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A BLOW-UP RESULT TO A DELAYED CAUCHY VISCOELASTIC PROBLEM

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ABSTRACT. In this paper, we consider a Cauchy problem for a nonlinear viscoelastic equation with delay. Under suitable conditions on the initial data and the relaxation function, in the whole space, we prove a finite-time blow-up result.

1. Introduction. In this work, we are concerned with the following delayed Cauchy problem

$$
\begin{cases}
u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)\,ds & \\
\quad +\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = |u|^{p-1}\,u & x \in \mathbb{R}^n,\ t>0, \\
u_t\,(x,t-\tau) = f_0\,(x,t-\tau) & \text{in } (0,\tau)\,, \\
u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) & x\in\mathbb{R}^n,
\end{cases}
\tag{1.1}
$$

where $p>1$, μ_1 is a positive constant, μ_2 is a real number and $\tau>0$ represents the time delay. The function g is the relaxation function subjected to conditions to be specified and u_0, u_1 and f_0 are the initial data to be specified later. This type of problem arises in viscoelasticity and in systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann model.

Viscoelastic wave problems in bounded domains were considered by many authors. For instance, Messaoudi **[16]** considered the following initial-boundary value problem:

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$$(1.2)\quad \begin{cases} u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + u_t|u_t|^{m-2} = u|u|^{p-2} & \Omega \times (0,\infty), \\ u(x,t) = 0 & x \in \partial\Omega, \quad t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & x \in \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$, with a smooth boundary $\partial\Omega$, $p > 2$, $m \geq 1$ and

$$g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$$

is a positive nonincreasing function. He showed, under suitable conditions on g, that solutions with initial negative energy blow up in finite time if $p > m$ and continue to exist if $m \geq p$. This result was later pursued, by the same author [**18**], for certain solutions with positive initial energy. A similar result has also been obtained by Wu [**28**] using a different method.

In the absence of the viscoelastic term ($g = 0$), the problem has been extensively studied, and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$(1.3)\qquad u_{tt} - \Delta u + au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0,\infty),$$

$m, \gamma \geq 0$, it is well known that, for $a = 0$, the source term $bu|u|^\gamma$, $\gamma > 0$, causes finite time blow up of solutions with negative initial energy (see [**3**]). The interaction between the damping and the source terms was first considered by Levine [**10, 11**] in the linear damping case, $m = 0$. He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [**5**] extended Levine's result to the nonlinear damping case, $m > 0$. In their work, the authors introduced a different method and showed that solutions with negative energy continue to exist globally 'in time' if $m \geq p$, blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended to solutions with negative initial energy by Messaoudi [**15**] and others. For results of the same nature, the reader is referred to Levine and Serrin [**13**], Vitillaro [**27**] and Messaoudi and Said-Houari [**19**].

For problem (1.2) in $\mathbb{R}^n$, we mention, among others, the work of Levine Serrin and Park [**12**], Todorova [**25, 26**], Messaoudi [**17**], Zhou [**29**], Kafini and Messaoudi [**7**] and Kafini [**6**].

Time delays arise in many applications since, in most instances, physical, chemical, biological, thermal and economic phenomena naturally not only depend on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research. In many cases, it was shown that delay is a source of instability, and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. For instance, for the system

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) & \\ \qquad +a_0 u_t(x,t) + a u_t(x,t-\tau) = 0 & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0 & \text{in } \Gamma_0 \times (0,\infty), \\ (\partial u/\partial v)(x,t) = 0 & \text{in } \Gamma_1 \times (0,\infty), \end{cases} \tag{1.4}$$

it is well known, in the absence of delay ($a = 0$, $a_0 > 0$), that this system is exponentially stable, see [**14, 30**]. In the presence of delay ($a > 0$), Nicaise and Pignotti [**21**] examined system (1.4) and proved, under the assumption that the weight of the feedback with delay is smaller than that without delay, i.e., $0 < a < a_0$, that the energy is exponentially stable. However, in the opposite case, they could produce a sequence of delays for which the corresponding solution is instable. The same results have been obtained for the case of boundary delay. For the treatment of this problem in more general abstract form, see [**2**], and for analogous results in the case of time-varying delay, see [**20, 23, 24**]. When the delay term in (1.4) is replaced by the distributed delay

$$\int_{\tau_1}^{\tau_2} a(s) u_t(x,t-s)\, ds,$$

exponential stability results have been obtained in [**22**] under the condition

$$\int_{\tau_1}^{\tau_2} a(s)\, ds < a_0.$$

Delay results with different systems can also be found in [**8, 9**].

