is defined where $d(\Phi_0, B, 0)$ denotes the topological degree of Φ_0 at 0 relative to B. Thus, it is natural to study the existence of solutions of a system of the form (3.12) by means of topological degree theory.

The following basic lemma on calculating the (topological) index of certain vector fields is due to J. Cronin [20], [21, pp. 45–46].

LEMMA 3.1. *Suppose that a vector field $\Psi = \{\psi^1, \cdots, \psi^n\}$ mapping a neighborhood of $\xi = 0$ into C^n is of the form $\psi^i(\xi) = P^i_{k_i}(\xi) + Q^i(\xi)$, where $P^i_{k_i}$ is a homogeneous polynomial of degree k_i and Q^i is a continuous function such that $|Q^i(\xi)|/|\xi|^{k_i} \to 0$ as $|\xi| \to 0$, and suppose that the vector field $P = \{P^1_{k_1}, \cdots, P^n_{k_n}\}$ vanishes only at $\xi = 0$. Then Ψ has an isolated zero at $\xi = 0$ and the index of Ψ at 0 is given by $i(\Psi, 0) = \prod_{j=1}^n k_j$.*

Since the functions $\varphi^i(\xi, 0)$ defined by (3.12) are of the form $P^i_{k_i} + Q^i$, the above lemma implies

THEOREM 3.4. *Suppose that the vector field $\Phi = \{\varphi^1, \cdots, \varphi^n\}$ defined by* (3.12) *is such that $\varphi^i(\xi, 0) = P^i_{k_i}(\xi) + Q^i(\xi)$, where the*

$P^i_{k_i}$ and Q^i satisfy the conditions of Lemma 3.1. *Then there exists a positive number b such that for each σ satisfying $|\sigma| < b$ the system* (3.12) *has at least one solution $\xi(\sigma)$ in C^n.*

PROOF. As a consequence of Lemma 3.1, the index of $\Phi_0(\xi) = \{\varphi^1(\xi, 0), \cdots, \varphi^n(\xi, 0)\}$ at 0 is defined and $i(\Phi_0, 0) = \prod_{j=1}^n k_j$.

Let B be an open ball centered at $\xi = 0$ with radius so small that Φ_0 has no zeros on $\bar{B}$ except at $\xi = 0$, and let ∂B denote the boundary of B. Since the φ^i are assumed to be continuous near $|\xi| = \sigma = 0$, there exists a constant b such that Φ does not vanish on ∂B when $|\sigma| < b$. The following well-known argument now provides solutions of (3.12). Since the functions φ^i are continuous on $\bar{B} \times \{|\sigma| < b\}$, it follows for each fixed σ satisfying $|\sigma| < b$ that Φ is homotopic to Φ_0 so that (e.g., see [**21**, p. 31])

$$d(\Phi, B, 0) = d(\Phi_0, B, 0) = \prod_{j=1}^n k_j \neq 0;$$

therefore, by the basic existence theorem of topological degree theory (e.g., see [**21**, p. 32]), for each fixed σ satisfying $|\sigma| < b$ there is at least one point $\xi(\sigma)$ in B such that $\Phi(\xi(\sigma), \sigma) = 0$.

The following stronger result in this direction is due to J. Cronin [**20**], [**21**, p. 47].

THEOREM 3.5. *Suppose that the vector field $\Phi = \{\varphi^1, \cdots, \varphi^n\}$ defined by* (3.12) *is such that $\xi = 0$ is an isolated zero of the system $\varphi^i(\xi, 0) \equiv P^i_{k_i}(\xi) + Q^i(\xi) = 0 \quad (i = 1, \cdots, n)$, where $P^i_{k_i}$ is a homogeneous polynomial of degree k_i and Q^i is a continuous function such that $|Q^i(\xi)|/|\xi|^{k_i} \to 0$ as $|\xi| \to 0$. Then there exists a positive number b such that for each σ satisfying $|\sigma| < b$ the system* (3.12) *has at least one solution $\xi(\sigma)$ in C^n.*

The proof of this theorem is an immediate consequence of a stronger version of Lemma 3.1 also due to J. Cronin [**20**], [**21**, p. 47]; namely, under the hypotheses in Theorem 3.5, the index at 0 of the vector field $\Psi = \{P^1_{k_1} + Q^1, \cdots, P^n_{k_n} + Q^n\}$ satisfies the inequality $i(\Psi, 0) \geqq \prod_{j=1}^n k_j$.

Let us observe here that, as a consequence of Remark 2.2, both of the above theorems together with Theorem 2.2 provide additional information concerning the existence of solutions of equation (**) in the case of a complex Hilbert space $\mathscr{H}$. However, not only do the solutions obtained in this manner fail, in general, to be continuous near $\sigma = 0$, but also the number of such solutions obtained is, in general, too small. In order to partially overcome this latter difficulty, J. Cronin [**19**, p. 212] defined the concept of a "good many points" and established a result for a restricted class of equations of the form (**) which says, roughly, that for a "good many" points z near $z = 0$ the absolute value of the topological index provides a lower bound for the number of distinct solutions of (**) (see also Theorem 3.3).

A different method of counting the number of solutions of some nonlinear equations in $\mathscr{H}$ is presented in §§4B and 4C wherein one relates the number of solutions in $\mathscr{H}$ to the number of certain kinds of fixed points of the nonlinear operator N.

4. **Solution of the branching equation in R^n.** In this section we are concerned with the solution of the branching equation in R^n. The emphasis is again on constructive ways of solving the branching equation, however, due to the inherent difficulty of determining only real solutions of such an equation, the results obtained are not as definitive as the results of §3.

A. *Algebraic methods and Newton's polygon for several variables.* Let us assume initially that the operators L and N in equation

$$(**) \qquad Lw + N(w, z) = 0$$

satisfy (LFB) and (NAB) in a real Hilbert space $\mathscr{H}$. Then the branching equation is equivalent to a system

$$\begin{aligned}\Phi^i(\xi, \sigma) = &\sum_{|l| \geqq 2} b^i_{l_1 \cdots l_n} \xi_1^{l_1} \cdots \xi_n^{l_n} \\ &+ \sum_{|l| \geqq 0} \xi_1^{l_1} \cdots \xi_n^{l_n} \sum_{p \geqq 1} c^{ip}_{l_1 \cdots l_n} \sigma^p \\ = &\ 0 \qquad (i = 1, \cdots, n),\end{aligned} \tag{4.1}$$

where the Φ^i are analytic in a ball $|\xi|^2 + \sigma^2 < a^2$ in R^{n+1}, and $z = \sigma z_0$, $\|z_0\| = 1$. Let $\varphi^i_{k_i}$ denote the homogeneous polynomial in $\Phi^i(\xi, 0)$ of lowest degree k_i which does not vanish identically, and let $R(\varphi)$ denote the resultant of the $\varphi^i_{k_i}$ $(i = 1, \cdots, n)$. The following result is a consequence of Theorem 3.1.

THEOREM 4.1. *Suppose that the resultant $R(\varphi)$ of the homogeneous polynomials $\varphi^i_{k_i}$ $(i = 1, \cdots, n)$ does not vanish, and that $M = \prod_{i=1}^n k_i$ is an odd integer. Then there exist positive constants b and c such that for $|\xi| < b$ and $|\sigma| < c$ the system* (4.1) *has at least one continuous real solution $\xi(\sigma)$ with $\xi(0) = 0$.*

In order to establish the result, one considers the system (4.1) as a system of equations in C^n with real coefficients. Since $R(\varphi) \neq 0$, it follows from Theorem 3.1 that for $|\xi| < b$ and $|\sigma| < c$ there are exactly $M = \prod_{i=1}^{n} k_i$ continuous solutions $\xi^l(\sigma)$ satisfying $\xi^l(0) = 0$ $(l = 1, 2, \cdots, M)$. However, since the coefficients of Φ are real, if σ is real and $(\zeta(\sigma), \sigma)$ is such a solution of (4.1), then $(\bar{\zeta}(\sigma), \sigma)$ is also a solution of (4.1) so that for real σ the complex solutions occur in pairs. Thus, since M is odd, there must be at least one continuous real solution $\xi(\sigma)$ of (4.1) satisfying $\xi(0) = 0$.

Clearly, if k is odd in Theorem 3.2, one again obtains the existence of at least one continuous real solution $\xi(\sigma)$ of (4.1) satisfying $\xi(0) = 0$.

On the other hand, the Poincaré-Osgood theorem of §3A is not directly applicable since the number of solutions obtained thereby is not, in general, odd (i.e., if M is an even integer there may or may not be real solutions of (4.1)). Thus, since in some applications showing that a solution of (**) does not exist is also of importance, the above algebraic methods by themselves are not sufficient to resolve the problem so that one needs other supplementary methods. One such supplementary method, which can sometimes be used to determine the exact number of real solutions of the branching equation, is the following Newton polygon method for several variables.

