JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 30, Number 3, Fall 2018

UNIFORM EXPONENTIAL STABILITY AND APPLICATIONS TO BOUNDED SOLUTIONS OF INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES

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ABSTRACT. Let $\mathbb{X}$ be a Banach space. Let A be the generator of an immediately norm continuous C_0-semigroup defined on $\mathbb{X}$. We study the uniform exponential stability of solutions of the Volterra equation

$$u'(t) = Au(t) + \int_0^t a(t-s)Au(s)\,ds, \quad t \geq 0, \ u(0) = x,$$

where a is a suitable kernel and $x \in \mathbb{X}$. Using a matrix operator, we obtain some spectral conditions on A that ensure the existence of constants $C, \omega > 0$ such that $\|u(t)\| \leq Ce^{-\omega t}\|x\|$, for each $x \in D(A)$ and all $t \geq 0$. With these results, we prove the existence of a uniformly exponential stable resolvent family to an integro-differential equation with infinite delay. Finally, sufficient conditions are established for the existence and uniqueness of bounded mild solutions to this equation.

1. Introduction. In the classical theory of heat conduction, it is assumed that the internal energy and the heat flux depend linearly on the temperature u and on its gradient ∇u. If we denote by $u = u(x,t)$ the temperature at position $x \in \mathcal{D}$ and time t (where $\mathcal{D}$ is a bounded subset of $\mathbb{R}^n$) and the heat flux by $\mathbf{q} = q(x,t)$, then the temperature u and the heat flux $\mathbf{q}$ are related by a constitutive relation, known as *Fourier's law* for the heat flux, given by

$$\mathbf{q} = -\lambda \nabla u,$$

2010 AMS *Mathematics subject classification.* Primary 47D06, Secondary 45D05, 45N05, 47J35, 45M10.

Keywords and phrases. Exponential stability, C_0-semigroups, mild solutions, almost periodic, Volterra equations, heat equation with memory.

The first author was supported by the NSF of China, grant No. 11361032 and FRFCUC, grant No. JB160713. The second author was supported by Fondecyt Iniciación, grant No. 11130619.

Received by the editors on April 14, 2016, and in revised form on May 8, 2017.

DOI:10.1216/JIE-2018-30-3-347

where $\lambda > 0$ is the coefficient of thermal diffusion. Assuming that $u_t = -\text{div}\mathbf{q}$, we obtain the diffusion equation

$$u_t = \lambda \Delta u. \tag{1.1}$$

Under these conditions, the classical heat equation (1.1) describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. The problem of heat conduction in material with memory was first discussed by Coleman and Gurtin **[11]**, Gurtin and Pipkin **[15]** and Nunziato **[25]**, among others. Gurtin and Pipkin **[15]** consider, after a linearization, that the density $e(x,t)$ of the internal energy and the heat flux $\mathbf{q}$ are related by

$$e(x,t) = \nu u(x,t) + \int_{-\infty}^{t} b(t-s)u(x,s)\,ds, \tag{1.2}$$

and

$$\mathbf{q}(x,t) = -\int_{-\infty}^{t} a(t-s)\nabla u(x,s)\,ds \tag{1.3}$$

where $\nu \neq 0$ (known as the heat capacity) and a, b are positive relaxation functions. On the other hand, Coleman and Gurtin **[11]** consider the heat flux as a perturbation of the Fourier law, that is,

$$\mathbf{q}(x,t) = -\gamma\nabla u(x,t) - \int_{-\infty}^{t} a(t-s)\nabla u(x,s)\,ds, \tag{1.4}$$

where $\gamma > 0$ is the constant of thermal conduction. The heat relaxation function a is assumed to be in $L^1(\mathbb{R}_+)$. A typical choice of a is $a(t) = \alpha t^{\mu-1}/(\Gamma(\mu))e^{-\beta t}$, where $\alpha < 0$, $\beta > 0$ and $\mu > 0$, see **[23, 28]** and the references therein.

With the equations (1.2), (1.3) and (1.4) at hand, we obtain the heat equation with memory

$$\begin{aligned}\nu\partial_t u(x,t) = \gamma\Delta u(x,t) + \int_{-\infty}^{t} a(t-s)\Delta u(x,s)\,ds \\ + f(x,t,u(x,t)), \quad t \in \mathbb{R},\end{aligned}$$

where $f(x,t,u)$ is the energy supply which may depend upon the temperature. This equation can be written in the abstract form

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t,u(t)), \quad t \in \mathbb{R}, \tag{1.5}$$

where $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a closed linear operator defined on a Banach space $\mathbb{X}$, $a \in L^1(\mathbb{R}_+)$ and f is a suitable function. In this paper, we study the problem of existence, uniqueness and regularity of solutions for equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \in \mathbb{R}, \tag{1.6}$$

where $a(t) := \alpha t^{\mu-1}/(\Gamma(\mu))e^{-\beta t}$, $\alpha, \beta, \mu \in \mathbb{R}$. Concretely, we prove that, under appropriate assumptions on α, β, μ, A and on the forcing function f, there exists a unique mild solution to (1.6) which behaves in the same way that f does. To do this, first we study the uniform exponential stability of solutions to the Volterra equation

$$\begin{cases} u'(t) = Au(t) + \int_{0}^{t} a(t-s)Au(s)\,ds & t \geq 0, \\ u(0) = x, \end{cases} \tag{1.7}$$

where A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$ and $x \in \mathbb{X}$. The problem of existence of uniformly exponentially stable solutions to problem (1.7) has been studied by many researchers, see **[9, 10, 14, 16]** and the references therein.

Under appropriate assumptions on α, β, μ and A, we prove that the solution u to equation (1.7) is uniformly exponentially stable, that is, there exist $C, \omega > 0$ such that, for each $x \in D(A)$, the solution $\|u(t)\| \leq Ce^{-\omega t}\|x\|$ for all $t \geq 0$. It is remarkable that some examples show that, with small changes on the parameters α, β or μ, we can lose the uniform exponential stability of solutions to eq. (1.7), see Example 2.5 below. With this result, we are able to prove that equation (1.6) has a unique mild solution, given by

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

where $\{S(t)\}_{t\geq 0} \subseteq \mathcal{B}(\mathbb{X})$ is the *resolvent family* generated by A, $\mathcal{B}(\mathbb{X})$ denotes the space of bounded linear operators from $\mathbb{X}$ into $\mathbb{X}$ endowed with the operator topology. We note here that $S(t)$ is given by $S(t)x := u(t;x) = u(t)$, where $u(t)$ is the unique solution of problem (1.7).