Our aim in the present work is to extend the existing results established for the wave equation to our delayed Cauchy problem. To our knowledge, it is the first time a delayed Cauchy problem has been discussed. To achieve our goal, some conditions must be imposed on the relaxation function g and to the initial data as well. The paper is organized as follows. In Section 2, we present the transformation of the problem as well as assumptions on g in addition to the existence result. Section 3 is devoted to the statement and the proof of the main result.

2. Preliminaries. In this section, we transform the equation in (1.1) to a system, using an idea of **[21]** and introduce the associated energy. The reader is referred to **[1]** for the existence of solutions to nonlinear problems with delay.

Therefore, we introduce the new variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), \quad x \in \mathbb{R}^n,\ \rho \in (0,1),\ t > 0.$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \mathbb{R}^n,\ \rho \in (0,1),\ t > 0.$$

Then, problem (1.1) takes the form

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) & \\ \quad + \int_0^t g(t-s)\Delta u(x,s)\,ds & \\ \quad +\mu_1 u_t(x,t) + \mu_2 z(x,1,t) = |u(x,t)|^{p-1}u(x,t) & \text{in } \mathbb{R}^n \times (0,\infty), \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 & \text{in } \mathbb{R}^n \times (0,1) \\ & \qquad \times(0,\infty), \\ z(x,\rho,0) = f_0(x,-\rho\tau) & \text{in } \mathbb{R}^n \times (0,1), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2.1}$$

We introduce the "modified" energy functional

$$\begin{aligned} E(t) := {} & \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)||\nabla u(t)||_2^2 + \frac{1}{2}||u_t||_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ & + \frac{\xi}{2}\int_{\mathbb{R}^n}\int_0^1 z^2(x,\rho,t)\,d\rho\,dx - \frac{1}{p+1}||u(t)||_{p+1}^{p+1}, \end{aligned} \tag{2.2}$$

where

$$(g \circ v)(t) = \int_0^t g(t-\tau)||v(t) - v(\tau)||_2^2\,d\tau,$$

for $t \geq 0$ and

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|), \quad \mu_1 > |\mu_2|. \tag{2.3}$$

The next lemma shows that the associated energy of the problem decreases under the condition $\mu_1 > |\mu_2|$ and the assumption

(G1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a differentiable function such that

$$1 - \int_0^\infty g(s)\, ds = l > 0, \quad g'(t) \le 0,\ t \ge 0.$$

Lemma 2.1. *Let u be the solution of* (2.1)*. Then, there exists a positive constant C_0 such that*

$$E'(t) \le -C_0 \Big[\int_{\mathbb{R}^n} \left(u_t^2 + z^2(x,1,t)\right)\, dx - (g' \circ \nabla u) + g(s)\|\nabla u\|_2^2 \Big] \le 0. \tag{2.4}$$

Proof. Multiplying equation $(2.1)_1$ by u_t and integrating over $\mathbb{R}^n$ and $(2.1)_2$ by $(\xi/\tau)z$ and integrating over $(0,1)\times\mathbb{R}^n$ with respect to ρ and x, summing up, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\frac{1}{2}||u_t||_2^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\, ds\right)\right. \\ &\qquad \left. ||\nabla u(t)||_2^2 + \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p+1}\|u\|_{p+1}^{p+1}\right) \\ &\qquad + \frac{\xi}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\int_0^1 z^2(x,\rho,t)\, d\rho\, dx \\ &= -\mu_1 \int_{\mathbb{R}^n} u_t^2 dx + \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2} g(t)\|\nabla u\|_2^2 \\ &\qquad - \frac{\xi}{\tau}\int_{\mathbb{R}^n}\int_0^1 z z_\rho(x,\rho,t)\, d\rho\, dx \\ &\qquad - \mu_2 \int_{\mathbb{R}^n} u_t z(x,1,t)\, dx. \end{aligned} \tag{2.5}$$

Now, we estimate the last two terms of the right-hand side in (2.5) as follows:

$$\begin{aligned} &-\frac{\xi}{\tau}\int_{\mathbb{R}^n}\int_0^1 z z_\rho(x,\rho,t)\, d\rho\, dx \\ &\quad = -\frac{\xi}{2\tau}\int_{\mathbb{R}^n}\int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,t)\, d\rho\, dx \end{aligned}$$

$$= \frac{\xi}{2\tau} \int_{\mathbb{R}^n} (z^2(x,0,t) - z^2(x,1,t))\, dx$$
$$= \frac{\xi}{2\tau} \left(\int_{\mathbb{R}^n} u_t^2 dx - \int_{\mathbb{R}^n} z^2(x,1,t)\, dx \right)$$

and

$$-\mu_2 \int_{\mathbb{R}^n} u_t z(x,1,t) dx \leq \frac{|\mu_2|}{2} \left(\int_{\mathbb{R}^n} u_t^2\, dx + \int_{\mathbb{R}^n} z^2(x,1,t)\, dx \right).$$