Let us assume for the remainder of §4A that L and N satisfy only (LF) and (NS), so that the branching equation is equivalent to a system

$$
\begin{aligned}
\varphi^i(\xi, \sigma) &= a^i\sigma + \sum_{r+s=2}^{q} \varphi^i_{rs}(\xi)\sigma^s + \rho^i(\xi, \sigma) \\
&= 0 \qquad (i = 1, \cdots, n),
\end{aligned} \tag{4.2}
$$

where the $\varphi^i_{rs}(\xi)$ are homogeneous polynomials of degree r in ξ, and the ρ^i are continuous in a neighborhood $\mathcal{U}$ of $|\xi| = \sigma = 0$ and satisfy (2.22) and (2.23). The following Newton polygon method for such a system is due to Graves [27]. We consider here only the case $\sigma > 0$; the case $\sigma < 0$ is treated in a similar way by first replacing σ by $-\sigma$ in (4.2).

Suppose that $\varphi_{rs} = \{\varphi^1_{rs}, \cdots, \varphi^n_{rs}\}$, that the φ_{rs} satisfy $\varphi_{r0}(\xi) \equiv 0$ $(r = 2, 3, \cdots, k-1)$ and $\varphi_{k0}(\xi) \not\equiv 0$ for $\xi \neq 0$, and that at least one of the φ_{rs} with $r < k$ and $s > 0$ does not vanish identically. Then the line joining the point $(k, 0)$ to the point (r, s) has negative slope. Denoting the maximum such slope by $-p/q$, substituting

$$
\xi = \theta^p\beta \quad \text{and} \quad \sigma = \theta^q \tag{4.3}
$$

into (4.2), and dividing by θ^{kp}, one obtains the system

$$\begin{aligned}\psi^i(\beta,\theta) = &\sum_{(r,s)\in l_0} \varphi^i_{rs}(\beta) + \theta^{t_1} \sum_{(r,s)\in l_1} \varphi^i_{rs}(\beta) \\ &+ \theta^{t_2} \sum_{(r,s)\in l_2} \varphi^i_{rs}(\beta) + \cdots + \rho^i(\theta^p\beta, \theta^q)/\theta^{kp} \\ &= 0 \qquad (i = 1, \cdots, n),\end{aligned} \tag{4.4}$$

where l_0 is the line through the point $(k, 0)$ with slope $-p/q$, l_1 is the closest parallel line containing plotted points, l_2 is the next closest parallel line, etc. The following theorem is useful in determining the exact number of real solutions of (4.2) (see [27, p. 152]).

THEOREM 4.2. (a) *Every continuous solution* $\beta = \beta(\theta)$ *of* (4.4) *yields a continuous solution* $\xi = \xi(\sigma)$ *of* (4.2) *by means of the substitutions in* (4.3). (b) *If the system*

$$\sum_{(r,s)\in l_0} \varphi^i_{rs}(\beta) = 0 \qquad (i = 1, \cdots, n) \tag{4.5}$$

has only a finite number of solutions, then every solution $\xi = \xi(\sigma)$ *of* (4.2) *continuous near* $\sigma = 0$ $(\sigma \geqq 0)$ *with* $\xi(0) = 0$ *is obtained by means of the substitutions* (4.3) *from a solution* $\beta = \beta(\theta)$ *of* (4.4) *continuous near* $\theta = 0$ $(\theta \geqq 0)$.

The proof of a related result is sketched in §4B.

As an application of this last theorem, let us consider the boundary value problem

$$\begin{aligned}-\Delta w - \mu_{12} w + \sigma\alpha + \beta w^2 &= 0 \qquad \text{in } \Omega, \\ w &= 0 \qquad \text{on } \partial\Omega,\end{aligned} \tag{4.6}$$

which was discussed also in §2 (see (2.10)); a detailed analysis of essentially the same example is to be found in [27]. We recall that, by Theorem 2.2 and the discussion of this boundary value problem in §2, finding small solutions of (4.6) is equivalent to finding sufficiently small solutions of the system

$$\varphi^i(\xi,\sigma) = (\sigma\alpha + \beta(\xi_1 u_1 + \xi_2 u_2 + v(\xi,\sigma))^2, u_i) = 0 \qquad (i = 1, 2), \tag{4.7}$$

where $\{u_1, u_2\} \equiv \{(2/\pi)\sin x_1 \sin 2x_2, (2/\pi)\sin 2x_1 \sin x_2\}$ is an orthonormal basis for the two-dimensional null space $\mathfrak{N}$ of $L = A - \mu_{12} I$. By the method of successive substitutions, (4.7) can be written as

$$\begin{aligned}\varphi^i(\xi,\sigma) = &\ \sigma(\alpha, u_i) + \varphi^i_{20}(\xi) + \sigma\varphi^i_{11}(\xi) \\ &+ \sigma^2\varphi^i_{02}(\xi) + \rho^i = 0 \qquad (i = 1, 2),\end{aligned} \tag{4.8}$$

where $\varphi_{20}^{i}(\xi) = (\beta(\xi_1 u_1 + \xi_2 u_2)^2, u_i)$. Let us now consider the following two special cases (see also [27, p. 157]).

Case I. Suppose that $\alpha = (2/\pi)u_1 \sin x_1 \sin 3x_2$ and $\beta = 4\pi^2 u_1$. Some simple calculations yield $\varphi_{20}^{1} = 9\xi_1{}^2 + 4\xi_2{}^2$, $\varphi_{20}^{2} = 8\xi_1\xi_2$, $(\alpha, u_1) \equiv a^2 = 2^9/63\pi^3$ and $(\alpha, u_2) = 0$. Thus, the system (4.8) becomes

$$
\begin{aligned}
9\xi_1{}^2 + 4\xi_2{}^2 + a^2\sigma + \sigma\varphi_{11}^{1} + \sigma^2\varphi_{02}^{1} + \rho^1 &= 0, \\
8\xi_1\xi_2 + \sigma\varphi_{11}^{2} + \sigma^2\varphi_{02}^{2} + \rho^2 &= 0.
\end{aligned} \tag{4.9}
$$

The line l_0 contains the points $(2, 0)$ and $(0, 1)$ so that, under the substitution $\xi_i = \sigma^{1/2}\beta_i$ $(\sigma > 0)$, the system (4.9) reduces to

$$
\begin{aligned}
9\beta_1{}^2 + 4\beta_2{}^2 + a^2 + \sigma^{1/2}\zeta^1 &= 0, \\
8\xi_1\xi_2 + \sigma^{1/2}\zeta^2 &= 0.
\end{aligned} \tag{4.10}
$$

Setting $\sigma = 0$ in (4.10) we see that there are no real solutions and, hence, no continuous real solutions of the boundary value problem (4.6) for small positive values of σ. On the other hand, since the substitution $\xi_i = |\sigma|^{1/2}\beta_i$ $(\sigma < 0)$ changes a^2 into $-a^2$ in (4.10), the elementary implicit function theorem now determines four real solutions of (4.10) which, in turn, generate four real solutions of the boundary value problem (4.6) for sufficiently small negative values of σ.

Case II. Suppose that $\alpha = (2/\pi)u_2 \sin x_1 \sin 3x_2$ and $\beta = 4\pi^2 u_1$. Then the φ_{20}^{i} are the same as in Case I, $(\alpha, u_1) = 0$, and $(\alpha, u_2) \equiv b$, $b > 0$, so that in this case (4.9) is replaced by

$$
\begin{aligned}
9\xi_1{}^2 + 4\xi_2{}^2 + \sigma\varphi_{11}^{1} + \sigma^2\varphi_{02}^{1} + \rho^1 &= 0, \\
8\xi_1\xi_2 + b\sigma + \sigma\varphi_{11}^{1} + \sigma^2\varphi_{02}^{1} + \rho^2 &= 0.
\end{aligned} \tag{4.11}
$$

Since (4.11) has no real solutions under the substitution $\xi_i = |\sigma|^{1/2}\beta_i$, the boundary value problem (4.6) has no real solutions for σ near $\sigma = 0$.

The above example suggests that finding real solutions of an equation such as (**) is, in general, a difficult problem; that is, by varying somewhat only the coefficient α in (4.6), this simple boundary value problem may have no solutions or it may have as many as four solutions. However, we will see in the next section that some of the difficulties which arise in solving (4.6) apparently do so because one is seeking solutions right at the eigenvalue μ_{12} of the linearized problem rather than at a nearby value of $\mu \neq \mu_{12}$.

B. *Topological degree methods.* Let us assume initially that the operators L and N in equation

$$(*) \qquad Lw + N(w, z) = \eta w$$

satisfy (LSA) and (NS) so that the branching equation $(2)_{\mathrm{sa}}$ in §2 is equivalent to a system of the form

$$(4.12) \qquad \left(-\eta \sum_{j=1}^{n} \xi_j u_j + N\left(\sum_{j=1}^{n} \xi_j u_j + v, \sigma z_0 \right), u_i \right) = 0 \qquad (i = 1, \cdots, n),$$

where $\{u_1, \cdots, u_n\}$ is an orthonormal basis for $\mathfrak{N}(A - \lambda_0 I)$, $z = \sigma z_0$ ($\|z_0\| = 1$), and $v = v(\xi, \sigma, \eta)$ is the unique solution of $(1)_{\mathrm{sa}}$ as determined in Theorem 2.1.

