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ESSENTIAL NORM OF GENERALIZED WEIGHTED COMPOSITION OPERATORS FROM H^∞ TO THE LOGARITHMIC BLOCH SPACE

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ABSTRACT. In this paper, we give some estimates of the essential norm for generalized weighted composition operators from H^∞ to the logarithmic Bloch space. Moreover, we give a new characterization for the boundedness, compactness and essential norm of the generalized weighted composition operator from H^∞ to the logarithmic Bloch space.

1. Introduction. Let X, Y be Banach spaces and $\|\cdot\|_{X\to Y}$ denote the operator norm. The essential norm of a bounded linear operator $T : X \to Y$ is its distance to the set of compact operators K mapping X into Y, that is,

$$\|T\|_{e,X\to Y} = \inf\{\|T-K\|_{X\to Y} : K \text{ is compact}\}.$$

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of analytic functions on $\mathbb{D}$. Let $H^\infty = H^\infty(\mathbb{D})$ denote the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. We say that an $f \in H(\mathbb{D})$ belongs to the logarithmic Bloch space, denoted by $\mathcal{LB}$, if

$$\|f\|_{\log} = \sup_{z\in\mathbb{D}} (1-|z|^2) \log \frac{e}{1-|z|^2} |f'(z)| < \infty.$$

It is easy to see that $\mathcal{LB}$ is a Banach space with the norm $\|f\|_{\mathcal{LB}} = |f(0)| + \|f\|_{\log}$. For some related spaces and operators on them, see, for example, [**16, 17, 19, 22**]. From [**30**], we see that $\mathcal{LB} \bigcap H^\infty$ is the

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space of multipliers of the Bloch space $\mathcal{B}$. Here,

$$\mathcal{B} = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|^2) < \infty\}.$$

A weight $v : \mathbb{D} \to \mathbb{R}_+$ is called *radial* if $v(z) = v(|z|)$ for all $z \in \mathbb{D}$. Assume that v is radial; we say that an $f \in H(\mathbb{D})$ belongs to the weighted type space, denoted by H_v^∞, if

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

H_v^∞ is a Banach space with the above norm. The associated weight $\widetilde{v}$ of v is defined by

$$\widetilde{v}(z) = \frac{1}{\sup\{|f(z)| : f \in H_v^\infty, \|f\|_v \le 1\}}, \quad z \in \mathbb{D}.$$

It is easy to check that $\widetilde{v}_\alpha(z) = v_\alpha(z)$ when $v = v_\alpha(z) = (1-|z|^2)^\alpha (0 < \alpha < \infty)$, see e.g., [**14**]. When $v = v_\alpha(z)$, we will denote H_v^∞ by $H_{v_\alpha}^\infty$. When $v = (1-|z|^2)\log e/1-|z|^2$, we will denote H_v^∞ and $\|f\|_v$ by $H_{v_{\log}}^\infty$ and $\|f\|_{\log}$, respectively.

Let φ be an analytic self-map of $\mathbb{D}$ and $u \in H(\mathbb{D})$. The *weighted composition operator*, denoted by uC_φ, is defined by

$$(uC_\varphi f)(z) = u(z)\cdot f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

When $u = 1$, we get the composition operator, denoted by C_φ. See [**2**] for the theory of composition operators.

Let $\mathbb{N}_0$ denote the set of all nonnegative integers. Let $n \in \mathbb{N}_0$. The generalized weighted composition operator, denoted by $D_{\varphi,u}^n$, is defined as follows.

$$(D_{\varphi,u}^n f)(z) = u(z)\cdot f^{(n)}(\varphi(z)), \quad f \in H(\mathbb{D}),\ z \in \mathbb{D}.$$

This operator was introduced by Zhu [**31**]. When $n = 0$, $D_{\varphi,u}^n = uC_\varphi$. When $n = 1$ and $u(z) = \varphi'(z)$, then $D_{\varphi,u}^n = DC_\varphi$, which was studied in [**3, 7, 8, 9, 10, 18, 20**]. When $u(z) = 1$, then $D_{\varphi,u}^n = C_\varphi D^n$, which was studied in [**26**]. See, for example, [**6, 21, 23, 24, 28, 31, 32, 33, 34**] for the study of generalized weighted composition operators on various function spaces.