Hence, we obtain

$$\begin{aligned} \frac{dE(t)}{dt} \leq & -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \int_{\mathbb{R}^n} u_t^2\, dx \\ & - \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right) \int_{\mathbb{R}^n} z^2(x,1,t)\, dx \\ & + (g' \circ \nabla u) - g(t) \|\nabla u\|_2^2. \end{aligned}$$

Using (2.3), we have, for some $C_0 > 0$,

$$E'(t) \leq -C_0 \left[\int_{\mathbb{R}^n} (u_t^2 + z^2(x,1,t))\, dx - (g' \circ \nabla u) + g(t) \|\nabla u\|_2^2 \right] \leq 0. \quad \square$$

Now, we state without proof, a local existence result, which can be established by combining the arguments of [**2**] and [**4**].

Proposition 2.2. *Assume that* (G1) *holds and*

$$1 < p \leq \frac{n}{n-2}, \tag{2.6}$$

whenever $n \geq 3$ *and* $p \geq 1$, *whenever* $n = 1, 2$. *Then, for any initial data*

$$u_0 \in H^1(\mathbb{R}^n), \quad u_1 \in L^2(\mathbb{R}^n),$$

with compact supports, problem (1.1) *has a unique local solution*

$$\begin{aligned} & u \in C([0,T); H^1(\mathbb{R}^n)), \\ & u_t \in C([0,T); L^2(\mathbb{R}^n)) \cap L^2([0,T) \times \mathbb{R}^n), \end{aligned}$$

for T *small enough.*

Remark 2.3. (G1) is necessary to guarantee the hyperbolicity of system (1.1).

In order to state and prove our main result, we need the following.

Lemma 2.4 ([**29**]). *Suppose that* Ψ *is a twice continuously differentiable function satisfying*

$$\begin{aligned} \Psi''(t) + \Psi'(t) &\geq C_0(t+L)^{\beta}\Psi^{1+\alpha}(t), \quad t > 0, \\ \Psi(0) > 0, &\qquad \Psi'(0) \geq 0, \end{aligned} \tag{2.7}$$

where $C_0, L > 0$, $-1 < \beta \leq 0$, $\alpha > 0$ *are constants. Then,* Ψ *blows up in finite time.*

3. Blow up.

Our main result is as follows.

Theorem 3.1. *Assume that* (G1) *and* (2.6) *hold. Assume further that*

$$\int_0^t g(s)\,ds < \frac{2p-2}{2p-1}. \tag{3.1}$$

Then, for any initial data $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$, *with compact supports satisfying*

$$\begin{aligned} E(0) = \frac{1}{2}\|u_1\|_2^2 + \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{1}{p+1}\|u_0\|_{p+1}^{p+1} \\ + \frac{\xi}{2}\int_{\mathbb{R}^n}\int_0^1 f_0(x, -\rho\tau)\,d\rho\,dx < 0 \end{aligned} \tag{3.2}$$

and

$$\int_{\mathbb{R}^n} u_0 u_1\,dx \geq 0, \tag{3.3}$$

the corresponding solution blows up in finite time.

Remark 3.2. An example of a function g which satisfies (G1) and (3.1) is

$$g(t) = e^{-\alpha t} \quad \text{for any } \alpha > \frac{2p-1}{2p-2}.$$

Proof of Theorem 3.1. From Lemma 2.1 and (3.2), we have

$$E(t) \leq E(0) < 0. \tag{3.4}$$

In order to apply Lemma 2.4, we define

$$\Psi(t)=\frac{1}{2}\int_{\mathbb{R}^n}|u(x,t)|]^2dx+\int_0^t\left[\frac{\mu_1}{2}||u||_2^2-NE(s)\right]ds, \tag{3.5}$$

where $N>0$ is to be specified later.