Although the methods of this section can be applied also to systems such as (4.12), for the sake of simplicity we consider here only the special case corresponding to $\sigma = 0$ (however, see [38], [58]). In fact, since the hypothesis (NS) implies

$$(4.13) \qquad N(w, 0) = \sum_{r=2}^{q} F_{r0}(w, 0) + G(w, 0),$$

we consider in this section an equation of the form

$$(\dagger) \qquad (A - \lambda_0 I)w + T(w) + R(w) = \eta w$$

where $\eta = \lambda - \lambda_0$, $L = A - \lambda_0 I$ satisfies (LSA), and the nonlinear operators T and R satisfy the following hypothesis:

(TH) $T: \mathfrak{D}(A) \to \mathcal{H}$ is a homogeneous polynomial of degree k ($k \geqq 2$) such that if $w_i \in \mathfrak{D}(A)$ and $\theta \in R^m$ then

$$T\left(\sum_{j=1}^{m} \theta_j w_j \right) = \sum_{|l|=k} \theta_1^{\,l_1} \cdots \theta_m^{\,l_m} T_{l_1 \cdots l_m}(w_1, \cdots, w_m)$$

where $T_{l_1 \cdots l_m}$ is a mapping from $\mathfrak{D}(A) \times \cdots \times \mathfrak{D}(A)$ into $\mathcal{H}$ which is independent of $\theta_1, \cdots, \theta_m$;

(TD) for each element $w \in \mathfrak{D}(A)$ there exists a linear transformation D_w and a transformation E_w, both mapping $\mathfrak{D}(A)$ into $\mathcal{H}$ and satisfying

(a) $T(w + h) - T(w) = D_w(h) + E_w(h)$, $h \in \mathfrak{D}(A)$,

(b) there is a constant $\tilde{d}$ such that $\|D_w(h)\| \leqq \tilde{d} |||w|||^{k-1} |||h|||$,

(c) $\|E_w(h)\| \leqq \tilde{Q}(|||w|||, |||h|||) |||h|||$ where $\tilde{Q}: R^2 \to [0, \infty)$ satisfies

$\lim_{|x|\to 0} \tilde{Q}(x_1, x_2) = 0$ and $|||w||| = \|Lw\| + \|w\|$, $w \in \mathfrak{D}(A)$, denotes the L-norm on $\mathfrak{D}(A)$;

(RH) $R : \mathfrak{D}(A) \cap \mathcal{O} \to \mathcal{H}$ is a higher order operator in the sense that $\|R(w)\| = o(|||w|||^k)$ as $|||w||| \to 0$, and R satisfies also a local Lipschitz condition on $\mathfrak{D}(A)$ of the type described in (NC).

Let us remark in connection with these last hypotheses that (RH), in particular, is assumed throughout the remainder of this section. Also, the nonlinear term in the Hartree equation discussed in §2 provides a nontrivial example of an operator satisfying (TH) and (TD).

Under these assumptions, the equation $(1)_{sa}$ of §2 becomes

$$v + K[-\eta v + PT(u + v) + PR(u + v)] = 0, \tag{$(1)_{sa}$}$$

where P is the projection operator of $\mathcal{H}$ onto $\mathfrak{N}(A - \lambda_0 I)^{\perp}$, and the branching equation $(2)_{sa}$ is equivalent to the system

$$-\eta\xi_i + \left(T\left(\sum_{j=1}^{n} \xi_j u_j\right), u_i\right) + r^i(\xi, \eta) = 0 \qquad (i = 1, \cdots, n), \tag{4.14}$$

where

$$r^i(\xi,\eta) = \left(T\left(\sum_{j=1}^{n} \xi_j u_j + v\right) - T\left(\sum_{j=1}^{n} \xi_j u_j\right) + R\left(\sum_{j=1}^{n} \xi_j u_j + v\right), u_i \right) \qquad (i = 1, \cdots, n) \tag{4.15}$$

and $v = v(\xi, \eta)$ is the unique solution of $(1)_{sa}$ as determined in Theorem 2.1. The following lemma shows in what sense $v(\xi, \eta)$ is a higher order term.

LEMMA 4.1. *There exist constants b, d and C such that, for $|\xi| = \|u\| < b$ and $|\eta| < d$, $|||v||| \leqq C|\xi|^k$.*

The proof is immediate. If v satisfies $(1)_{sa}$ then $v \in \mathfrak{D}(A)$ and

$$|||v||| \leqq (1 + \|K\|)[\,|\eta|\,\|v\| + \tilde{Q}(|||v|||, \|u\|)\,|||v||| + \tilde{d}\|u\|^{k-1}|||v||| + \|T(u)\| + \|R(u + v)\|]. \tag{4.16}$$

Hence, if one now chooses b and d sufficiently small then

$$|||v||| \leqq 2(1 + \|K\|)[\,\|T(u)\| + \|R(u + v)\|]. \tag{4.17}$$

Since T and R satisfy (TH) and (RH), the inequality $|||v||| \leqq \|u\|$ and (4.17) imply the desired bound for $|||v|||$.

REMARK 4.1. As a consequence of Lemma 4.1 and the above hypotheses on T and R, one easily sees that the functions $r^i(\xi, \eta)$ in (4.15) are continuous near $|\xi| = \eta = 0$, and are also higher order in the sense that if $|\eta| < d$ then $r^i(\xi, \eta)/|\xi|^k \to 0$ as $|\xi| \to 0$.

The form of the system (4.14) and the homogeneity of T suggest for $\eta > 0$ the substitution

$$(4.18) \qquad \xi_i = \eta^\alpha \beta_i, \qquad \alpha = 1/(k-1) \qquad (i = 1, \cdots, n)$$

under which one obtains the system

$$(4.19) \qquad \begin{aligned} F^i(\beta_i, \eta) = & -\beta_i + \left(T\left(\sum_{j=1}^n \beta_j u_j \right), u_i \right) \\ & + \eta^{-k\alpha} r^i(\eta^\alpha \beta, \eta) = 0 \qquad (i = 1, \cdots, n). \end{aligned}$$

This last system is, of course, the analog of (4.4). For negative η, one sets $\xi = |\eta|^\alpha \beta$ and proceeds in an analogous way.

In order to simplify the statement of some of our results, it is convenient to recall that $Q = I - P$ is the projection of $\mathcal{H}$ onto $\mathfrak{N}(A - \lambda_0 I)$ and introduce the following additional hypothesis on T:

(Tnd) QT is nondegenerate on $\mathfrak{N}(A - \lambda_0 I)$; i.e., if $u \in \mathfrak{N}(A - \lambda_0 I)$ and $u \neq 0$ then $QT(u) \neq 0$.

The following result (e.g., see [58, p. 51]) is a supplement to Theorem 4.2 and is useful in determining the exact number of real solutions of (4.14).

THEOREM 4.3. (a) *Every continuous solution* $\beta = \beta(\eta)$ *of* (4.19) *yields a continuous solution of* (4.14) *by means of* (4.18). (b) *Suppose that T satisfies* (TH), (TD), *and* (Tnd), *and suppose that the system*

$$(4.20) \quad f^i(\beta) = -\beta_i + \left(T\left(\sum_{j=1}^n \beta_j u_j \right), u_i \right) = 0 \qquad (i = 1, \cdots, n)$$

has only a finite number of solutions. Then every solution $\xi = \xi(\eta)$ *of* (4.14) *continuous near* $\eta = 0$ $(\eta \geqq 0)$ *with* $\xi(0) = 0$ *is obtained by means of* (4.18) *from a solution* $\beta = \beta(\eta)$ *of* (4.19) *continuous near* $\eta = 0$ $(\eta \geqq 0)$ *with* $\beta(0) \neq 0$.

PROOF. Our proof of part (b) closely parallels Graves' proof [27] of Theorem 4.2 except for the important first step of showing that, under our assumptions on T, if $\xi = \xi(\eta)$ is a nontrivial continuous solution of (4.14) with $\xi(0) = 0$, and if $\beta = \beta(\eta)$ is defined by (4.18), then $\beta(\eta)$ is bounded as $\eta \to 0^+$. Let us suppose for the sake of contradiction that $\beta(\eta)$ is unbounded as $\eta \to 0^+$. Then there is a sequence $\{\eta_l\}$ such that $\eta_l \to 0^+$ and $\eta_l/|\xi(\eta_l)|^{k-1} \to 0$. If one now sets $\xi(\eta_l) = |\xi(\eta_l)|\,\theta^l$, where $\theta^l = \{\theta_1^l, \cdots, \theta_n^l\}$ is a unit vector in $\boldsymbol{R}^n$, then there is a subsequence $\{\theta^l\}$ such that $\theta^l \to \theta$ in $\boldsymbol{R}^n$, $|\theta| = 1$, and the corresponding subsequences $\{\eta_l\}$ and $\{\xi(\eta_l)\}$ satisfy

$$\eta\xi_i = \left(T\left(\sum_{j=1}^{n} \xi_j u_j \right), u_i \right) + r^i(\xi, \eta) \qquad (i = 1, \cdots, n). \tag{4.21}$$

The homogeneity of T now implies

$$\frac{\eta_l}{|\xi(\eta_l)|^{k-1}}\,\theta_i^{\,l} = \left(T\left(\sum_{j=1}^{n} \theta_j^{\,l} u_j \right), u_i \right) + \frac{r^i}{|\xi(\eta_l)|^k} \qquad (i = 1, \cdots, n). \tag{4.22}$$

Therefore, letting $l \to \infty$, one finds by the use of Remark 4.1 that

$$0 = \left(T\left(\sum_{j=1}^{n} \theta_j u_j \right), u_i \right) \qquad (i = 1, \cdots, n). \tag{4.23}$$

The last equation and assumption (Tnd) imply $\sum_{j=1}^{n} \theta_j u_j = 0$ which is a contradiction.