The compactness of $C_\varphi : \mathcal{B} \to \mathcal{B}$ was studied by many authors, see [**12, 25, 27, 29**]. In [**27**], Wulan, Zheng and Zhu proved that $C_\varphi : \mathcal{B} \to \mathcal{B}$

is compact if and only if

$$\lim_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = \lim_{j\to\infty} \|C_\varphi I^j\|_{\mathcal{B}} = 0,$$

where $I^j(z) = z^j$. In **[29]**, Zhao extended the result in **[27]** to the case of Bloch type spaces and showed that

$$\|C_\varphi\|_{e,\mathcal{B}\to\mathcal{B}} = \frac{e}{2} \limsup_{j\to\infty} \|\varphi^j\|_{\mathcal{B}} = \frac{e}{2} \limsup_{j\to\infty} \|C_\varphi I^j\|_{\mathcal{B}}.$$

In **[26]**, Wu and Wulan proved that $C_\varphi D^n : \mathcal{B} \to \mathcal{B}$ is compact if and only if

$$\lim_{j\to\infty} \|C_\varphi D^n I^j\|_{\mathcal{B}} = 0.$$

The boundedness, compactness and essential norm of composition operator and its generalizations on $\mathcal{B}$ were studied, for example, in **[5, 11, 12, 13, 15, 25, 26, 27, 29, 33, 35]**.

In **[1]**, the authors studied the boundedness and compactness of the operator $uC_\varphi : H^\infty \to \mathcal{LB}$. In **[34]**, Zhu studied the operator $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. Among others, she obtained the following result.

Theorem A. *Let $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded. Then, the following statements are equivalent.*

(a) $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ *is compact.*

(b) $\lim_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} = 0$.

(c) $\lim_{|\varphi(a)|\to 1} \|D^n_{\varphi,u} f_{\varphi(a)}\|_{\mathcal{LB}} = 0$ *and* $\lim_{|\varphi(a)|\to 1} \|D^n_{\varphi,u} h_{\varphi(a)}\|_{\mathcal{LB}} = 0$, *where*

$$f_a(z) = \frac{1-|a|^2}{1-\overline{a}z} \quad \text{and} \quad h_a(z) = \frac{(1-|a|^2)^2}{(1-\overline{a}z)^2}, \quad a,\ z \in \mathbb{D}.$$

(d)

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\log(e/1-|z|^2)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} = 0$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\log(e/1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^{n}} = 0.$$

Motivated by Theorem A, the purpose of this paper is to give some estimates of the essential norm for the operator $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. Moreover, we give a new characterization for the boundedness, compactness and essential norm of $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$.

Throughout this paper, we say that $P \lesssim Q$ if there exists a constant C such that $P \le CQ$. The symbol $P \approx Q$ means that $P \lesssim Q \lesssim P$.

2. Essential norm of $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. In this section, we give some estimates for the essential norm of the operator $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. For this purpose, we need the following lemma.

Lemma 2.1 ([**25,** Lemma 3.3])**.** *Let $X, Y \subset H(\mathbb{D})$ be two Banach spaces, and let $T : X \to Y$ be a linear operator. Assume that the following conditions are satisfied*:

(i) *the point evaluation functionals on Y are bounded*;

(ii) *for every bounded sequence in X, there is a subsequence which converges uniformly on compact subsets of $\mathbb{D}$ to an element of X*;

(iii)*if a sequence $\{f_j\}$ in X uniformly converges on compact subsets of $\mathbb{D}$ to zero, then $\{Tf_j\}$ converges uniformly on compact subsets of $\mathbb{D}$ to zero.*

Then, $T : X \to Y$ is a compact operator if and only if $\{Tf_j\}$ converges to zero in the norm of Y for each bounded sequence $\{f_j\}$ in X such that $f_j \to 0$ uniformly on compact subsets of $\mathbb{D}$.

Theorem 2.2. *Let $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded. Then,*

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \approx \max\{A, B\} \approx \max\{E, F\} \approx \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}},$$

where

$$A := \limsup_{|a|\to 1} \|D^n_{\varphi,u} f_a\|_{\mathcal{LB}}, \qquad B := \limsup_{|a|\to 1} \|D^n_{\varphi,u} h_a\|_{\mathcal{LB}},$$

$$E := \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\log(e/1-|z|^2)|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}},$$

$$F := \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\log(e/1-|z|^2)|u'(z)|}{(1-|\varphi(z)|^2)^n}.$$

Proof of Theorem 2.2. First, we prove that

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \lesssim \max\{A,B\} \quad \text{and} \quad \|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \lesssim \max\{E,F\}.$$