Therefore,

$$\begin{aligned}\Psi'(t)&=\int_{\mathbb{R}^n}uu_t\,dx+\frac{\mu_1}{2}||u||_2^2-NE(t),\\ \Psi''(t)&=\int_{\mathbb{R}^n}uu_{tt}\,dx+\int_{\mathbb{R}^n}|u_t|^2dx+\mu_1\int_{\mathbb{R}^n}uu_t\,dx-NE'(t).\end{aligned}\tag{3.6}$$

It is clear, from (3.2)–(3.6), that

$$\Psi(0)>0,\qquad \Psi'(0)\geq 0.$$

We then use equation (2.1) to estimate

$$\begin{aligned}\int_{\mathbb{R}^n}uu_{tt}\,dx=&-\int_{\mathbb{R}^n}|\nabla u|^2dx\\ &+\int_{\mathbb{R}^n}\nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\ &-\mu_1\int_{\mathbb{R}^n}uu_t\,dx-\mu_2\int_{\mathbb{R}^n}uz(x,1,t)\,dx\\ &+\int_{\mathbb{R}^n}|u|^{p+1}dx.\end{aligned}\tag{3.7}$$

Using

$$\begin{aligned}&\int_{\mathbb{R}^n}\nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx\\ &\quad=\int_0^t g(t-s)\int_{\mathbb{R}^n}\nabla u(t)\cdot[\nabla u(s)-\nabla u(t)]\,dx\,ds\\ &\qquad\qquad+\left(\int_0^t g(s)\,ds\right)\int_{\mathbb{R}^n}|\nabla u(t)|^2dx,\end{aligned}\tag{3.8}$$

equations (3.6)–(3.8) give

$$
\begin{aligned}
\Psi''(t) + \Psi'(t) = &-\left(1 - \int_0^t g(s)\,ds\right) \int_{\mathbb{R}^n} |\nabla u|^2 dx \\
&- \int_0^t g(t-s) \int_{\mathbb{R}^n} \nabla u(t) \cdot [\nabla u(t) - \nabla u(s)]\,dx\,ds \\
&+ \int_{\mathbb{R}^n} u u_t\,dx - \mu_2 \int_{\mathbb{R}^n} u z(x,1,t)\,dx \\
&+ \int_{\mathbb{R}^n} |u|^{p+1} dx \\
&+ \int_{\mathbb{R}^n} |u_t|^2 dx + \frac{\mu_1}{2} ||u||_2^2 - NE(t) - NE'(t).
\end{aligned} \tag{3.9}
$$

Then, we use Young's inequality to estimate terms in (3.9) as follows.

The second term, for any $\delta > 0$,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \nabla u(t) \int_0^t g(t-s)[\nabla u(t) - \nabla u(s)]\,ds\,dx \\
&\quad \leq \delta \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{4\delta} \int_{\mathbb{R}^n} \left| \int_0^t g(t-s)[\nabla u(t) - \nabla u(s)]\,ds \right|^2 dx \\
&\quad \leq \delta \int_{\mathbb{R}^n} |\nabla u(t)|^2 dx + \frac{1}{4\delta} \left(\int_0^t g(s)\,ds \right) (g \circ \nabla u);
\end{aligned} \tag{3.10}
$$

the third term, for any $\delta_1 > 0$,

$$
\int_{\mathbb{R}^n} u u_t\,dx \leq \delta_1 \int_{\mathbb{R}^n} u^2 dx + \frac{1}{4\delta_1} \int_{\mathbb{R}^n} u_t^2\,dx, \tag{3.11}
$$

and the fourth term, for any $\delta_1 > 0$,

$$
\mu_2 \int_{\mathbb{R}^n} u z(x,1,t)\,dx \leq |\mu_2| \left(\delta_1 \int_{\mathbb{R}^n} u^2 dx + \frac{1}{4\delta_1} \int_{\mathbb{R}^n} z^2(x,1,t)\,dx \right) \tag{3.12}
$$

are as shown above. By combining (2.4), (3.4), (3.9), (3.11) and (3.12), we get

$$\begin{aligned}\Psi''(t)+\Psi'(t) \geq & -\left(1+\delta-\int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ & +\int_{\mathbb{R}^n}|u|^{p+1}dx-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ\nabla u) \\ & +\left(\frac{\mu_1}{2}-\delta_1(1+|\mu_2|)\right)\int_{\mathbb{R}^n}u^2dx \\ & +\left(NC_0-\frac{|\mu_2|}{4\delta_1}\right)\int_{\mathbb{R}^n}z^2(x,1,t)\,dx \\ & +\left(1+NC_0-\frac{1}{4\delta_1}\right)\int_{\mathbb{R}^n}u_t^2\,dx.\end{aligned}\tag{3.13}$$