As in the proof of Graves [27] one now sees that the compact set S of limiting values of $\beta(\eta)$ as $\eta \to 0^+$ is connected and, by (4.19) and Remark 4.1, the points β in S satisfy (4.20). Therefore, S consists of a single point β^0 and if we set $\beta(0) = \beta^0$ then $\beta(\eta)$ is continuous at 0^+. Finally, $\beta(0) = \beta^0$ cannot be zero because (4.22) and $\lim_{\eta \to 0^+} \eta/|\xi|^{k-1} = \infty$ would imply that $(T(w), w)$ is unbounded on the unit sphere in $\mathfrak{N}(A - \lambda_0 I)$.

REMARK 4.2. A similar result holds for $\eta \leqq 0$ with (4.20) replaced by

$$g^i(\beta) = \beta_i + \left(T\left(\sum_{j=1}^{n} \beta_j u_j \right), u_i \right) = 0 \qquad (i = 1, \cdots, n). \tag{4.24}$$

As a consequence of Theorem 4.3 we have the following

COROLLARY. *Suppose that the hypotheses of part* (b) *in Theorem* 4.3 *are satisfied. Then* (4.14) *has a nontrivial solution* $\xi = \xi(\eta)$ *continuous near* $\eta = 0$ $(\eta \geqq 0)$ *with* $\xi(0) = 0$ *only if the operator* QT *has a nontrivial isolated fixed point.*

In fact, if $\xi = \xi(\eta)$ is a continuous nontrivial solution of (4.14) satisfying $\xi(0) = 0$, then it is necessarily of the form $\xi = \eta^\alpha\beta$, where $\beta = \beta(\eta)$

is a solution of (4.19) which is continuous near $\eta = 0$ $(\eta \geqq 0)$ with $\beta(0) = (\beta_1{}^0, \cdots, \beta_n{}^0) \neq 0$. Since (4.19) is equivalent to

$$\left(T\left(\sum_{j=1}^{n} \beta_j u_j\right) - \sum_{j=1}^{n} \beta_j u_j, u_i \right) + \eta^{-k\alpha} r^i(\eta^\alpha \beta(\eta), \eta) = 0 \tag{4.25}$$
$$(i = 1, \cdots, n),$$

letting $\eta \to 0^+$ and using Remark 4.1 and (4.20) one easily sees that $QT(U) = U$ where U is isolated and $U = \sum_{j=1}^{n} \beta_j{}^0 u_j \neq 0$.

In view of the above corollary, a natural question which motivates our approach throughout the remainder of this section is the following: When is this "necessary" condition on QT also a "sufficient" one for the existence of a nontrivial real solution of (4.14) near $\eta = 0$ $(\eta \geqq 0)$?

Since (4.20) can be rewritten as

$$(4.20)' \quad f^i(\beta) = \left(QT\left(\sum_{j=1}^{n} \beta_j u_j\right) - \sum_{j=1}^{n} \beta_j u_j, u_i \right) = 0 \quad (i = 1, \cdots, n),$$

one of the more useful answers to the above question appears to be

THEOREM 4.4. *Suppose that T satisfies* (TH) *and* (TD), *and suppose that β^0 is a nontrivial isolated zero of f in* (4.20) [*resp., g in* (4.24)] *such that the* (*topological*) *index of f at β^0* [*resp., g at β^0*] *does not vanish. Then there exists a positive number δ such that for each η satisfying $0 < \eta < \delta$* [*resp., $-\delta < \eta < 0$*] *the system* (4.14) *has at least one nontrivial real solution $\xi(\eta)$ satisfying $|\xi(\eta)| \to 0$ as $\eta \to 0^+$* [*resp., $\eta \to 0^-$*].

The proof is similar to that of Theorem 3.4. Namely, since for each fixed η satisfying $0 < \eta < \delta$ (δ sufficiently small) the continuous vector field $F = \{F^1, \cdots, F^n\}$ defined by (4.19) is homotopic to f, the basic existence theorem of topological degree theory implies that there is at least one point $\beta(\eta)$ near β^0 such that $\beta(\eta)$ is a solution of (4.19) and $\xi(\eta) = \eta^\alpha \beta(\eta) = O(\eta^\alpha)$ is a solution of (4.14).

REMARK 4.3. If the vector field f in (4.20) has a zero at β^0 and the Jacobian $|\partial f/\partial \beta|$ does not vanish at β^0, then it is well known that β^0 is an isolated zero of f and the index of f at β^0 does not vanish. On the other hand, since the index of f at β^0 may be defined and nonvanishing even though $|\partial f/\partial \beta|$ vanishes at β^0, the last theorem provides a useful extension of the implicit function theorem in $\boldsymbol{R}^n$.

The following result on calculating the topological index of certain vector fields in $\boldsymbol{R}^n$ is implicitly contained in J. Cronin [**21**, pp. 42–50].

LEMMA 4.2. *Suppose that a vector field* $\Psi = \{\psi^1, \cdots, \psi^n\}$ *mapping a neighborhood of* $\xi = 0$ *into* $\mathbf{R}^n$ *is of the form* $\psi^i(\xi) = P^i_{k_i}(\xi) + Q^i(\xi)$, *where* $P^i_{k_i}$ *is a homogeneous polynomial of degree* k_i *and* Q^i *is a continuous function such that* $Q^i/|\xi|^{k_i} \to 0$ *as* $|\xi| \to 0$, *and suppose that the vector field* $P = \{P^1_{k_1}, \cdots, P^n_{k_n}\}$ *vanishes only at* $\xi = 0$. *Then* Ψ *has an isolated zero at* $\xi = 0$ *and the index of* Ψ *at* 0 *satisfies* $i(\Psi, 0) = \prod_{i=1}^n k_i \pmod 2$.

As an application of Theorem 4.4 to finding solutions of equation (†) we have the following branching theorem which is a generalization of a result of V. M. Krasnosel'skiĭ [41]; the result in [41] imposes a certain "simple root" condition and also complete continuity assumptions on L, T and R.

THEOREM 4.5. *Suppose that* $\mathfrak{N}(A - \lambda_0 I)$ *is two-dimensional and that* T *satisfies* (TH), (TD) *and* (Tnd) *with* k *even in* (TH). *Then there exists a positive number* δ *such that for* $0 < |\lambda - \lambda_0| < \delta$ *the equation* (†) *has at least one nontrivial solution* $w = w(\lambda)$ *satisfying* $\|w(\lambda)\| \to 0$ *as* $\lambda \to \lambda_0$.

REMARK. If k is odd there may be no nontrivial solutions of an equation of the form (†) as in the example [**40**, p. 193] where $\mathcal{H} = \mathbf{R}^2$, $A = I$, $\lambda_0 = 1$, $R = 0$ and

$$T(x_1, x_2) = \{-x_2(x_1^2 + x_2^2), x_1(x_1^2 + x_2^2)\}.$$

PROOF. It will be sufficient to consider only the case $\eta = \lambda - \lambda_0 \geqq 0$ as the proof in the case $\eta \leqq 0$ is similar.

In order to obtain a nontrivial fixed point of the operator QT, it is convenient to first construct a special orthonormal basis $\{v_1, v_2\}$ for $\mathfrak{N}(A - \lambda_0 I)$ which satisfies also $(T(v_1), v_2) = 0$. Let $\{u_1, u_2\}$ be any orthonormal basis for $\mathfrak{N}(A - \lambda_0 I)$ and set

$$g(t) = (T(u_1 + tu_2), tu_1 - u_2). \tag{4.26}$$

Since (TH) implies that g is a polynomial of odd degree $(k + 1)$, there is a real zero t_0 of g such that

$$g(t) = (t - t_0)^l G(t), \tag{4.27}$$

where l is odd and $G(t_0) \neq 0$. If we now set

$$\begin{aligned} v_1 &= (u_1 + t_0 u_2)/(1 + t_0^2)^{1/2}, \\ v_2 &= (t_0 u_1 - u_2)/(1 + t_0^2)^{1/2}, \end{aligned} \tag{4.28}$$

then $\{v_1, v_2\}$ is also an orthonormal basis for $\mathfrak{N}(A - \lambda_0 I)$ and $(T(v_1), v_2) = (1 + t_0^2)^{-(k+1)/2} g(t_0) = 0$.