Let $\rho\in[0,1)$. Define $K_\rho : H(\mathbb{D})\to H(\mathbb{D})$ by $(K_\rho f)(z) = f_\rho(z) = f(\rho z)$, $f\in H(\mathbb{D})$. It is clear that K_ρ is compact on H^∞ and $\|K_\rho\|_{H^\infty\to H^\infty}\le 1$ and $f_\rho\to f$ uniformly on compact subsets of $\mathbb{D}$ as $\rho\to 1$. Let $\{\rho_j\}\subset(0,1)$ be a sequence such that $\rho_j\to 1$ as $j\to\infty$. Then, for all positive integers j, the operator $D^n_{\varphi,u}K_{\rho_j} : H^\infty\to\mathcal{LB}$ is compact. Hence,

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \le \limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j}\|_{H^\infty\to\mathcal{LB}}. \tag{2.1}$$

Therefore, we only need prove that

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j}\|_{H^\infty\to\mathcal{LB}} \lesssim \max\{A,B\}$$

and

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j}\|_{H^\infty\to\mathcal{LB}} \lesssim \max\{E,F\}.$$

Let $f\in H^\infty$ be such that $\|f\|_\infty\le 1$. Then,

$$\begin{aligned} &\|(D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j})f\|_{\mathcal{LB}} \\ &\quad = |u(0)f^{(n)}(\varphi(0)) - \rho_j^n u(0) f^{(n)}(\rho_j\varphi(0))| + \|u\cdot(f-f_{\rho_j})^{(n)}\circ\varphi\|_{\log}. \end{aligned} \tag{2.2}$$

It is obvious that

$$\lim_{j\to\infty} |u(0)f^{(n)}(\varphi(0)) - \rho_j^n u(0) f^{(n)}(\rho_j\varphi(0))| = 0. \tag{2.3}$$

In addition,

$$
\begin{aligned}
(2.4)\quad & \limsup_{j\to\infty} \|u\cdot (f-f_{\rho_j})^{(n)}\circ\varphi\|_{\log} \\
&\le \limsup_{j\to\infty} \sup_{|\varphi(z)|\le\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2} \\
&\times |(f-f_{\rho_j})^{(n+1)}(\varphi(z))||\varphi'(z)||u(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2} \\
&\times |(f-f_{\rho_j})^{(n+1)}(\varphi(z))||\varphi'(z)||u(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)|\le\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2} \\
&\times |(f-f_{\rho_j})^{(n)}(\varphi(z))||u'(z)| \\
&\quad + \limsup_{j\to\infty} \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|(f-f_{\rho_j})^{(n)}(\varphi(z))||u'(z)| \\
&= R_1+R_2+R_3+R_4,
\end{aligned}
$$

where $N\in\mathbb{N}$ is large enough such that $\rho_j\ge 1/2$ for all $j\ge N$,

$$
(2.5)\quad R_1 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\le\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2} \\
|(f-f_{\rho_j})^{(n+1)}(\varphi(z))||\varphi'(z)||u(z)|,
$$

$$
(2.6)\quad R_2 := \limsup_{j\to\infty} \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2} \\
|(f-f_{\rho_j})^{(n+1)}(\varphi(z))||\varphi'(z)||u(z)|,
$$

$$
R_3 := \limsup_{j\to\infty} \sup_{|\varphi(z)|\le\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|(f-f_{\rho_j})^{(n)}(\varphi(z))||u'(z)|,
$$

and

$$
R_4 := \limsup_{j\to\infty} \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|(f-f_{\rho_j})^{(n)}(\varphi(z))||u'(z)|.
$$

Since $D^n_{\varphi,u}: H^\infty\to\mathcal{LB}$ is bounded, by taking the test functions z^n and

z^{n+1}, see **[34]**, we obtain that $u \in \mathcal{LB}$ and

$$M := \sup_{z\in\mathbb{D}} (1-|z|^2)\log\frac{e}{1-|z|^2}|\varphi'(z)||u(z)| < \infty.$$

Since $\rho_j^{n+1} f_{\rho_j}^{(n+1)} \to f^{(n+1)}$ uniformly on compact subsets of $\mathbb{D}$ as $j\to\infty$, we get

$$R_1 \le M \limsup_{j\to\infty} \sup_{|w|\le\rho_N} |f^{(n+1)}(w) - \rho_j^{n+1} f^{(n+1)}(\rho_j w)| = 0. \tag{2.7}$$