The existence and regularity of bounded mild solutions to equation (1.6) in the case $\mu = 1$ have been studied by many researchers, see **[2, 6, 7, 10, 21, 28]** and the references therein.

This paper is organized as follows. In Section 2, we find conditions on α, β, μ and A that ensure the uniform exponential stability of solutions to (1.7). Several examples are examined. In Section 3, we study the existence of bounded mild solutions to the linear case in equation (1.6). Using the results from Section 2, we find a resolvent family associated to equation (1.6), which is uniformly exponentially stable. Assuming that A generates an immediately norm continuous C_0-semigroup, we are able to give a simply spectral condition on A in order to guarantee the existence of bounded and continuous solutions to equation (1.6) in the linear case. In Section 4, we present the results for the semilinear equation (1.5). There, using the previous results on the linear case and the Banach contraction principle, we present some existence results of solutions that are directly based upon the data of the problem. We finish this paper with an example, to show the feasibility of the abstract results.

2. Uniform exponential stability of solutions to an abstract Volterra equation. In this section, we study the uniform exponential stability of solutions to the homogeneous abstract Volterra equation

$$\begin{cases} u'(t) = Au(t) + \displaystyle\int_0^t a(t-s)Au(s)\,ds & t \geq 0, \\ u(0) = x, \end{cases} \tag{2.1}$$

where $a(t) := \alpha e^{-\beta t}(t^{\mu-1}/\Gamma(\mu))$, $t > 0$, $\alpha \in \mathbb{R}$, $\beta > 0$, $\mu \geq 1$, A generates a C_0-semigroup on a Banach space $(\mathbb{X}, \|\cdot\|)$ and $x \in \mathbb{X}$. We say that a solution of (2.1) is *uniformly exponentially stable* if there exist $\omega > 0$ and $C > 0$ such that, for each $x \in D(A)$, the corresponding solution $u(t)$ satisfies

$$||u(t)|| \leq Ce^{-\omega t}\|x\|, \quad t \geq 0. \tag{2.2}$$

Definition 2.1. Let $\mathbb{X}$ be a Banach space. A strongly continuous function $T : \mathbb{R}_+ \to \mathcal{B}(\mathbb{X})$ is said to be *immediately norm continuous* if $T : (0, \infty) \to \mathcal{B}(\mathbb{X})$ is continuous.

As in **[10]**, to study the uniform exponential stability of solutions to (2.1), we introduce a matrix operator, as follows. We define the operator $\mathcal{A}|_D$ on $\mathcal{X} := \mathbb{X} \times M$ by

$$\mathcal{A}|_D \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} A & \delta_0 \\ B & d/ds \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix}, \tag{2.3}$$

with domain $D(\mathcal{A}|_D) = D(A) \times M$, where $Bx := a(\cdot)Ax$, $\delta_0 f = f(0)$ and $M := \{s^{\mu-1}e^{-\beta s}x : x \in \mathbb{X}\}$. Observe that M is a closed subspace of $L^p(\mathbb{R}_+, \mathbb{X})$ for all $p \geq 1$. Now, we study some spectral properties of $\mathcal{A}|_D$.

Proposition 2.2. *Let* $a(t) := \alpha e^{-\beta t}(t^{\mu-1}/\Gamma(\mu))$, *where* $\alpha \neq 0$, $\beta > 0$ *and* $\mu \geq 1$. *The following assertions hold*:

- (i) $-\beta$ *is an eigenvalue of* $\mathcal{A}|_D$, *that is*, $-\beta \in \sigma_p(\mathcal{A}|_D)$ *if and only if* $0 \in \sigma_p(A)$;
- (ii) *if* $\alpha/\beta^\mu = -1$, *then* $0 \in \sigma_p(\mathcal{A}|_D)$;
- (iii) *if* $\alpha/\beta^\mu \neq -1$ *and* $\lambda \notin \{-\beta, (-\alpha)^{1/\mu} - \beta\}$, *then* $\lambda \in \sigma_p(\mathcal{A}|_D)$ *if and only if* $\lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \sigma_p(A)$.

Proof.

(i) Consider the equation

$$(\lambda - \mathcal{A}|_D) \begin{pmatrix} x \\ f \end{pmatrix} = 0,$$

which is equivalent to the initial value problem

$$\begin{cases} (\lambda - A)x - f(0) = 0 \\ -\alpha e^{-\beta s}(s^{\mu-1}/\Gamma(\mu))Ax + \lambda f(s) - f'(s) = 0. \end{cases} \tag{2.4}$$

If $\lambda = -\beta$, then, from (2.4), we obtain

$$\begin{cases} (-\beta - A)x - f(0) = 0 \\ f(s) = e^{-\beta s}f(0) - \alpha e^{-\beta s}(s^\mu/\Gamma(\mu+1))Ax. \end{cases} \tag{2.5}$$

Thus, there exists a nonzero $f(s) \in M$ if and only if $Ax = 0$ for some nonzero $x \in D(A)$. Therefore, $-\beta \in \sigma_p(\mathcal{A}|_D)$ if and only if $0 \in \sigma_p(A)$.

On the other hand, if $\lambda \neq -\beta$, then, from (2.4), we obtain (using the method of variation of parameters) that $f(0) = \alpha/(\lambda+\beta)^\mu Ax$, and

$$f(s) = f(0)e^{\lambda s} - \frac{e^{\lambda s}}{\Gamma(\mu)}\frac{\alpha}{(\lambda+\beta)^\mu}\int_0^{(\lambda+\beta)s} v^{\mu-1}e^{-v}Ax\,dv. \tag{2.6}$$

The first equation in (2.4) yields

$$\frac{\alpha}{(\lambda+\beta)^\mu}Ax = (\lambda - A)x. \tag{2.7}$$

If $x = 0$, then, from (2.7), we get $f(0) = 0$, and therefore, from (2.6), we conclude that $f(s) = 0$ for all s. From (2.7), we conclude that $\lambda \in \sigma_p(\mathcal{A}|_D)$ if and only if there exists an $x \neq 0$ such that

$$\left[\lambda - \left(1 + \frac{\alpha}{(\lambda+\beta)^\mu}\right)A\right]x = 0. \tag{2.8}$$

(ii) If $\alpha/\beta^\mu = -1$, then, by (2.8) (with $\lambda = 0$) we conclude that $0 \in \sigma_p(\mathcal{A}|_D)$.