At this stage, we choose δ_1 small enough such that

$$\frac{\mu_1}{2}-\delta_1(1+|\mu_2|)>0$$

and N large enough such that

$$\begin{aligned}NC_0-\frac{|\mu_2|}{4\delta_1}&>0,\\ 1+NC_0-\frac{1}{4\delta_1}&>0.\end{aligned}$$

Therefore, (3.13) becomes

$$\begin{aligned}\Psi''(t)+\Psi'(t) \geq & -\left(1+\delta-\int_0^t g(s)\,ds\right)\|\nabla u\|^2 \\ & -\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)(g\circ\nabla u)+\int_{\mathbb{R}^n}|u|^{p+1}dx.\end{aligned}\tag{3.14}$$

Now, we exploit (2.2) to substitute for $\|\nabla u\|^2$. Thus, (3.14) takes the form

$$\begin{aligned}\Psi''(t)+\Psi'(t) \geq & -2\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}E(t)+\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}||u_t||_2^2 \\ & +\left[\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\right](g\circ\nabla u)(t)\end{aligned}\tag{3.15}$$

$$+\left[1-\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}\frac{2}{p+1}\right]||u||_{p+1}^{p+1}.$$

At this point, we choose $\delta>0$ so that

$$\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}-\frac{1}{4\delta}\left(\int_0^t g(s)\,ds\right)\geq 0$$

and

$$1-\frac{(1+\delta-\int_0^t g(s)\,ds)}{(1-\int_0^t g(s)\,ds)}\frac{2}{p+1}>0.$$

This is, of course, possible by (3.1). Thus, (3.15) becomes, for $\gamma>0$,

$$\Psi''(t)+\Psi'(t)\geq\gamma||u||_{p+1}^{p+1}. \tag{3.16}$$

Now, we use Hölder's inequality to estimate $||u||_{p+1}^{p+1}$, as follows.

(3.17)

$$\int_{\mathbb{R}^n}|u|^2dx\leq\left(\int_{\mathbb{R}^n}|u|^{p+1}dx\right)^{2/(p+1)}\left(\int_{B(t+L)}1\,dx\right)^{(p-1)/(p+1)},$$

where $L>0$ is such that

$$\operatorname{supp}\{u_0(x),u_1(x)\}\subset B(L),$$

and $B(t+L)$ is the ball, with radius $t+L$ centered at the origin. If we call W_n the volume of the unit ball, then

$$\int_{\mathbb{R}^n}|u|^{p+1}dx\geq\left(\int_{\mathbb{R}^n}|u|^2dx\right)^{(p+1)/2}(W_n(t+L)^n)^{(1-p)/2}. \tag{3.18}$$

From the definition of $\Psi(t)$ in (3.5), we have

$$2\Psi(t)=\int_{\mathbb{R}^n}|u(x,s)|^2dx+\int_0^t[\mu_1||u||_2^2-2NE(s)]\,ds.$$

Therefore,

$$\begin{aligned}(2\Psi(t))^{(p+1)/2}&=\left[\int_{\mathbb{R}^n}|u(x,t)|^2dx+\int_0^t(\mu_1||u||_2^2-2NE(s))\,ds\right]^{(p+1)/2}\\&\leq 2^{(p-1)/2}\left[\left(\int_{\mathbb{R}^n}|u(x,t)|^2dx\right)^{(p+1)/2}\right.\end{aligned}$$

$$+\left(\int_0^t (\mu_1 ||u||_2^2 - 2NE(s))\, ds\right)^{(p+1)/2}\Bigg],$$

which implies that
(3.19)

$$\begin{aligned}\left(\int_{\mathbb{R}^n} |u(x,t)|^2 dx\right)^{(p+1)/2} &\geq 2(\Psi(t))^{(p+1)/2} \\ &\quad - \left(\int_0^t (\mu_1 ||u||_2^2 - 2NE(s))\, ds\right)^{(p+1)/2} \\ &\geq (\Psi(t))^{(p+1)/2}.\end{aligned}$$

Finally, estimation (3.16) and (3.19) imply that

$$\Psi''(t) + \Psi'(t) \geq \gamma(\Psi(t))^{(p+1)/2}(W_n(t+L)^n)^{(1-p)/2}.$$

It is easy to verify that the requirements of Lemma 2.4 are satisfied with

$$C_0 = \gamma(W_n)^{(1-p)/2} > 0,$$
$$-1 < \beta = \frac{n(1-p)}{2} < 0, \quad \alpha = \frac{p-1}{2} > 0.$$

Therefore, Ψ blows up in finite time. This completes the proof. $\square$

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