Similarly, from the fact that $u \in \mathcal{LB}$, we obtain

$$R_3 \le \|u\|_{\mathcal{LB}} \limsup_{j\to\infty} \sup_{|w|\le\rho_N} |f^{(n)}(w) - \rho_j^{n} f^{(n)}(\rho_j w)| = 0. \tag{2.8}$$

Since $\|f\|_\infty \le 1$, we get

(2.9)

$$\begin{aligned}
&\sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|(f-f_{\rho_j})^{(n+1)}(\varphi(z))||\varphi'(z)||u(z)| \\
&\quad\lesssim \frac{\|f\|_\infty}{n!\rho_N^{n+1}} \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|\varphi'(z)||u(z)|\frac{n!|\varphi(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}} \\
&\quad\lesssim \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|\varphi'(z)||u(z)|\frac{n!|\varphi(z)|^{n+1}}{(1-|\varphi(z)|^2)^{n+1}} \\
&\quad\lesssim \sup_{|a|>\rho_N} \left\| D^n_{\varphi,u}\left(f_a - \frac{1}{n+1}h_a\right)\right\|_{\log} \\
&\quad\lesssim \sup_{|a|>\rho_N} \left\|D^n_{\varphi,u} f_a\right\|_{\mathcal{LB}} + \sup_{|a|>\rho_N} \left\|D^n_{\varphi,u} h_a\right\|_{\mathcal{LB}}.
\end{aligned}$$

Letting $N\to\infty$, we obtain

$$R_2 \lesssim A + B \lesssim \max\{A, B\}. \tag{2.10}$$

From (2.9), we see that

$$R_2 \lesssim E. \tag{2.11}$$

After a calculation, we have

(2.12)

$$\sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|(f-f_{\rho_j})^{(n)}(\varphi(z))||u'(z)|$$

$$\lesssim \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)| \times \frac{\|f\|_\infty}{n+2}\frac{n!|\varphi(z)|^n}{(1-|\varphi(z)|^2)^n}$$

$$\lesssim \sup_{|\varphi(z)|>\rho_N} (1-|z|^2)\log\frac{e}{1-|z|^2}|u'(z)| \times \frac{1}{n+2}\frac{n!|\varphi(z)|^n}{(1-|\varphi(z)|^2)^n}$$

$$\lesssim \sup_{|a|>\rho_N} \left\| D^n_{\varphi,u}\left(f_a - \frac{1}{n+2}h_a\right)\right\|_{\log}$$

$$\lesssim \sup_{|a|>\rho_N} \left\|D^n_{\varphi,u} f_a\right\|_{\mathcal{LB}} + \sup_{|a|>\rho_N} \left\|D^n_{\varphi,u} h_a\right\|_{\mathcal{LB}}.$$

Letting $N\to\infty$, we get

$$R_4 \lesssim A+B \lesssim \max\{A,B\}. \tag{2.13}$$

From (2.12),

$$R_4 \lesssim F. \tag{2.14}$$

Thus, by (2.2)–(2.8), (2.10) and (2.13), we get

$$\begin{aligned} &\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j}\|_{H^\infty\to\mathcal{LB}} \\ &\quad = \limsup_{j\to\infty} \sup_{\|f\|_\infty\le 1} \|(D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j})f\|_{\mathcal{LB}} \\ &\quad = \limsup_{j\to\infty} \sup_{\|f\|_\infty\le 1} \|u\cdot(f-f_{\rho_j})^{(m)}\circ\varphi\|_{\log} \lesssim \max\{A,B\}. \end{aligned} \tag{2.15}$$

From (2.2)–(2.8), (2.11) and (2.14), we obtain

$$\limsup_{j\to\infty} \|D^n_{\varphi,u} - D^n_{\varphi,u}K_{\rho_j}\|_{H^\infty\to\mathcal{LB}} \lesssim \max\{E,F\}. \tag{2.16}$$

Therefore, by (2.1), (2.15) and (2.16), we obtain

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \lesssim \max\{A,B\} \quad \text{and} \quad \|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \lesssim \max\{E,F\}.$$

Now, we prove that

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim \max\{A,B\} \quad \text{and} \quad \|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim \max\{E,F\}.$$

Let $a\in\mathbb{D}$. It is easy to see that $f_a, h_a \in H^\infty$ and $\|f_a\|_\infty \lesssim 1$, $\|g_a\|_\infty \lesssim 1$ for all $a\in\mathbb{D}$ and f_a, g_a uniformly converge to 0 on compact subsets of $\mathbb{D}$ as $|a|\to 1$. Thus, for any compact operator $K: H^\infty\to\mathcal{LB}$, from Lemma 2.1, we obtain $\lim_{|a|\to 1}\|Kf_a\|_{\mathcal{LB}} = 0$ and $\lim_{|a|\to 1}\|Kh_a\|_{\mathcal{LB}} = 0$.