(iii) Suppose that $\alpha/\beta^\mu \neq -1$ and $\lambda \notin \{-\beta, (-\alpha)^{1/\mu} - \beta\}$. From (2.8), we have

$$\begin{aligned} 0 &= \left[\lambda - \left(1 + \frac{\alpha}{(\lambda+\beta)^\mu}\right)A\right]x \\ &= \left(1 + \frac{\alpha}{(\lambda+\beta)^\mu}\right)\left[\lambda\frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu + \alpha} - A\right]x, \end{aligned}$$

and therefore, $\lambda \in \sigma_p(\mathcal{A}|_D)$ if and only if $\lambda(\lambda+\beta)^\mu/((\lambda+\beta)^\mu + \alpha) \in \sigma_p(A)$. □

Lemma 2.3. *Let* $a(t) := \alpha e^{-\beta t}(t^{\mu-1}/\Gamma(\mu))$, *where* $\alpha \neq 0$, $\beta > 0$ *and* $\mu \geq 1$. *If* $\lambda \neq (-\alpha)^{1/\mu} - \beta$ *and* $\lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \rho(A)$, *then* $\lambda \in \rho(\mathcal{A}|_D)$.

Proof. Consider the eigenequation

$$(\lambda - \mathcal{A}|_D)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} y \\ g \end{pmatrix}, \tag{2.9}$$

which is equivalent to the initial value problem

$$(2.10)\qquad \begin{cases}(\lambda - A)x - f(0) = y\\ -\alpha e^{-\beta s}(s^{\mu-1}/\Gamma(\mu))Ax + \lambda f(s) - f'(s) = g(s).\end{cases}$$

Multiplying by $e^{-\lambda s}$ the second equation in (2.10) and integrating from $s = 0$ to $s = \infty$, we obtain

$$\frac{(\lambda+\beta)^\mu + \alpha}{(\lambda+\beta)^\mu}\left[\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha} - A\right]x = \widehat{g}(\lambda) + y.$$

By hypothesis, we conclude that

$$x = \frac{(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha}\left[\frac{\lambda(\lambda+\beta)^\mu}{(\lambda+\beta)^\mu+\alpha} - A\right]^{-1}[\widehat{g}(\lambda) + y].$$

In order to find f in equation (2.9), we solve the initial value problem (2.10), and we obtain

$$f(s) = \left(-\int_0^s g(v)e^{-\lambda v}dv - \int_0^s \alpha e^{-\beta v}\frac{v^{\mu-1}}{\Gamma(\mu)}e^{-\lambda v}Ax\,dv\right)e^{\lambda s}.$$

Therefore, $\lambda \in \rho(\mathcal{A}|_D)$. □

The next theorem is the main result in this section, and it is an extension of [**10,** Theorem 3.4].

Theorem 2.4. *Let $\alpha \neq 0$, $\beta > 0$ and $\mu \geq 1$ such that* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$. *Assume that*

(i) *A generates an immediately norm continuous C_0-semigroup on a Banach space X;*
(ii) $\sup\{\mathrm{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu+\alpha)^{-1} \in \sigma(A)\} < 0$.

Then, the solutions to problem (2.1) *are uniformly exponentially stable.*

Proof. An easy computation shows that the space $M := \{s^{\mu-1}e^{-\beta s}x : x \in \mathbb{X}\}$ satisfies the hypothesis in [**10,** Theorem 2.9]. By Lemma 2.3, we obtain that

$$\sigma(\mathcal{A}|_D) \subset \big\{\lambda \in \mathbb{C} : \lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu+\alpha)^{-1} \in \sigma(A)\big\} \cup \{(-\alpha)^{1/\mu} - \beta\}.$$

Therefore, the solutions to (2.1) are uniformly exponentially stable by [**10,** Theorem 2.9]. □

Example 2.5. Take $\mathbb{X} = \mathbb{C}$ and $A = -I$ in the problem (2.1).

(1) If $\alpha = 1$, $\beta = 1$ and $\mu = 2$, then $a(t) = te^{-t}$ and the solution to (2.1) is given by

$$u(t) = \left(\frac{1}{3}e^{-2t} + \frac{2}{3}e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)x,$$

and therefore, $\|u(t)\| \to 0$ as $t \to \infty$.

(2) If $\alpha = -1$, $\beta = 1$, $\mu = 2$, then $a(t) = -te^{-t}$ and the solution to (2.1) is given by

$$u(t) = \left(\frac{1}{3} + \frac{2}{3}e^{-3t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)\right)x.$$

Therefore, $\|u(t)\| \to 1/3$ as $t \to \infty$.

(3) If $\alpha = 2$, $\beta = 2$ and $\mu = 2$, then $a(t) = 2te^{-2t}$, and the solution to (2.1) is

$$u(t) = \left(\frac{1}{5}e^{-3t} + \frac{2}{5}e^{-t}(2\cos t + \sin t)\right)x.$$

Then, $\|u(t)\| \to 0$ as $t \to \infty$.

(4) If $\alpha = -4$, $\beta = 2$ and $\mu = 2$, then $a(t) = -4te^{-2t}$, and the solution to (2.1) is

$$u(t) = \left(\frac{1}{2} + \frac{1}{14}e^{-5t/2}\left(7\cos\left(\frac{\sqrt{7}t}{2}\right) + \sqrt{7}\sin\left(\frac{\sqrt{7}t}{2}\right)\right)\right)x,$$

and therefore, $\|u(t)\| \to 1/2$ as $t \to \infty$.

(5) If $\alpha = -1/2$, $\beta = 1$ and $\mu = 3$, then $a(t) = -(t^2/4)e^{-t}$, and numerical computation shows that the solution to (2.1) satisfies $\|u(t)\| \to 0$ as $t \to \infty$.

(6) If $\alpha = -2$, $\beta = 1$ and $\mu = 3$, then $a(t) = -t^2e^{-t}$, and numerical computation shows that the solution to (2.1) satisfies $\|u(t)\| \to \infty$ as $t \to \infty$.