In terms of the special orthonormal basis $\{v_1, v_2\}$ the basic system of equations (4.20) becomes

$$f(\beta) = -\beta_i + \left(T \left(\sum_{j=1}^{2} \beta_j v_j \right), v_i \right) \qquad (i = 1, 2), \tag{4.29}$$

and similar changes occur in (4.14) and (4.19). If we now set $\beta_2 = 0$ in (4.29), it follows that $f^2(\beta_1, 0) = \beta_1{}^k(T(v_1), v_2)$ vanishes for all β_1 so that the system (4.29) reduces on $\beta_2 = 0$ to

$$f^1(\beta_1, 0) = -\beta_1 + \beta_1{}^k(T(v_1), v_1). \tag{4.30}$$

Since (Tnd) and $(T(v_1), v_2) = 0$ imply $(T(v_1), v_1) \neq 0$, we see that $\beta^0 = (\rho, 0)$ is a nontrivial solution of (4.29) where $\rho = (T(v_1), v_1)^{-\alpha}$ and $\alpha = 1/(k - 1)$. Thus, in order to complete the proof by means of Theorem 4.4, it will be sufficient to show that the index $i(f, \beta^0)$ of f at β^0 is defined and $i(f, \beta^0) \neq 0$.

Let us first transform f into a vector field $\Phi = \{\varphi^1, \varphi^2\}$ under the change of variables $\beta_1 = x_1 + \rho$ and $\beta_2 = x_2$. Then (TH) and some elementary calculations yield

$$\varphi^1(x) = f^1(x_1 + \rho, x_2) = (k - 1)x_1 + ax_2 + Q^1(x), \tag{4.31}$$

where a is a constant and Q^1 is a continuous function such that $Q^1/|x| \to 0$ as $|x| \to 0$. Similarly, (TH) and $(T(v_1), v_2) = 0$ imply

$$\varphi^2(x) = -x_2 + \sum_{m=1}^{k} b_m x_2{}^m (x_1 + \rho)^{k-m}, \tag{4.32}$$

where the b_m are constants. Clearly, if the index $i(\Phi, 0)$ at $x = 0$ is defined and $i(\Phi, 0) \neq 0$, then $i(f, \beta^0)$ is defined and $i(f, \beta^0) \neq 0$. However, since the vector field Φ may have a degenerate linear part at $x = 0$, $i(\Phi, 0)$ may also be somewhat difficult to calculate.

Let us therefore introduce another vector field $\Psi = \{\psi^1, \psi^2\}$ defined on $|x| < \rho$ where $\psi^1 = \varphi^1$ and

$$\psi^2(x) = \varphi^2(x) - x_2\varphi^1(x)/(x_1 + \rho). \tag{4.33}$$

We note first of all that Φ has an isolated zero at $x = 0$ if and only if Ψ does. Moreover, since $\Phi = -c\Psi$ $(c > 0)$ implies $\varphi^1 = 0$, which together with (4.33) implies $\psi^2 = \varphi^2$ so that $\varphi^2 = 0$ also, it follows that if Φ has an isolated zero at $x = 0$ then Φ and Ψ are nowhere opposing near $x = 0$. Therefore, by the Poincaré-Bohl theorem (e.g., see [21,

p. 32]), the indices of Φ and Ψ at $x = 0$ are equal whenever they are defined.

In order to calculate $i(\Psi, 0)$ we observe that (4.27) and some elementary calculations yield [**60**, Lemma 2]

$$(T(v_1 + \tau v_2), \tau v_1 - v_2) = \tau^l H(\tau) \tag{4.34}$$

where l is odd and H is a polynomial of degree $(k + 1 - l)$ with $H(0) \neq 0$. Therefore

$$\begin{aligned} f^2 - \frac{\beta_2}{\beta_1} f^1 &= \beta_1{}^k \left(T\left(v_1 + \frac{\beta_2}{\beta_1} v_2\right), - \frac{\beta_2}{\beta_1} v_1 + v_2 \right) \\ &= - \beta_1{}^{k-l} \beta_2{}^l H\left(\frac{\beta_2}{\beta_1}\right), \end{aligned} \tag{4.35}$$

so that ψ^2 is, in fact, of the form

$$\psi^2 = bx_2{}^l + Q^2(x) \tag{4.36}$$

where l is odd, $b = -\rho^{k-l} H(0) \neq 0$, and Q^2 is a continuous function such that $Q^2/|x|^l \to 0$ as $|x| \to 0$. Since the vector field $\{(k-1)x_1 + ax_2, bx_2{}^l\}$ vanishes only at $x = 0$, it now follows from (4.31), (4.36) and Lemma 4.1 that Ψ has an isolated zero at $x = 0$ and $i(\Psi, 0) = l \pmod 2$ $\neq 0$. Thus $i(f, \beta^0) \neq 0$ which completes the proof.

Since the above reasoning may be applied to each real zero of odd order of the polynomial g defined in (4.26), we have, for example, the following

COROLLARY. *Suppose in addition to the hypotheses in Theorem* 4.5 *that the polynomial* g *in* (4.26) *has* M (*distinct*) *real zeros of odd order. Then there exists a positive number* δ *such that for* $0 < |\lambda - \lambda_0| < \delta$ *the equation* (†) *has at least* M *nontrivial solutions which tend to* $w = 0$ *as* λ *tends to* λ_0.

In the case where k is *odd* in (TH), one may again formulate a branching theorem in terms of the number of real zeros of odd order of the polynomial g in (4.26); however, as shown by the example in the remark following Theorem 4.5 where $g(t) = -(t^2 + 1)^2$, the polynomial g in this case does not necessarily have any real zeros.

Finally, let us remark that a result such as Theorem 4.5 is of some interest because even in the case of compact operators A and T it is not a direct consequence of other standard theorems seeing as the dimension of the null space $\mathfrak{N}(A - \lambda_0 I)$ is not odd and T is not assumed to be a gradient operator.

C. *Gradient operators.* We consider in this section an equation of the form

$$(\dagger) \qquad (A - \lambda_0 I)w + T(w) + R(w) = \eta w,$$

where $\eta = \lambda - \lambda_0$, $L = A - \lambda_0 I$ satisfies (LSA), and, in addition to (TH), (TD) and (RH), T satisfies the following hypothesis:

(TG) T is the (strong) gradient on $\mathfrak{D}(A)$ of the functional $(1/(k+1))(T(w), w)$; i.e., there exists a functional r such that, for all $w, h \in \mathfrak{D}(A)$,

$$(T(w+h), w+h) - (T(w), w) = (k+1)(T(w), h) + r(w, h)$$

where $r(w, h)/|||h||| \to 0$ as $|||h||| \to 0$.

By the usual "Euler Identity" for homogeneous polynomials (e.g., see [**56**, p. 272]), hypothesis (TG) is equivalent to assuming that T is the (strong) gradient of some functional τ which is homogeneous of degree $(k+1)$.

Although we assume throughout this section that T is the (strong) gradient of some functional τ, we do not assume that τ is weakly continuous (or even everywhere defined) so that those variational methods based on Ljusternik-Schnirelmann category theory (e.g., see [**8**]), the genus of a set (see [**40**]), and Morse theory in Hilbert spaces (e.g., see [**48**], [**57**]) would not necessarily apply to equation (†).

Instead of the more restrictive nondegeneracy assumption (Tnd) it is convenient to assume that the set $C_S = \{u \in S : (T(u), u) > 0\}$ is nonempty where here and in the sequel S denotes the unit sphere in $\mathfrak{N} \equiv \mathfrak{N}(A - \lambda_0 I)$. Let us remark that if $(T(u), u) \not\equiv 0$ on S and k is even in (TH), then C_S is always nonempty so that Theorem 4.6 below holds whenever $(T(u), u) \not\equiv 0$ on S. On the other hand, if $(T(u), u) \leqq 0$, $u \in$ S, and k is odd, then Theorem 4.7 below continues to hold but with the interval $\lambda_0 < \lambda < \lambda_0 + \delta$ replaced by $\lambda_0 - \delta < \lambda < \lambda_0$.

The following lemma [**58**, Lemma 3] will be useful in determining nontrivial fixed points of the operator QT (i.e., nontrivial zeros of f in (4.20)).

LEMMA 4.3. *Suppose that $\mathfrak{N}$ is n-dimensional, that T satisfies* (TH), (TD) *and* (TG), *and that $(T(v_1), v_1) = a$ is a positive relative maximum of $(T(u), u)|_S$ (the restriction of the functional $(T(u), u)$ to* S). *Then there exists an orthonormal basis $\{v_1, \cdots, v_n\}$ for $\mathfrak{N}$ which satisfies*

$$(4.37) \qquad (T(v_1), v_j) = 0 \qquad (j = 2, 3, \cdots, n),$$

$$(4.38) \qquad (D_{v_1}(v_p), v_q) = 0 \qquad (p \neq q),$$

$$(4.39)\qquad \begin{aligned}(D_{v_1}(v_n), v_n) &\leqq (D_{v_1}(v_{n-1}), v_{n-1}) \leqq \cdots \\ &\leqq (D_{v_1}(v_2), v_2) \leqq a.\end{aligned}$$

Let us note in connection with Lemma 4.3 that since $\mathfrak{N}$ is finite-dimensional and the set C_S is nonempty, $(T(u), u)|_S$ has at least one positive relative maximum so that such an element $v_1 \in S$ always exists. A similar lemma holds if $(T(v_1), v_1)$ is a positive relative minimum of $(T(u), u)|_S$ except that the inequalities in (4.39) are reversed.