Hence,

$$\begin{aligned}\|D^n_{\varphi,u} - K\|_{H^\infty\to\mathcal{LB}} &\gtrsim \limsup_{|a|\to 1} \|(D^n_{\varphi,u} - K)f_a\|_{\mathcal{LB}} \\ &\geq \limsup_{|a|\to 1} \|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} - \limsup_{|a|\to 1} \|K f_a\|_{\mathcal{LB}} = A,\end{aligned}$$

and similarly, $\|D^n_{\varphi,u} - K\|_{H^\infty\to\mathcal{LB}} \gtrsim B$. Therefore,

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} = \inf_K \|D^n_{\varphi,u} - K\|_{H^\infty\to\mathcal{LB}} \gtrsim \max\{A, B\}.$$

Let $\{z_j\}_{j\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_j)| \to 1$ as $j \to \infty$. Set

$$p_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \overline{\varphi(z_j)}z)} - \frac{1}{1+n}\frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \overline{\varphi(z_j)}z)^2}$$

and

$$q_j(z) = \frac{1 - |\varphi(z_j)|^2}{(1 - \overline{\varphi(z_j)}z)} - \frac{1}{n+2}\frac{(1 - |\varphi(z_j)|^2)^2}{(1 - \overline{\varphi(z_j)}z)^2}.$$

Also, p_j and q_j belong to H^∞ and uniformly converge to 0 on compact subsets in $\mathbb{D}$. Moreover,

$$p_j^{(n)}(\varphi(z_j)) = 0, \quad |p_j^{(n+1)}(\varphi(z_j))| = n!\frac{|\varphi(z_j)|^{n+1}}{(1 - |\varphi(z_j)|^2)^{n+1}}$$

and

$$|q_j^{(n)}(\varphi(z_j))| = \frac{n!}{n+2}\frac{|\varphi(z_j)|^n}{(1 - |\varphi(z_j)|^2)^n}, \quad q_j^{(n+1)}(\varphi(z_j)) = 0.$$

Thus, for any compact operator $T : H^\infty \to \mathcal{LB}$, we obtain

$$\begin{aligned}\|D^n_{\varphi,u} - T\|_{H^\infty\to\mathcal{LB}} &\gtrsim \limsup_{j\to\infty} \|D^n_{\varphi,u}(p_j)\|_{\mathcal{LB}} - \limsup_{j\to\infty} \|T(p_j)\|_{\mathcal{LB}} \\ &\gtrsim \limsup_{j\to\infty} \frac{(1 - |z_j|^2)\log\frac{e}{1-|z_j|^2}|u(z_j)||\varphi'(z_j)||\varphi(z_j)|^{n+1}}{(1 - |\varphi(z_j)|^2)^{n+1}}\end{aligned}$$

and

$$\|D^n_{\varphi,u} - T\|_{H^\infty\to\mathcal{LB}} \gtrsim \limsup_{j\to\infty} \|D^n_{\varphi,u}(q_j)\|_{\mathcal{LB}} - \limsup_{j\to\infty} \|T(q_j)\|_{\mathcal{LB}}$$
$$\gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)\log e/1-|z_j|^2|u'(z_j)||\varphi(z_j)|^n}{(1-|\varphi(z_j)|^2)^n}.$$

Hence,

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} = \inf_T \|D^n_{\varphi,u} - T\|_{H^\infty\to\mathcal{LB}}$$
$$\gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)\log e/1-|z_j|^2|u(z_j)||\varphi'(z_j)|}{(1-|\varphi(z_j)|^2)^{n+1}} = E$$

and

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim \limsup_{j\to\infty} \frac{(1-|z_j|^2)\log e/1-|z_j|^2|u'(z_j)}{(1-|\varphi(z_j)|^2)^n} = F.$$

Therefore, $\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim \max\{E,F\}$.