Remark 2.6. Easy computation shows that the parameters α, β and μ in cases Example 2.5 (1), (3), (5) satisfy the hypotheses in Theorem 2.4, whereas, in cases (2), (4), (6), the hypotheses are not fulfilled. On the other hand, several examples in case $\mu = 1$ can be found in [**10**].

Example 2.7. Consider the problem

(2.11)
$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) + \dfrac{1}{2}\displaystyle\int_0^t (t-s)e^{-(t-s)/3}\dfrac{\partial^2 u}{\partial x^2}(s,x)\,ds \quad t \geq 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(0) = u_0. \end{cases}$$

Let $\mathbb{X} = L^2[0,\pi]$, and define $A := d^2/dx^2$, with domain $D(A) = \{g \in H^2[0,\pi] : g(0) = g(\pi) = 0\}$. Then, (2.11) can be converted into the abstract form (2.1) with $\alpha = 1/2$, $\beta = 1/3$ and $\mu = 2$. It is well known that A generates an analytic (and, hence, immediately norm continuous) C_0-semigroup $T(t)$ on $\mathbb{X}$. Moreover, $\sigma(A) = \sigma_p(A) = \{-n^2 : n \in \mathbb{N}\}$. Since we must have $\lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu+\alpha)^{-1} \in \sigma(A)$, we need to solve the equations

$$\frac{\lambda(\lambda+1/3)^2}{(\lambda+1/3)^2+1/2} = -n^2,$$

obtaining that the solutions are given by

$$\begin{aligned} \lambda_{n,1} &= \frac{-(2+3n^2)}{9} - \frac{2^{2/3}a_n}{c_n} + \frac{1}{9\sqrt[3]{4}}c_n, \\ \lambda_{n,2} &= \frac{-(2+3n^2)}{9} + \frac{(1+\sqrt{3}i)a_n}{2^{1/3}c_n} - \frac{1}{18\sqrt[3]{4}}(1-\sqrt{3}i)c_n, \\ \lambda_{n,3} &= \frac{-(2+3n^2)}{9} + \frac{(1-\sqrt{3}i)a_n}{2^{1/3}c_n} - \frac{1}{18\sqrt[3]{4}}(1+\sqrt{3}i)c_n, \end{aligned}$$

for all $n \geq 1$, where

$$a_n := -\frac{1}{9} + \frac{2}{3}n^2 - n^4,$$

and

$$c_n := \Big(4 - 765n^2 + 108n^4 - 108n^6 + 27n\sqrt{-8+801n^2-216n^4+216n^6}\Big)^{1/3}.$$

It is easy to see that

$$\sup\left\{\mathrm{Re}\lambda : \lambda\left(\lambda+\frac{1}{3}\right)^2\left(\left(\lambda+\frac{1}{3}\right)^2+\frac{1}{2}\right)^{-1}\in\sigma(A)\right\}<0.$$

Therefore, from Theorem 2.4, we conclude that the solution u to problem (2.11) is uniformly exponentially stable.

Remark 2.8. If $\beta = 0$ in equation (2.1), then we obtain the fractional differential equation

$$u'(t) = Au(t) + \alpha\int_0^t \frac{(t-s)^{\mu-1}}{\Gamma(\mu)}Au(s)\,ds, \quad t \geq 0, \tag{2.12}$$

with the initial condition $u(0) = x$, where

$$\int_0^t \frac{(t-s)^{\mu-1}}{\Gamma(\mu)}v(s)\,ds$$

corresponds to the Riemann-Liouville fractional integral of order μ, see **[24]**. By using the properties of fractional calculus, see **[19,** Chapter 2**]**, the equation (2.12) can be written as

$$D_t^{\mu+1}u(t) = AD_t^{\mu}u(t) + Au(t), \quad t \geq 0, \tag{2.13}$$

where D_t^{μ} denotes the Caputo fractional derivative of order μ. The existence of mild solutions to equations in the form of (2.13) has been widely considered by several authors in the last few years, see for instance, **[3, 8, 12, 17, 22]** and the references therein.

On the other hand, if u is a T-periodic function, then Au and u' are also T-periodic functions. However, the Riemann-Liouville integral

$$\alpha\int_0^t \frac{(t-s)^{\mu-1}}{\Gamma(\mu)}Au(s)\,ds$$

cannot be a T-periodic function, see **[4, 5]**, except in the case $\alpha = 0$. The existence of periodic functions and almost periodic functions, among others, to an integro-differential equation will be considered in the next section.

3. Bounded mild solutions for equations with infinite delay. In this section, we study bounded solutions for the linear integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad t \in \mathbb{R}, \tag{3.1}$$

where $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$, A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$, $\alpha, \beta, \mu \in \mathbb{R}$ are given and f is a bounded and continuous function. To begin our study, we note in the next proposition that, under the given hypothesis on A, it is possible to construct for (3.1) a strongly continuous family of bounded and linear operators that commutes with A and satisfies certain *resolvent equations*. This class of strongly continuous families has been extensively studied in the literature of abstract Volterra equations, see e.g., **[28]** and the references therein.

Proposition 3.1. *Let* $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$ *where* $\alpha \neq 0$, $\beta > 0$ *and* $\mu \geq 1$ *and* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$. *Assume that*

(a) *A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$;*
(b) $\sup\{\mathrm{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^{\mu}((\lambda+\beta)^{\mu}+\alpha)^{-1} \in \sigma(A)\} < 0.$

Then, there exists a uniformly exponentially stable and strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{X})$ such that $S(t)$ commutes with A, that is, $S(t)D(A) \subset D(A)$ and $AS(t)x = S(t)Ax$ for all $x \in D(A)$, $t \geq 0$ and

$$S(t)x = x + \int_{0}^{t} b(t-s)AS(s)x\,ds, \quad \text{for all } x \in \mathbb{X},\ t \geq 0, \tag{3.2}$$

where $b(t) := 1 + \int_0^t a(s)\,ds$, $t \geq 0$.