REMARK 4.4. In terms of the special basis $\{v_1, \cdots, v_n\}$ the basic equations (4.20) become

$$(4.40)\qquad f^i(\beta) = -\beta_i + \left(T\left(\sum_{j=1}^{n} \beta_j v_j \right), v_i \right) \qquad (i = 1, \cdots, n).$$

Therefore, since (4.37) implies that $f^j(\beta) \equiv 0$ $(j = 2, \cdots, n)$ on $\beta_2 = \beta_3 = \cdots = \beta_n = 0$, the system (4.40) reduces to the single equation

$$(4.41)\qquad f^1(\beta_1, 0, \cdots, 0) = -\beta_1 + \beta_1{}^k(T(v_1), v_1)$$

and, hence, the system (4.40) always has at least one nontrivial solution $(a^{-\alpha}, 0, \cdots, 0)$ [if k is odd there are at least two nontrivial solutions $(\pm a^{-\alpha}, 0, \cdots, 0)$].

REMARK 4.5. Let $\beta^0 = (a^{-\alpha}, 0, \cdots, 0)$ be a nontrivial zero of f as determined in Remark 4.4. Then Lemma 4.3 and some calculations using the differential D_w of $T(w)$ imply [58, p. 54]

$$(4.42)\qquad \left| \frac{\partial f}{\partial \beta} \right|_{\beta = \beta^0} = \prod_{j=2}^{n} [(D_{v_1}(v_j), v_j) - a].$$

Thus, as a consequence of (4.39), f has a degenerate linear part at $\beta = \beta^0$ if and only if there is an integer $l, 2 \leqq l \leqq n$, such that $(D_{v_1}(v_s), v_s) = a$ $(s = 2, 3, \cdots, l)$. The integer l is a convenient measure of the degeneracy of f at $\beta = \beta^0$.

In view of Lemma 4.3 and Remark 4.4, it is clear that each positive relative extrema of $(T(u), u)|_S$ generates a nontrivial fixed point of the operator QT. Thus, it is natural to ask the question: Does the number of nontrivial fixed points of QT that correspond to positive relative extrema of $(T(u), u)|_S$ give a lower bound for the number of nontrivial solutions of equation (†) near $\lambda = \lambda_0$? Some partial answers to this question are obtained in the present section. For example, as indicated by the following two theorems, the answer is "essentially yes" when the null space $\mathfrak{N}$ is two-dimensional [59, §3].

THEOREM 4.6. *Suppose that* $\mathfrak{N}(A - \lambda_0 I)$ *is two-dimensional, that* T *satisfies* (TH), (TD), *and* (TG), *and that* k *is even in* (TH). *Suppose that there are* M *points at which* $(T(u), u)|_S$ *has a positive relative extremum. Then there exists a positive number* δ *such that for* $0 < |\lambda - \lambda_0| < \delta$ *the equation* (†) *has at least* M *nontrivial solutions* $w_i = w_i(\lambda)$ *satisfying* $\lim_{\lambda \to \lambda_0} \|w_i(\lambda)\| = 0$ $(i = 1, \cdots, M)$.

PROOF. Let us indicate the main points of the proof in the case when $k = 2$ to illustrate, in particular, the role played by the positive relative extrema values of $(T(u), u)|_S$; a proof of the general case may be found in [59]. It will be sufficient to consider the case when $\eta = \lambda - \lambda_0 \geqq 0$ and v_1 corresponds to a positive relative maximum of $(T(u), u)|_S$, as the proof in the other cases is similar. In particular then we may introduce the special basis $\{v_1, v_2\}$ of Lemma 4.3 and determine as in Remark 4.4 a nontrivial solution $\beta^0 = (a^{-1}, 0)$ of the basic system (4.40). Thus, in order to complete the proof by means of Theorem 4.4, it will be sufficient to show that the index $i(f, \beta^0)$ is defined and does not vanish.

It is convenient to first transform f into a vector field $\Phi = \{\varphi^1, \varphi^2\}$ under the change of variables $\beta_1 = x_1 + a^{-1}$ and $\beta_2 = x_2$. Then (TH), (TG) and some simple calculations yield

$$\begin{aligned}\varphi^1(x) &= f^1(x_1 + a^{-1}, x_2)\\ &= x_1 + a{x_1}^2 + \tfrac{1}{2}(D_{v_1}(v_2), v_2){x_2}^2,\end{aligned} \tag{4.43}$$

$$\begin{aligned}\varphi^2(x) &= f^2(x_1 + a^{-1}, x_2)\\ &= [a^{-1}(D_{v_1}(v_2), v_2) - 1]\,x_2\\ &\quad + (D_{v_1}(v_2), v_2)x_1x_2 + b{x_2}^2,\end{aligned} \tag{4.44}$$

where $b = (T(v_2), v_2)$. As a consequence of Remarks 4.3 and 4.5, it is clear that if $(D_{v_1}(v_2), v_2) < a$ then $x = 0$ is an isolated zero of Φ and $i(\Phi, 0) \neq 0$. Thus, we need consider only the case where $(D_{v_1}(v_2), v_2) = a$ so that

$$\varphi^1(x) = x_1 + a{x_1}^2 + \tfrac{1}{2}a{x_2}^2, \tag{4.45}$$

$$\varphi^2(x) = ax_1x_2 + b{x_2}^2, \tag{4.46}$$

and Φ has a degenerate linear part at $x = 0$.

Let us introduce another vector field $\Psi = \{\psi^1, \psi^2\}$ where $\psi^1 = \varphi^1$ and

$$\psi^2(x) = \varphi^2(x) - ax_2\varphi^1(x) = b{x_2}^2 - \tfrac{1}{2}a^2{x_2}^3 - a^2{x_1}^2x_2. \tag{4.47}$$

Since Φ has an isolated zero at $x = 0$ if and only if Ψ does, and, by an argument similar to that in the paragraph following (4.33), Φ and Ψ are nowhere opposing near $x = 0$, it follows from the Poincaré-Bohl theorem that the indices of Φ and Ψ are equal whenever they are defined. However, by (4.45), (4.47) and Lemma 4.2, if $b = 0$ then the index $i(\Psi, 0)$ is defined and $i(\Psi, 0) \neq 0$, whereas if $b \neq 0$ then $i(\Psi, 0)$ may be zero. Thus, in order to complete the proof, we now show that b necessarily vanishes.

By employing the given assumptions on T, and the conditions $(T(v_1), v_2) = 0$ and $(D_{v_1}(v_2), v_2) = a$, one first obtains the identity

$$(T(v_1 + \sigma v_2), v_1 + \sigma v_2) = a + \tfrac{3}{2}\, a\sigma^2 + b\sigma^3. \tag{4.48}$$

The inequality $(1+\sigma^2)^{3/2} < 1 + \frac{3}{2}\sigma^2 + \frac{3}{8}\sigma^4$ $(\sigma \neq 0)$ then implies

$$(T(v_1 + \sigma v_2), v_1 + \sigma v_2) > a(1 + \sigma^2)^{3/2} + \sigma^3(b - \tfrac{3}{8}\, a\sigma). \tag{4.49}$$

Hence, if $b \neq 0$ then for all σ sufficiently small and satisfying $b\sigma > 0$ we have $(T(v_1 + \sigma v_2),\ v_1 + \sigma v_2) > a(1 + \sigma^2)^{3/2}$. But, since $(v_1 + \sigma v_2)/(1 + \sigma^2)^{1/2}$ belongs to S and $a = (T(v_1), v_1)$ is a relative maximum for $(T(u), u)$ on S, this is a contradiction. Thus, $b = 0$ which completes the proof.

Remark 4.6. Suppose that k is even in (TH) and $(T(u), u)|_S$ attains a positive relative maximum at v_1. Then $(T(u), u)|_S$ assumes a negative relative minimum at $-v_1$ and $(T(-v_1), -v_1)^{-\alpha} = -(T(v_1), v_1)^{-\alpha}$, $\alpha = 1/(k-1)$. Thus, since $(T(v_1),\ v_1)^{-\alpha}v_1 = (T(-v_1), -v_1)^{-\alpha}(-v_1)$, the solutions constructed in Theorem 4.6 corresponding to v_1 and $-v_1$ may not be distinct so that we count there only the number of positive relative extrema points.