Finally, we prove that

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \approx \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}. \tag{2.17}$$

For $a \in \mathbb{D}$,

$$f_a(z) = (1-|a|^2)\sum_{k=0}^\infty \overline{a}^k z^k, \qquad h_a(z) = (1-|a|^2)^2 \sum_{k=0}^\infty \frac{\Gamma(k+2)}{k!\Gamma(2)} \overline{a}^k z^k.$$

Since $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded, we see from **[34]** that

$$\sup_{1\le j<\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}} < \infty.$$

For any positive integer $j \ge n$,

$$\begin{aligned}
&\|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} \\
&\quad\le (1-|a|^2)\sum_{k=0}^\infty |a|^k \|D^n_{\varphi,u} I^k\|_{\mathcal{LB}} \\
&\quad\lesssim (1-|a|^2)\sum_{k=0}^{j-1} |a|^k \|D^n_{\varphi,u} I^k\|_{\mathcal{LB}} + (1-|a|^2)\sum_{k=j}^\infty |a|^k \|D^n_{\varphi,u} I^k\|_{\mathcal{LB}} \\
&\quad\lesssim j(1-|a|^2) \sup_{0\le k\le j-1} \|D^n_{\varphi,u} I^k\|_{\mathcal{LB}} + \sup_{k\ge j} \|D^n_{\varphi,u} I^k\|_{\mathcal{LB}}.
\end{aligned}$$

By letting $|a| \to 1$ in the last estimate for a fixed j, and then letting $j \to \infty$ in such obtained inequality, we obtain

$$\limsup_{|a|\to 1} \|D^n_{\varphi,u} f_a\|_{\mathcal{LB}} \lesssim \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}.$$

Similarly,

$$\limsup_{|a|\to 1} \|D^n_{\varphi,u} h_a\|_{\mathcal{LB}} \lesssim \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}.$$

Therefore,

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \approx \max\{A, B\} \lesssim \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}.$$

On the other hand, for each $j \in \mathbb{N}_0$, $I^j \in H^\infty$ with $\|I^j\|_\infty = 1$, and the sequence $\{I^j\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$. Hence for any compact operator $K : H^\infty \to \mathcal{LB}$, by Lemma 2.1, we have $\lim_{j\to\infty} \|KI^j\|_{\mathcal{LB}} = 0$. Hence,

$$\|D^n_{\varphi,u} - K\|_{H^\infty\to\mathcal{LB}} \geq \limsup_{n\to\infty} \|(D^n_{\varphi,u} - K)I^j\|_{\mathcal{LB}} \geq \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}.$$

Therefore,

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} = \inf_K \|D^n_{\varphi,u} - K\|_{H^\infty\to\mathcal{LB}} \geq \limsup_{j\to\infty} \|D^n_{\varphi,u} I^j\|_{\mathcal{LB}}.$$

This completes the proof of Theorem 2.2. □

By Theorem 2.2, we immediately get the following result.

Corollary 2.3. *Let φ be an analytic self-map of $\mathbb{D}$ such that $C_\varphi : H^\infty \to \mathcal{LB}$ is bounded. Then*:

$$\begin{aligned}
&\|C_\varphi\|_{e,H^\infty\to\mathcal{LB}} \\
&\quad \approx \limsup_{|a|\to 1} \|C_\varphi f_a\|_{\mathcal{LB}} \approx \limsup_{|a|\to 1} \|C_\varphi h_a\|_{\mathcal{LB}} \\
&\quad \approx \limsup_{|\varphi(z)|\to 1} \frac{(1-|z|^2)\log e/1-|z|^2|\varphi'(z)|}{1-|\varphi(z)|^2} \approx \limsup_{n\to\infty} \|\varphi^n\|_{\mathcal{LB}}.
\end{aligned}$$

3. A new characterization of boundedness and compactness of $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$. In this section, we also give a new characterization for the boundedness, compactness and essential norm of the operator

$D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ by using the above two integral operators. For this purpose, we need the following lemmas.