Proof. For $t \geq 0$ and $x \in \mathbb{X}$, define $S(t)x := u(t;x) = u(t)$, where $u(t)$ is the unique solution of equation (2.1). See **[13,** page 449, Corollary 7.22] for the existence of such a solution and its strong continuity. We will see that $S(\cdot)x$ satisfies the resolvent equation (3.2). Since $S(t)x$ is the solution of (2.1), we have that $S(t)x$ is differentiable and satisfies

$$S'(t)x = AS(t)x + \int_{0}^{t} a(t-s)AS(s)x\,ds. \tag{3.3}$$

Integrating (3.3), we conclude from Fubini's theorem that

$$\begin{aligned} S(t)x - x &= \int_0^t AS(s)x\,ds + \int_0^t \int_0^s a(s-\tau)AS(\tau)x\,d\tau\,ds \\ &= \int_0^t AS(s)x\,ds + \int_0^t \int_\tau^t a(s-\tau)AS(\tau)x\,ds\,d\tau \\ &= \int_0^t AS(s)x\,ds + \int_0^t \int_0^{t-\tau} a(v)AS(\tau)x\,dv\,d\tau \\ &= \int_0^t AS(s)x\,ds + \int_0^t (b(t-\tau)-1)AS(\tau)x\,d\tau \\ &= \int_0^t b(t-\tau)AS(\tau)x\,d\tau. \end{aligned}$$

The commutativity of $S(t)$ with A follows in the same manner as [**28,** pages 31, 32]. Finally, from Theorem 2.4, there exist $C, \omega > 0$ such that $\|S(t)\| \le Ce^{-\omega t}$ for all $t \ge 0$, that is, $\{S(t)\}_{t\ge 0}$ is uniformly exponentially stable. □

We recall that a function $u \in C^1(\mathbb{R};\mathbb{X})$ is called a strong solution of (3.1) on $\mathbb{R}$ if $u \in C(\mathbb{R}; D(A))$ and (3.1) holds for all $t \in \mathbb{R}$. If $u(t) \in \mathbb{X}$ instead of $u(t) \in D(A)$, and (3.1) holds for all $t \in \mathbb{R}$, we say that u is a *mild solution* of (3.1). Let

$$BC(\mathbb{X}) := \{f : \mathbb{R} \to \mathbb{X} : \|f\|_\infty := \sup_{t\in\mathbb{R}} \|f(t)\| < \infty\}$$

be the Banach space of all bounded and continuous functions.

For $T > 0$ fixed, $P_T(\mathbb{X})$ denotes the space of all vector-valued periodic functions, that is,

$$P_T(\mathbb{X}) := \{f \in BC(\mathbb{X}) : f(t+T) = f(t), \text{ for all } t \in \mathbb{R}\}.$$

We denote by $AP(\mathbb{X})$ the space of all almost periodic functions (in the sense of Bohr), which consists of all $f \in BC(\mathbb{X})$ such that, for every $\varepsilon > 0$, there exists an $l > 0$ such that, for every subinterval of $\mathbb{R}$ of length l, at least one point τ is contained such that $\|f(t+\tau)-f(t)\|_\infty \le \varepsilon$. A function $f \in BC(\mathbb{X})$ is said to be *almost automorphic* if, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$f(t) = \lim_{n\to\infty} g(t - s_n), \qquad \text{for each } t \in \mathbb{R}.$$

We denote by $AA(\mathbb{X})$ the Banach space of all almost automorphic functions.

On the other hand, the space of compact almost automorphic functions is the space of all functions $f \in BC(\mathbb{X})$ such that, for all sequence $(s'_n)_{n\in\mathbb{N}}$ of real numbers, there exists a subsequence $(s_n)_{n\in\mathbb{N}} \subset (s'_n)_{n\in\mathbb{N}}$ such that $g(t) := \lim_{n\to\infty} f(t + s_n)$ and $f(t) = \lim_{n\to\infty} g(t - s_n)$ uniformly over compact subsets of $\mathbb{R}$.

We note that $P_T(\mathbb{X})$, $AP(\mathbb{X})$, $AA(\mathbb{X})$ and $AA_c(\mathbb{X})$ are Banach spaces under the norm $||\cdot||_\infty$, and

$$P_T(\mathbb{X}) \subset AP(\mathbb{X}) \subset AA(\mathbb{X}) \subset AA_c(\mathbb{X}) \subset BC(\mathbb{X}).$$

Note, too, that all of these inclusions are proper. Now, we consider the set $C_0(X) := \{f \in BC(\mathbb{X}) : \lim_{|t|\to\infty} ||f(t)|| = 0\}$ and define the space of asymptotically periodic functions as $AP_T(\mathbb{X}) := P_T(\mathbb{X}) \oplus C_0(\mathbb{X})$. Analogously, we define the space of asymptotically almost periodic functions,

$$AAP(\mathbb{X}) := AP(\mathbb{X}) \oplus C_0(\mathbb{X}),$$

the space of asymptotically compact almost automorphic functions,

$$AAA_c(\mathbb{X}) := AA_c(\mathbb{X}) \oplus C_0(\mathbb{X}),$$

and the space of asymptotically almost automorphic functions,

$$AAA(\mathbb{X}) := AA(\mathbb{X}) \oplus C_0(\mathbb{X}).$$

We have the following natural proper inclusions

$$AP_T(\mathbb{X}) \subset AAP(\mathbb{X}) \subset AAA_c(\mathbb{X}) \subset AAA(\mathbb{X}) \subset BC(\mathbb{X}).$$

Again, all of these inclusions are proper.

Throughout, we will use the notation $\mathcal{N}(\mathbb{X})$ to denote any of the spaces $AP_T(\mathbb{X})$, $AAP(\mathbb{X})$, $AAA_c(\mathbb{X})$ and $AAA(\mathbb{X})$ defined above. Finally, we define the set $\mathcal{N}(\mathbb{R} \times \mathbb{X}; \mathbb{X})$, which consists of all functions $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ such that $f(\cdot, x) \in \mathcal{N}(\mathbb{X})$ uniformly for each $x \in K$, where K is any bounded subset of $\mathbb{X}$.

We have the following result in the abstract case.

Theorem 3.2. *Let* $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$*, where* $\alpha \neq 0$*,* $\beta > 0$*,* $\mu \geq 1$ *and* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$*. Assume that*

(a) *A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$;*
(b) $\sup\{\mathrm{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^{\mu}((\lambda+\beta)^{\mu}+\alpha)^{-1} \in \sigma(A)\} < 0.$

If f belongs to $\mathcal{N}(\mathbb{X})$, then the unique mild solution of problem (3.1) *belongs to $\mathcal{N}(\mathbb{X})$, which is given by*

$$u(t) = \int_{-\infty}^{t} S(t-s)f(s)\, ds, \quad t \in \mathbb{R},$$

where $\{S(t)\}_{t\geq 0}$ is given in Proposition 3.1.