Theorem 4.7. *Suppose that* $\mathfrak{N} \equiv \mathfrak{N}(A - \lambda_0 I)$ *is two dimensional, that* T *satisfies* (TH), (TD) *and* (TG), *and that* k *is odd in* (TH). *Suppose that*

$$(T(u), u)|_S \not\equiv \textit{constant} \quad (\textit{i.e.}, (T(u), u) \not\equiv \textit{constant}\ \|u\|^{k+1}, u \in \mathfrak{N}) \tag{4.50}$$

and that there are M *points at which* $(T(u), u)|_S$ *has a positive relative extremum. Then there exists a positive number* δ *such that for* $\lambda_0 < \lambda < \lambda_0 + \delta$ *the equation* (†) *has at least* M *nontrivial solutions* $w_i = w_i(\lambda)$ *satisfying* $\lim_{\lambda \to \lambda_0^+} \|w_i(\lambda)\| = 0$ $(i = 1, \cdots, M)$.

A proof of Theorem 4.7 in the general case of odd k $(k \geqq 3)$ may be found in [**59**]; a somewhat different proof of the case $k = 3$ may be found in Knightly and Sather [**38**, p. 73].

REMARK 4.7. Suppose that k is odd and $(T(u), u)|_S$ attains a relative extremum at v_1. Then $(T(u), u)|_S$ attains a relative extremum at $-v_1$ also so that M in Theorem 4.7 is always an even integer. Moreover, if $(T(u),u) \geqq 0$ and $(T(u), u) = 0$ only if $u = 0$, then condition (4.50) implies that the (absolute) maximum and minimum are distinct positive extrema of $(T(u), u)|_S$ so that the integer M in Theorem 4.7 satisfies also $M \geqq 4$.

REMARK 4.8. Suppose that the higher order operator R in equation (†) is "analytic". Then, under the assumptions of either Theorem 4.6 or Theorem 4.7, one can also show that a solution $w = w(\lambda)$ of (†) corresponding to a relative extrema $(T(v_1), v_1)$ of $(T(u), u)|_S$ is continuous in λ for $\lambda_0 < \lambda < \lambda_0 + \delta$ (e.g., see Theorem 4.1). Therefore, as a consequence of (4.18), Lemma 4.1 and Remark 4.4, such a solution also satisfies $\lim_{\lambda \to \lambda_0^+} \eta^{-\alpha} w = (T(v_1), v_1)^{-\alpha} v_1$ so that, near $\lambda = \lambda_0$ $(\lambda > \lambda_0)$, w is the form $\eta^\alpha (T(v_1), v_1)^{-\alpha} v_1 + W(\eta)$ where $W = o(\eta^\alpha)$ as $\eta \to 0^+$.

Due to the more involved analysis required, the results for $n > 2$ are less definitive than those obtained above for $n = 2$. However, one can establish some special results for $k = 2$ and $k = 3$ (see [**59**]).

As an application of Theorem 4.6 let us consider once again a boundary value problem of the type

$$(4.51) \qquad \begin{aligned} -\Delta w - \mu_0 w + w^2 &= (\mu - \mu_0) w && \text{in } \Omega, \\ w &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\mu_0 = 5$ and $\Omega = (0, \pi) \times (0, \pi)$. By using the inequality (2.11) and some direct calculations, one can easily show that $T(w) = w^2$ satisfies the hypotheses of Theorem 4.6. Thus, it follows from Theorem 4.6 (the space $\mathfrak{N}(A - \mu_0 I)$ is 2-dimensional here) that the boundary value problem (4.51) has at least as many solutions near $\mu = \mu_0$ as the number of positive extrema of the functional $(T(u), u)$ on S; in particular then there is always at least one solution of the problem.

Some results which are closely related to Theorem 4.6 and Theorem 4.7 have been obtained recently by Kirchgässner [**36**, Theorem 4]; in addition, Kirchgässner [**36**] has obtained other results for $k \geqq 3$ and $n \geqq 2$ provided that a certain "simple degeneracy" condition holds, and has also announced a result for $k = 2$ and $n \geqq 2$. The approach used in [**36**] is a constructive one and is based upon certain fixed point theorems.

The von Kármán equations and buckling problems. The nonlinear von Kármán equations have been the subject of a number of mathematical investigations during the last thirty years. These investigations have included the study of buckling problems as in the papers of Friedrichs and Stoker [**25**], Keller, Keller and Reiss [34], Bauer and Reiss [5], Berger and Fife [9], [**10**], Berger [7], Wolkowisky [75], and Knightly and Sather [38], as well as the study of various existence problems as in the papers of Morosov [49], Fife [22], and Knightly [37]. The reformulation of such problems as a single nonlinear operator equation in some appropriate Hilbert space as in Berger and Fife [9], [**10**] and Berger [7] has proved to be especially useful, and is the approach adopted for the buckling problems considered below.

(a) *Buckling of a clamped plate.* As an application of the above results, let us study the nonlinear deflections of a thin elastic plate that is clamped at the edges and subjected to certain edge loadings; the material outlined below is contained in Knightly and Sather [38] wherein the effects of normal loading are also considered.

Let Ω be a bounded region in the xy-plane (representing the shape of the plate) with boundary $\partial\Omega$ consisting of a finite number of smooth arcs but no cusps. We consider the following version of the von Kármán equations (e.g., see [37]):

$$\text{(vKa)} \qquad \Delta^2 f = -\tfrac{1}{2}[w, w],$$

$$\text{(vKb)} \qquad \Delta^2 w = \lambda[F, w] + [f, w],$$

where Δ is the Laplacian with respect to x and y, and

$$\text{(4.52)} \qquad [u, v] = u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}.$$

Here $w = w(x, y)$ is a measure of the deflection of the plate out of its plane, λ is a parameter measuring the edge loadings acting on $\partial\Omega$, and $f = f(x, y)$ is a certain stress function in the plate.

We assume throughout the application that the given function $F = F(x, y)$ satisfies $F \in C^3(\Omega)$, and that F and the first and second partials of F are all uniformly bounded in Ω.

The determination in Ω of solutions $w, f \in C^4(\Omega) \cap C^1(\bar{\Omega})$ of the two coupled nonlinear partial differential equations (vK) together with the "clamped plate boundary conditions"

$$\text{(4.53)} \qquad w = w_x = w_y = f = f_x = f_y = 0 \quad \text{on } \partial\Omega$$

will constitute a classical solution of problem CP.

Let $W = \mathring{W}^{2,2}(\Omega)$ be the (real) Hilbert space obtained by the completion of $C_0{}^\infty(\Omega)$ in the norm induced by the inner product

$$(u, v) = \int_\Omega (u_{xx}v_{xx} + 2u_{xy}v_{xy} + u_{yy}v_{yy})\, dx\, dy. \tag{4.54}$$

In order to reformulate problem CP as an operator equation in W, we note first of all that for smooth functions φ and ψ in W an integration by parts yields

$$(f, \varphi) = -\tfrac{1}{2} b(w, w; \varphi), \tag{4.55a}$$

$$(w, \psi) = b(f, w; \psi) + \lambda c(w; \psi), \tag{4.55b}$$

where

$$b(u, v; \varphi) = \int_\Omega [(u_y v_{xy} - u_x v_{yy})\varphi_x + (u_x v_{xy} - u_y v_{xx})\varphi_y]\, dx\, dy, \tag{4.56}$$

$$c(u; \varphi) = \int_\Omega [F, u]\, \varphi\, dx\, dy. \tag{4.57}$$

The equations (4.55) suggest defining a generalized solution of problem CP to be a pair of functions w, f in W which satisfy (4.55) for all φ, ψ in W. It can then be shown (e.g., see [7], [9] that every classical solution is a generalized solution and, conversely, every generalized solution is a classical solution in Ω and at all sufficiently smooth portions of $\partial\Omega$. Thus, it will be sufficient for our purposes to determine a generalized solution of the problem CP.

Let us now indicate how the system of equations (4.55) can be reformulated as two uncoupled operator equations in W; the necessary details and the appropriate Sobolev inequalities may be found in [7], [9]. Since the functionals $b(u, v; \varphi)$ and $c(u; \varphi)$ are bounded and linear in φ, it follows from the Riesz representation theorem that there exists a bounded symmetric bilinear operator $B: W \times W \to W$, and a bounded linear operator $A: W \to W$ such that, for all φ in W,

$$b(u, v; \varphi) = (B(u, v), \varphi), \qquad u, v \in W, \tag{4.58}$$

$$c(u; \varphi) = (Au, \varphi), \qquad u \in W. \tag{4.59}$$

Using these equations one now easily sees that (4.55) can be rewritten as

$$f = -\tfrac{1}{2} B(w, w), \tag{4.60}$$

$$w - \lambda A w + T(w) = 0, \tag{4.61}$$

where the nonlinear operator $T: W \to W$ is given by

$$T(w) = \tfrac{1}{2} B(B(w, w), w). \tag{4.62}$$

Thus, in order to determine a generalized solution of problem CP, it will be sufficient to determine a solution in W of the single operator equation (4.61).

Let us remark here that one can even show that B (and hence T) is a compact operator on W; however, although the compactness of T plays an important role in obtaining the global branching results of Berger [7] and Berger and Fife [9], [10], it is not required for the local branching analysis given below.