Lemma 3.1 ([**5**]). *For $\alpha > 0$, we have $\lim_{k\to\infty} k^\alpha \|z^{k-1}\|_{v_\alpha} = (2\alpha/e)^\alpha$.*

Lemma 3.2 ([**14**]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then, the following statements hold.*

(a) *The weighted composition operator $uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if $\sup_{z\in\mathbb{D}} w(z)/(\widetilde{v}(\varphi(z)))|u(z)| < \infty$. Moreover, the following holds*:

$$\|uC_\varphi\|_{H^\infty_v \to H^\infty_w} = \sup_{z\in\mathbb{D}} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

(b) *Suppose that $uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty_v \to H^\infty_w} = \lim_{s\to 1^-} \sup_{|\varphi(z)|>s} \frac{w(z)}{\widetilde{v}(\varphi(z))}|u(z)|.$$

Lemma 3.3 ([**4**]). *Let v and w be radial, non-increasing weights tending to zero at the boundary of $\mathbb{D}$. Then, the following statements hold.*

(a) *$uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded if and only if*

$$\sup_{k\geq 0} \frac{\|u\varphi^k\|_w}{\|z^k\|_v} < \infty,$$

with the norm comparable to the above supremum.

(b) *Suppose that $uC_\varphi : H^\infty_v \to H^\infty_w$ is bounded. Then*

$$\|uC_\varphi\|_{e,H^\infty_v \to H^\infty_w} = \limsup_{k\to\infty} \frac{\|u\varphi^k\|_w}{\|z^k\|_v}.$$

Theorem 3.4. *Let $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$. Then, $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded if and only if*

$$\sup_{j\geq 1} j^n \|I_u(\varphi^j)\|_{\mathcal{LB}} < \infty \quad \textit{and} \quad \sup_{j\geq 1} j^n \|J_u(\varphi^{j-1})\|_{\mathcal{LB}} < \infty. \tag{3.1}$$

Here

$$I_u g(z) = \int_0^z g'(\xi)u(\xi)\,d\xi, \qquad J_u g(z) = \int_0^z g(\xi)u'(\xi)\,d\xi, \quad g \in H(\mathbb{D}).$$

Proof. By [**34,** Theorem 1], $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded if and only if

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log e/1-|z|^2|u(z)||\varphi'(z)|}{(1-|\varphi(z)|^2)^{n+1}} < \infty \tag{3.2}$$

and

$$\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\log e/1-|z|^2|u'(z)|}{(1-|\varphi(z)|^2)^{n}} < \infty. \tag{3.3}$$

By Lemma 3.2, we see that the inequality in (3.2) is equivalent to $u\varphi' C_\varphi : H^\infty_{v_{n+1}} \to H^\infty_{v_{\log}}$ being bounded. Hence, by Lemmas 3.1 and 3.3, we get

$$\sup_{j\geq 1} j^{1+n}\|u\varphi'\varphi^{j-1}\|_{v_{\log}} \approx \sup_{j\geq 1} \frac{\|u\varphi'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_{n+1}}} < \infty.$$

Also, the inequality in (3.3) is equivalent to $u'C_\varphi : H^\infty_{v_n} \to H^\infty_{v_{\log}}$ being bounded. By Lemmas 3.1 and 3.3, we obtain

$$\sup_{j\geq 1} j^{n}\|u'\varphi^{j-1}\|_{v_{\log}} \approx \sup_{j\geq 1} \frac{\|u'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_{n}}} < \infty.$$

It is clear that $I_u g(0) = 0$, $J_u g(0) = 0$, and

$$(I_u(\varphi^j)(z))' = ju(z)\varphi'(z)\varphi^{j-1}(z), \qquad (J_u(\varphi^{j-1})(z))' = u'(z)\varphi^{j-1}(z).$$

Hence, $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded if and only if

$$\sup_{j\geq 1} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}} = \sup_{j\geq 1} j^{n+1}\|u\varphi'\varphi^{j-1}\|_{v_{\log}} < \infty$$

and

$$\sup_{j\geq 1} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}} = \sup_{j\geq 1} j^{n}\|u'\varphi^{j-1}\|_{v_{\log}} < \infty.$$

The proof is complete. □

Theorem 3.5. *Let $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u}: H^\infty \to \mathcal{LB}$ is bounded. Then*:

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \approx \max\Big\{\limsup_{j\to\infty} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}}, \limsup_{j\to\infty} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}}\Big\}.$$

Proof. The lower estimate. From Theorem 2.2 and Lemmas 3.1 and 3.2, we have

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim E = \|u\varphi' C_\varphi\|_{e,H^\infty_{v_{n+1}}\to H^\infty_{v_{\log}}} = \limsup_{j\to\infty}\frac{\|u\varphi'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_{n+1}}}$$
$$\approx \limsup_{j\to\infty} j^{n+1}\|u\varphi'\varphi^{j-1}\|_{v_{\log}} = \limsup_{j\to\infty} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}},$$

and

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim F = \|u' C_\varphi\|_{e,H^\infty_{v_n}\to H^\infty_{v_{\log}}} = \limsup_{j\to\infty}\frac{\|u'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_n}}$$
$$\approx \limsup_{j\to\infty} j^n\|u'\varphi^{j-1}\|_{v_{\log}} = \limsup_{j\to\infty} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}}.$$