Proof. By Proposition 3.1, the family $\{S(t)\}_{t\geq 0}$ is uniformly exponentially stable, and therefore, u is well defined and belongs to $\mathcal{N}(\mathbb{X})$ (see [**20,** Theorem 3.3]). Since S satisfies the resolvent equation

$$S(t)x = x + \int_0^t b(t-s)AS(s)x\, ds, \quad x \in \mathbb{X},$$

where

$$b(t) = 1 + \int_0^t a(s)\, ds,$$

we have that b is differentiable, and the above equation shows that, for each $x \in \mathbb{X}$, $S'(t)x$ exists and

$$S'(t)x = AS(t)x + \int_0^t a(t-s)AS(s)x\, ds.$$

It remains to prove that u is a mild solution of (3.1). Since A is a closed operator, using Fubini's theorem, we have

$$\begin{aligned} u'(t) &= S(0)f(t) + \int_{-\infty}^{t} S'(t-s)f(s)\, ds \\ &= f(t) + \int_{-\infty}^{t} \left[AS(t-s)f(s) + \int_0^{t-s} a(t-s-\tau)AS(\tau)f(s)\, d\tau \right] ds \\ &= f(t) + \int_{-\infty}^{t} AS(t-s)f(s)\, ds + \int_{-\infty}^{t}\int_0^{t-s} a(t-s-\tau)AS(\tau)f(s)\, d\tau\, ds \end{aligned}$$

$$
\begin{aligned}
&= f(t) + Au(t) + \int_{-\infty}^{t} \int_{s}^{t} a(t-v) A S(v-s) f(s)\, dv\, ds \\
&= f(t) + Au(t) + \int_{-\infty}^{t} \int_{-\infty}^{v} a(t-v) A S(v-s) f(s)\, ds\, dv \\
&= f(t) + Au(t) + \int_{-\infty}^{t} a(t-v) \int_{-\infty}^{v} A S(v-s) f(s)\, ds\, dv \\
&= f(t) + Au(t) + \int_{-\infty}^{t} a(t-v) A u(v)\, dv.
\end{aligned}
$$

This concludes the proof. □

In the case of Hilbert spaces, we can refer to a result of You **[29]**, which characterizes norm continuity of C_0-semigroups, obtaining the following result.

Corollary 3.3. *Let A be the generator of a C_0-semigroup on a Hilbert space $\mathbb{H}$. Let $s(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma(A)\}$ be the spectral bound of A. Let $\alpha \neq 0$ and $\beta, \mu > 0$ such that $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$. Assume that:*

(a) $\lim_{\mu \in \mathbb{R}, |\mu| \to \infty} \|(\mu_0 + i\mu - A)^{-1}\| = 0$ *for some* $\mu_0 > s(A)$;
(b) $\sup\{\mathrm{Re}\lambda : \lambda(\lambda+\beta)^{\mu}((\lambda+\beta)^{\mu} + \alpha)^{-1} \in \sigma(A)\} < 0.$

If f belongs to $\mathcal{N}(\mathbb{H})$, then the unique mild solution of problem (3.1) *belongs to $\mathcal{N}(\mathbb{H})$.*

Remark 3.4. In the case $A = \rho I$, $\rho \in \mathbb{C}$, we obtain from (3.2), using Laplace transform, that, for each $x \in X$:

$$
\begin{aligned}
\widehat{S}_{\rho}(\lambda)x &= \frac{(\lambda+\beta)^{\mu} x}{\lambda(\lambda+\beta)^{\mu} - \rho(\lambda+\beta)^{\mu} - \alpha\rho} \\
&= \frac{(\lambda+\beta)^{\mu} x}{(\lambda+\beta)^{\mu+1} - (\lambda+\beta)^{\mu}(\rho+\beta) - \alpha\rho}.
\end{aligned}
\tag{3.4}
$$

If $\beta + \rho = 0$, then

$$
\widehat{S}_{\rho}(\lambda)x = \frac{(\lambda+\beta)^{\mu} x}{(\lambda+\beta)^{\mu+1} - \alpha\rho} = \frac{(\lambda+\beta)^{\mu+1-1} x}{(\lambda+\beta)^{\mu+1} - \alpha\rho},
$$

and therefore,

$$
S_{\rho}(t)x = e^{-\beta t} E_{\mu+1,1}(\alpha\rho t^{\mu+1})x, \tag{3.5}
$$

where $E_{a,b}(z)$ denotes the Mittag-Leffler function.

In the next result, we consider the case $0 < \mu < 1$.

Theorem 3.5. *Let $A := \rho I$, where $\rho \in \mathbb{R}$ is given. Suppose that $\beta = -\rho$, $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$ and $\alpha\rho < 0$ for $0 < \mu < 1$. Let $f \in \mathcal{N}(\mathbb{X})$. Consider the equation*

$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^{t} \frac{(t-s)^{\mu-1}}{\Gamma(\mu)} e^{-\beta(t-s)} u(s)\, ds + f(t), \quad t \in \mathbb{R}. \tag{3.6}$$

Then, equation (3.6) has a unique mild solution u which belongs to $\mathcal{N}(\mathbb{X})$ and is given by

$$u(t) = \int_{-\infty}^{t} S_\rho(t-s) f(s)\, ds, \quad t \in \mathbb{R}, \tag{3.7}$$

where $\{S_\rho(t)\}_{t\geq 0}$ is defined by (3.5).