The linearized eigenvalue problem associated with the generalized problem CP is

$$w - \lambda Aw = 0, \qquad w \in W. \tag{4.63}$$

Here, A is a selfadjoint compact linear operator so that the spectrum of A consists of an infinite number of discrete real eigenvalues λ_m of finite multiplicity with $|\lambda_m|$ tending to infinity as $m \to \infty$. Thus, for any such λ_m, the linear operator $I - \lambda_m A$ satisfies hypothesis (LSA) with $\mathfrak{D}(A) = W$.

Although the operator equation (4.61) is not of the same form as equation (†), the associated branching equation in $\mathfrak{N}(I - \lambda_m A)$ does have the same form as $(2)_{\text{sa}}$ with

$$\eta = \frac{\lambda}{\lambda_m} - 1, \tag{4.64}$$

so that the above analysis applies also to operator equations such as (4.61). Therefore, in order to obtain nontrivial solutions of (4.61) near $\lambda = \lambda_m > 0$, when $\mathfrak{N}(I - \lambda_m A)$ is two dimensional, we need to show only that the nonlinear operator T defined by (4.62) satisfies the hypotheses of Theorem 4.7. Clearly, since T is generated by a bounded, symmetric, bilinear operator B, T satisfies (TH) with $\mathfrak{D}(A) = W$ and $k = 3$. The operator T satisfies also (TD) with the A-norm replaced by the norm in W and

$$D_w(h) = 2B(w, B(w, h)) + B(h, B(w, w)), \tag{4.65}$$

$$E_w(h) = D_h(w) + T(h). \tag{4.66}$$

In order to show that T is the (strong) gradient of the functional $\frac{1}{4}(T(w), w)$, one first uses the "divergence structure" of the nonlinear term $[u, v]$ (e.g., see [7, p. 692]) to show that the form $(B(u, v), w) = \int_\Omega [u, v]\, w \, dx\, dy$ is symmetric for u, v, w in W, and one can then verify (TG) by direct calculation (see also [7, p. 696]). Since in addition one can show that $(T(w), w) = \|B(w, w)\|^2$ and that $(T(w), w) = 0$ only if $w = 0$ (see [7, p. 699]), it then follows from Theorem 4.7

and Remark 4.7 that there are at least four nontrivial generalized solutions of problem CP for $\lambda_m < \lambda < \lambda_m + \delta$ provided that $(T(u), u)|_S \not\equiv$ constant (i.e., $(T(u), u) \not\equiv$ constant $\|u\|^{k+1}$, $u \in \mathfrak{N}$). Thus, under certain assumptions, the methods of this section yield nontrivial generalized solutions of problem CP (i.e., buckled states of the clamped plate).

(b) *Buckling of a simply supported rectangular plate.* As a second application to buckling problems, let us study the nonlinear deflections of a thin elastic simply supported rectangular plate that is subjected to a constant compressive thrust applied normal to its two short edges; the formulation of the problem used is that of Bauer and Reiss [5].

Let $\Omega = \{(x, y) : 0 < x < 2, 0 < y < 1\}$ and let $\partial\Omega$ denote the boundary of Ω. We consider the following special case of the von Kármán equations (vK) (e.g., see [5, p. 607])

(vKa)′ $$\Delta^2 f = -\tfrac{1}{2}[w, w],$$

(vKb)′ $$\Delta^2 w = [w, f] - \lambda w_{xx},$$

subject to the appropriate boundary conditions for f (see [5, p. 607]), and the "simply supported boundary conditions" for w given by $w = \Delta w = 0$ on $\partial\Omega$.

The determination in Ω of solutions $w, f \in C^4(\Omega) \cap C^2(\bar{\Omega})$ of the coupled nonlinear equations (vK)′ together with the appropriate simply supported boundary conditions will constitute a classical solution of problem SSP.

Let $\hat{W}$ be the (real) Hilbert space obtained by the closure in the norm of the space $W^{2,2}(\Omega)$ of the set of smooth functions defined in $\bar{\Omega}$ and vanishing on $\partial\Omega$. Then, by using the approach in Berger and Fife [10], one can define a "generalized" solution (pair) of problem SSP which is a classical solution in Ω and on $\partial\Omega$ except at the corners. In addition, one can show [10, p. 231] that, in order to obtain a generalized solution of problem SSP, it is sufficient to determine a solution w in $\hat{W}$ of a single operator equation

(4.67) $$w - \lambda A w + C(w) = 0.$$

Here $A : \hat{W} \to \hat{W}$ is a selfadjoint compact linear operator, and $C : \hat{W} \to \hat{W}$ is a smooth nonlinear gradient operator such that C satisfies (TH) with $k = 3$, and $(C(w), w)\hat{} \geqq 0$ with equality only if $w = 0$, where $(u, v)\hat{}$ denotes an appropriate inner product in $\hat{W}$. The above indicated formulation of the generalized problem SSP is considerably more involved than that of the generalized problem CP, and for the details the reader is referred to [5], [**10**].

The linearized eigenvalue problem associated with the generalized problem SSP is

$$(4.68) \qquad w - \lambda A w = 0, \qquad w \in \hat{W}.$$

Since the linearized eigenvalue problem associated with the classical problem SSP is

$$(4.69) \qquad \begin{aligned} \Delta^2 w + \lambda w_{xx} &= 0 \quad \text{in } \Omega, \\ w = \Delta w &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which can be solved explicitly (e.g., see [**5**, p. 608]), one easily sees that A has eigenvalues $\lambda_{mn} = (\pi^2/4)(m + 4n^2/m)^2$ and corresponding eigenfunctions $u_{mn} = \sin(m\pi x/2)\sin n\pi y$ $(m, n = 1, 2, \cdots)$. In particular then $\lambda_{41} = 25\pi^2/4$ is a double eigenvalue so that the null space of $\mathfrak{N}(I - \lambda_{41}A)$ is two-dimensional.

It now follows from Theorem 4.7 and Remark 4.7 that for $\lambda_{41} < \lambda < \lambda_{41} + \delta$ there are at least four nontrivial generalized solutions of problem SSP provided that $(C(u), u)\hat{}\,|_S \not\equiv$ constant, where S is the unit circle in $\mathfrak{N}(I - \lambda_{41}A)$. Thus, for $\lambda_{41} < \lambda < \lambda_{41} + \delta$ the methods of this section yield at least four buckled states of the simply supported rectangular plate provided that $(C(u), u)\hat{}\,|_S \not\equiv$ constant.

REMARK 4.9. Let us take the potential energy of a buckled state of problem SSP to be (e.g., see [**5**, p. 608])

$$(4.70) \qquad E = \int_\Omega [(\Delta w)^2 - \lambda w_x^2 + (\Delta f)^2]\, dx\, dy.$$

In terms of a generalized solution of problem SSP and the operators A and C, the energy E becomes (e.g., see [**10**, pp. 230–232])

$$(4.71) \qquad E(w) = (w, w)\hat{} - \lambda(Aw, w)\hat{} + \tfrac{1}{2}(C(w), w)\hat{}$$

so that the potential energy of an unbuckled state is $E(0) = 0$ whereas the potential energy of a buckled state w_0 satisfies $E(w_0) = -\frac{1}{2}(C(w_0), w_0)\hat{} < 0$. Let v_1 in S be defined by

$$(4.72) \qquad \theta = (C(v_1), v_1)\hat{} = \min_{w \in S} (C(w), w)\hat{},$$

where S is the unit circle in $\mathfrak{N}(I - \lambda_{41}A)$, let V_1 in S be such that $\Theta = (C(V_1), V_1)\hat{}$ is any other relative extrema of $(C(w), w)\hat{}\,|_S$ satisfying $\Theta > \theta$, and let u and U be the corresponding buckled states as determined in Remark 4.8 which satisfy

$$\eta^{-1/2}u \to \theta^{-1/2}v_1 \quad \text{and} \quad \eta^{-1/2}U \to \Theta^{-1/2}V_1 \qquad \text{as } \lambda \to \lambda_{41}^{+}.$$

Then, since $\theta < \Theta$ implies

$$(C(\theta^{-1/2}v_1), \theta^{-1/2}v_1)^{\wedge} = 1/\theta > 1/\Theta = (C(\Theta^{-1/2}V_1), \Theta^{-1/2}V_1)^{\wedge}, \tag{4.73}$$

it follows that, for λ sufficiently close to λ_{41},

$$E(u) < E(U). \tag{4.74}$$

Thus, if one now assumes that the "Principle of Least Energy" holds (i.e., the plate selects a buckled state with minimum potential energy) then the results of this section may predict that the "preferred" buckled states of the plate are the states $\pm u$ which correspond to the absolute minimum of $(C(w), w)^{\wedge}$ over the unit circle S in the null space $\mathfrak{N}(I - \lambda_{41}A)$.

REMARK 4.10. By using some techniques which are related to, but somewhat simpler than, those in [**10**], it can even be shown that, for $\lambda_{41} < \lambda < \lambda_{41} + \delta$, there are *exactly* eight buckled states of problem SSP which depend analytically on $(\lambda - \lambda_{41})^{1/2}$. This last result, as well as some other constructive results on the existence of buckled states of cylindrical panels and spherical caps, will appear in some forthcoming joint work of the author with George H. Knightly.

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