Therefore, by the proof of Theorem 3.4, we obtain

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}} \gtrsim \max\{\limsup_{j\to\infty} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}}, \limsup_{j\to\infty} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}}\}.$$

Proof. The upper estimate. By Lemmas 3.1 and 3.3, we get

$$\|u\varphi' C_\varphi\|_{e,H^\infty_{v_{n+1}}\to H^\infty_{v_{\log}}}$$
$$= \limsup_{j\to\infty}\frac{\|u\varphi'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_{n+1}}} = \limsup_{j\to\infty}\frac{j^{n+1}\|u\varphi'\varphi^{j-1}\|_{v_{\log}}}{j^{n+1}\|z^{j-1}\|_{v_{n+1}}}$$
$$\approx \limsup_{j\to\infty} j^{n+1}\|u\varphi'\varphi^{j-1}\|_{v_{\log}} = \limsup_{j\to\infty} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}}$$

and

$$\|u' C_\varphi\|_{e,H^\infty_{v_n}\to H^\infty_{v_{\log}}} = \limsup_{j\to\infty}\frac{\|u'\varphi^{j-1}\|_{v_{\log}}}{\|z^{j-1}\|_{v_n}} = \limsup_{j\to\infty}\frac{j^n\|u'\varphi^{j-1}\|_{v_{\log}}}{j^n\|z^{j-1}\|_{v_n}}$$
$$\approx \limsup_{j\to\infty} j^n\|u'\varphi^{j-1}\|_{v_{\log}} = \limsup_{j\to\infty} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}}.$$

Using the estimates and the proof of Theorem 3.4, we have

$$\|D^n_{\varphi,u}\|_{e,H^\infty\to\mathcal{LB}}$$
$$\lesssim \|u\varphi' C_\varphi\|_{e,H^\infty_{v_{1+n}}\to H^\infty_{v_{\log}}} + \|u' C_\varphi\|_{e,H^\infty_{v_n}\to H^\infty_{v_{\log}}}$$
$$\lesssim \max\left\{\limsup_{j\to\infty} j^n\|I_u(\varphi^j)\|_{\mathcal{LB}}, \limsup_{j\to\infty} j^n\|J_u(\varphi^{j-1})\|_{\mathcal{LB}}\right\}.$$

This completes the proof. □

From Theorem 3.5, we immediately get the following result.

Theorem 3.6. *Let $n \in \mathbb{N}_0$, $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$ such that $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is bounded. Then. $D^n_{\varphi,u} : H^\infty \to \mathcal{LB}$ is compact if and only if*

$$\limsup_{j\to\infty} j^n \|I_u(\varphi^j)\|_{\mathcal{LB}} = 0 \quad \textit{and} \quad \limsup_{j\to\infty} j^n \|J_u(\varphi^{j-1})\|_{\mathcal{LB}} = 0.$$

From Theorems 3.4 and 3.5, we immediately obtain a new characterization for the boundedness and compactness for the operator $uC_\varphi : H^\infty \to \mathcal{LB}$.

Corollary 3.7. *Let $u \in H(\mathbb{D})$, and let φ be an analytic self-map of $\mathbb{D}$. Then, the following statements hold.*

(i) *$uC_\varphi : H^\infty \to \mathcal{LB}$ is bounded if and only if*

$$\sup_{j\ge 1} \|I_u(\varphi^j)\|_{\mathcal{LB}} < \infty \quad \textit{and} \quad \sup_{j\ge 1} \|J_u(\varphi^{j-1})\|_{\mathcal{LB}} < \infty.$$

(ii) *$uC_\varphi : H^\infty \to \mathcal{LB}$ is compact if and only if $uC_\varphi : H^\infty \to \mathcal{LB}$ is bounded and*

$$\limsup_{j\to\infty} \|I_u(\varphi^j)\|_{\mathcal{LB}} = 0 \quad \textit{and} \quad \limsup_{j\to\infty} \|J_u(\varphi^{j-1})\|_{\mathcal{LB}} = 0.$$

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