Proof. Since $A = \rho I$ generates an immediately norm continuous C_0-semigroup and $\sigma(A) = \{\rho\}$, we have that $\lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \sigma(A)$ if and only if $\lambda(\lambda+\beta)^\mu - \rho(\lambda+\beta)^\mu - \alpha\rho = 0$. We claim that $S_\rho(t)$ is integrable. In fact, by [**12,** Theorem 1], there exists a constant $C_1 > 0$ depending only upon α, ρ and μ such that

$$|t^\mu E_{\mu+1,1}(\alpha\rho t^{\mu+1})| = |t^{(\mu+1)-1} E_{\mu+1,1}(\alpha\rho t^{\mu+1})| \leq \frac{C_1}{1 + |\alpha\rho| t^{\mu+1}},$$

for all $t \geq 0$, $x \in X$. Therefore,

$$\begin{aligned} \int_0^\infty \|S_\rho(t)\|\, dt &\leq \int_0^\infty |e^{-\beta t} t^{-\mu} t^{(\mu+1)-1} E_{\mu+1,1}(\alpha\rho t^{\mu+1})|\, dt \\ &\leq C_1 \int_0^\infty \frac{e^{-\beta t} t^{-\mu}}{1 + |\alpha\rho| t^{\mu+1}}\, dt \\ &\leq C_1 \int_0^\infty e^{-\beta t} t^{-\mu} dt \\ &= \frac{C_1}{\beta^{1-\mu}} \Gamma(1-\mu). \end{aligned}$$

We conclude that $S_\rho(t)$ is integrable, proving the claim. Therefore, from Theorem 3.2 and [**20,** Theorem 3.3], there exists a unique mild solution of equation (3.6) which belongs to $\mathcal{N}(\mathbb{X})$ and is explicitly given by (3.7). □

Remark 3.6. Unfortunately (to the best of our knowledge) if $\beta+\rho \neq 0$, there is not a known formula to the inverse Laplace transform of S_ρ in (3.4).

Example 3.7. Let $\rho = -1$, $\mu = 1/2$, $\alpha = 1/2$, $\beta = 1$. Hence, by Theorem 3.5, for any $f \in \mathcal{N}(\mathbb{X})$, there exists a unique mild solution $u \in \mathcal{N}(\mathbb{X})$ of the equation

$$u'(t) = -u(t) - \int_{-\infty}^{t} \frac{(t-s)^{-1/2}}{2\Gamma(1/2)} e^{-(t-s)} u(s)\, ds + f(t), \quad t \in \mathbb{R}, \tag{3.8}$$

given by

$$u(t) = \int_{-\infty}^{t} e^{-(t-s)} E_{3/2,1}\Big(- \frac{(t-s)^{3/2}}{2}\Big) f(s)\, ds, \quad t \in \mathbb{R},$$

since, $S_{-1}(t) = e^{-t} E_{3/2,1}(-t^{3/2}/2)$.

Remark 3.8. If we consider the *limit* case $\beta = 0$ in equation (3.1), we obtain the multi-term fractional differential equation

$$D^{\mu+1}u(t) = AD^{\mu}u(t) + \alpha Au(t) + F(t), \quad t \in \mathbb{R}, \tag{3.9}$$

where, here, D^μ denotes the Weyl fractional derivative (see **[24]**). Equations in the form of (3.9) have been widely studied by several authors (see **[1, 18, 26, 27]** and the references therein).

4. The semilinear case. In this section, we study the existence and uniqueness of solutions in $\mathcal{N}(\mathbb{X})$ for the semilinear integro-differential equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} a(t-s)Au(s)\, ds + f(t, u(t)), \quad t \in \mathbb{R}, \tag{4.1}$$

where $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$, A generates an immediately norm continuous C_0-semigroup on a Banach space $\mathbb{X}$, $\alpha, \beta, \mu \in \mathbb{R}$ are given and $f \in \mathcal{N}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$.

Definition 4.1. A function $u : \mathbb{R} \to \mathbb{X}$ is said to be a *mild* solution to equation (4.1) if

$$u(t) = \int_{-\infty}^{t} S(t-s) f(s, u(s))\, ds,$$

for all $t \in \mathbb{R}$, where $\{S(t)\}_{t \geq 0}$ is given in Proposition 3.1.

Theorem 4.2. *Let* $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$ *where* $\alpha \neq 0$, $\beta > 0$, $\mu \geq 1$ *and* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$. *Assume that*:

(a) *A generates an immediately norm continuous* C_0*-semigroup on a Banach space* $\mathbb{X}$;

(b) $\sup\{\mathrm{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^{\mu}((\lambda+\beta)^{\mu}+\alpha)^{-1} \in \sigma(A)\} < 0$.

If $f \in \mathcal{N}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *satisfies*

$$||f(t,u) - f(t,v)|| \leq L||u-v||, \tag{4.2}$$

for all $t \in \mathbb{R}$ *and* $u, v \in \mathbb{X}$, *where* $L < \omega/C$, *where* C *and* ω *are given by Proposition* 3.1. *Then, equation* (4.1) *has a unique mild solution* $u \in \mathcal{N}(\mathbb{X})$.

Proof. Define the operator

$$F : \mathcal{N}(\mathbb{X}) \longrightarrow \mathcal{N}(\mathbb{X})$$

by

$$(F\phi)(t) := \int_{-\infty}^{t} S(t-s) f(s, \phi(s))\, ds, \quad t \in \mathbb{R}, \tag{4.3}$$

where $\{S(t)\}_{t \geq 0}$ is given in Proposition 3.1. By **[20,** Theorems 3.3, 4.1], F is well defined and belongs to $\mathcal{N}(\mathbb{X})$. By Proposition 3.1, there exist $\omega > 0$ and $C > 0$ such that $||S(t)|| \leq Ce^{-\omega t}$ for all $t \geq 0$. For $\phi_1, \phi_2 \in \mathcal{N}(\mathbb{X})$ and $t \in \mathbb{R}$, we have: is a contraction; thus

$$\begin{aligned} ||(F\phi_1)(t) - (F\phi_2)(t)|| &\leq \int_{-\infty}^{t} ||S(t-s)[f(s,\phi_1(s)) - f(s,\phi_2(s))]||\, ds \\ &\leq \int_{-\infty}^{t} L\|S(t-s)\| \cdot \|\phi_1(s) - \phi_2(s)\|\, ds \end{aligned}$$

$$\leq L\|\phi_1 - \phi_2\|_\infty \int_0^\infty \|S(r)\|\, dr$$
$$\leq \frac{LC}{\omega}\|\phi_1 - \phi_2\|_\infty.$$

This proves that F is a contraction; thus, by the Banach fixed point theorem, there exists a unique $u \in \mathcal{N}(\mathbb{X})$ such that $Fu = u$. □

In Hilbert spaces, we have the following result.

Corollary 4.3. *Let A be the generator of a C_0-semigroup on a Hilbert space $\mathbb{H}$. Let $\alpha \neq 0$, $\beta > 0$ and $\mu \geq 1$ be such that* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$. *Assume that*:

(a) $\lim_{\mu\in\mathbb{R},|\mu|\to\infty} \|(\mu_0 + i\mu - A)^{-1}\| = 0$ *for some* $\mu_0 > s(A)$;
(b) $\sup\{\mathrm{Re}\lambda : \lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \sigma(A)\} < 0$.

If $f \in \mathcal{N}(\mathbb{R}\times\mathbb{H},\mathbb{H})$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{H}$, where $L < \omega/C$. Then, equation (4.1) *has a unique mild solution $u \in \mathcal{N}(\mathbb{H})$.*

In the special case $A = \rho I$, we obtain the following consequence of Theorem 3.5.

Corollary 4.4. *Let $A := \rho I$, where $\rho \in \mathbb{R}$ is given. Suppose that $\beta = -\rho$,* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$ *and $\alpha\rho < 0$ for $0 < \mu < 1$. Let $f \in \mathcal{N}(\mathbb{R}\times\mathbb{X},\mathbb{X})$. Consider the equation*

(4.4)
$$u'(t) = \rho u(t) + \rho\alpha \int_{-\infty}^t \frac{(t-s)^{\mu-1}}{\Gamma(\mu)} e^{-\beta(t-s)} u(s)\, ds + f(t,u(t)), \quad t \in \mathbb{R}.$$

If f satisfies $||f(t,u) - f(t,v)|| \leq L||u - v||$, for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $L < \beta^{1-\mu}/(C_1\Gamma(1-\mu))$, and C_1 is given in the proof of Theorem 3.5, *then equation* (4.4) *has a unique solution $u \in \mathcal{N}(\mathbb{X})$, given by*

$$u(t) = \int_{-\infty}^t S_\rho(t-s) f(s,u(s))\, ds, \quad t \in \mathbb{R}, \tag{4.5}$$

where $\{S_\rho(t)\}_{t\geq 0}$ is defined in (3.5).

Theorem 4.5. *Let* $a(t) := \alpha(t^{\mu-1}/\Gamma(\mu))e^{-\beta t}$*, where* $\alpha \neq 0$*,* $\beta > 0$*,* $\mu \geq 1$ *and* $\mathrm{Re}((-\alpha)^{1/\mu} - \beta) < 0$*. Assume that:*

(a) *A generates an immediately norm continuous* C_0*-semigroup on a Banach space* $\mathbb{X}$*;*
(b) $\sup\{\mathrm{Re}\lambda, \lambda \in \mathbb{C} : \lambda(\lambda+\beta)^\mu((\lambda+\beta)^\mu + \alpha)^{-1} \in \sigma(A)\} < 0$*.*

If $f \in \mathcal{N}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ *satisfies*

$$||f(t,u) - f(t,v)|| \leq L_f(t)||u-v||,$$

for all $t \in \mathbb{R}$ *and* $u, v \in \mathbb{X}$*, where* $L_f \in L^1(\mathbb{R})$*. Then, equation* (4.1) *has a unique mild solution* $u \in \mathcal{N}(\mathbb{X})$*.*

Proof. The proof follows similarly to that of [**21,** Theorem 4.2]. □

We conclude this paper with the following application.

Example 4.6. Consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{\partial^2 u}{\partial x^2}(t,x) - \displaystyle\int_{-\infty}^{t} \dfrac{(t-s)}{2\Gamma(2)} \\ \qquad\qquad \times e^{-(t-s)} \dfrac{\partial^2 u}{\partial x^2}(s,x)\, ds + f(t,u(t)) \\ u(0,t) = u(\pi,t) = 0, \end{cases} \tag{4.6}$$

with $x \in [0,\pi]$ and $t \in \mathbb{R}$. Let $\mathbb{X}, A$ and $D(A)$ be as in Example 2.7. Then, we can write the problem (4.6) in the abstract form (4.1) with $\alpha = -1/2$, $\beta = 1$ and $\mu = 2$. Since A generates an immediately continuous norm, with $\sigma(A) = \{-n^2 : n \in \mathbb{N}\}$, then the solutions to equation

$$\frac{\lambda(\lambda+1)^2}{(\lambda+1)^2 - (1/2)} = -n^2,$$

are given by

$$\lambda_{n,1} = \frac{-(n^2+2)}{3} - \frac{6a_n}{c_n} + \frac{1}{6}c_n,$$

$$\lambda_{n,2} = \frac{-(n^2+2)}{3} + 3\frac{a_n}{c_n} - \frac{c_n}{12} + \frac{\sqrt{3}}{2}\left(\frac{1}{6}c_n + 6\frac{a_n}{c_n}\right)i$$

$$\lambda_{n,3} = \frac{-(n^2+2)}{3} + 3\frac{a_n}{c_n} - \frac{c_n}{12} - \frac{\sqrt{3}}{2}\left(\frac{1}{6}c_n + 6\frac{a_n}{c_n}\right)i$$

for all $n \geq 1$, where

$$a_n = -\frac{1}{9} + \frac{2}{9}n^2 - \frac{1}{9}n^4,$$

and

$$c_n := \left(8 + 30n^2 + 24n^4 - 8n^6 + 6n\sqrt{24 + 9n^2 + 72n^4 - 24n^6}\right)^{1/3}.$$

Simple computation shows that

$$\sup\left\{\mathrm{Re}\lambda : \lambda(\lambda+1)^2\left((\lambda+1)^2 - \frac{1}{2}\right)^{-1} \in \sigma(A)\right\} < 0.$$

Therefore, from Proposition 3.1, we conclude that there exists a strongly continuous family of operators $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{X})$ such that $\|S(t)\| \leq Ce^{-\omega t}$ for some $C, \omega > 0$. Thus, if $f \in \mathcal{N}(\mathbb{R} \times \mathbb{X}, \mathbb{X})$ satisfies

$$||f(t,u) - f(t,v)|| \leq L||u - v||,$$

for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $L < \omega/C$, then equation (4.6) has a unique mild solution $u \in \mathcal{N}(\mathbb{X})$, by Theorem 4.2. On the other hand, if $||f(t,u) - f(t,v)|| \leq L_f(t)||u - v||$, for all $t \in \mathbb{R}$ and $u, v \in \mathbb{X}$, where $L_f \in L^1(\mathbb{R})$, then, by Theorem 4.5, equation (4.6) has a unique mild solution $u \in \mathcal{N}(\mathbb{X})$.

Acknowledgments. The authors thank the anonymous referees for their carefully reading of the manuscript and for making suggestions which improved the previous version of this paper.

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