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BLOW-UP OF SOLUTIONS FOR SEMILINEAR FRACTIONAL SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider the Cauchy problem in $\mathbb{R}^N$, $N \geq 1$, for the semi-linear Schrödinger equation with fractional Laplacian. We present the local well-posedness of solutions in $H^{\alpha/2}(\mathbb{R}^N)$, $0 < \alpha < 2$. We prove a finite-time blow-up result, under suitable conditions on the initial data.

1. Introduction. We study the initial-value problem for the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u = \Lambda^\alpha u + \lambda|u|^p & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(x,0) = f(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the pseudo-differential operator $\Lambda^\alpha := (-\Delta)^{\alpha/2}$ with $0 < \alpha < 2$ is defined by the Fourier transformation: $\widehat{\Lambda^\alpha u}(\xi) = |\xi|^\alpha \widehat{u}(\xi)$. Moreover, we assume that $T > 0$, $p > 1$, $u = u(x,t)$ is a complex-valued unknown function, $\lambda \in \mathbb{C} \setminus \{0\}$ and $f = f(x) \in H^{\alpha/2}(\mathbb{R}^N)$ is a given complex-valued function.

In recent years, the study of fractional calculus and fractional integrodifferential equations applied to physics and other areas has grown, see **[8, 12, 13]** and the references therein. Meltzler and Klafter discussed recent developments in the description of anomalous diffusion with the fractional dynamics approach in **[12, 13]** where many fractional partial differential equations are asymptotically derived from Lévy random walk models, a natural generalization of the Brownian walk models. Inspired by the Feynman path approach to quantum mechanics, Laskin used the path integral over Lévy-like quantum mechan-

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ical paths to obtain a fractional Schrödinger equation, which extends a classical result that the path integral over Brownian trajectories leads to the standard Schrödinger equations, (see [**10, 11**]). There are also papers that address fractional Schrödinger equations and their applications, see e.g., [**5, 16**].

When $\alpha = 2$, i.e.,

$$\begin{cases} i\partial_t u + \Delta u = \lambda |u|^p & (t,x) \in (0,T) \times \mathbb{R}^N, \\ u(x,0) = f(x) & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

it is well known, see [**3**], that local well-posedness holds for (1.2) in $H^1(\mathbb{R}^N)$ if $1 < p < 1 + (4/(N-2)_+)$. Moreover, it is also known that the local solutions can be globally extended for some small data when p is larger than the Strauss exponent p_s, which is the positive root of $Np^2 - (N+2)p - 2 = 0$, see [**2**]. However, there have been no results on global existence for $p \leq p_s$. In 2013, Ikeda and Wakasugi [**7**] proved a small-data blow-up result for (1.2) when $1 < p \leq 1 + 2/N$. For more information on the semilinear Schrödinger equations without gauge invariance, we refer the reader to [**6**].

The main goal in this paper is to generalize the blow-up result of Ikeda and Wakasugi [**7**] to the fractional Schrödinger equations (1.1). The local existence is accomplished by the Banach fixed point theorem, using semigroup theory and Stone's theorem on the fractional operator $A = -i(-\Delta)^{\alpha/2}$, which is the infinitesimal generator of a C_0 group of unitary operator on L^2, see [**3**]. The method used to prove the blow-up result is the test function method. This method was introduced by Baras and Kersner [**1**] in 1987 and developed by Zhang [**17**], Pohozaev and Mitidieri [**14**] in 2001. It was also used by Kirane, et al., [**9**] in 2002.

The paper is organized as follows. In Section 2, we present local existence of solutions for (1.1) with some properties. Section 3 contains the blow-up result of solutions for (1.1).

2. Local existence. This section is dedicated to showing the local existence and uniqueness of mild solutions of problem (1.1). Let $Au = -i(-\Delta)^{\alpha/2}u$. By applying Stone's theorem [**15,** theorem 1.10.8], we conclude that A is the infinitesimal generator of a C_0 group of unitary operators $S(t)$, $-\infty < t < \infty$, on $L^2(\mathbb{R}^N)$. We begin by giving the following definition.

Definition 2.1 (Mild solution)**.** Let $f \in H^{\alpha/2}(\mathbb{R}^N)$, $0 < \alpha < 2$, $p > 1$ and $T > 0$. We say that $u \in C([0,T], H^{\alpha/2}(\mathbb{R}^N))$ is a mild solution of problem (1.1) if u satisfies the following integral equation:

$$u(t) = S(t)f - i\lambda \int_0^t S(t-s)|u(s)|^p \, ds. \tag{2.1}$$

We set

$$p_0 = \begin{cases} \infty & \text{if } n = 1, \\ 1 + \dfrac{2(\alpha-1)}{\alpha(2-\alpha)} & \text{if } n = 2, \\ 1 + \dfrac{n(\alpha-1)}{(n-1)(n-\alpha)} & \text{if } n \geq 3. \end{cases}$$

Theorem 2.2 (Local existence)**.** *Given $f \in H^{\alpha/2}(\mathbb{R}^N)$, $\lambda \in \mathbb{C}\backslash\{0\}$, $0 < \alpha < 2$ and $1 < p < 1 + (2\alpha/(N-\alpha)_+)$, there exist $T > 0$ and a mild solution $u \in C([0,T], H^{\alpha/2}(\mathbb{R}^N))$ of (1.1). Moreover, if $1 < \alpha < 2$ and $1 < p < p_0$, then the solution u is unique, and therefore, there exist a maximal time $T_{\max} > 0$ and a unique mild solution $u \in C([0,T_{\max}), H^{\alpha/2}(\mathbb{R}^N))$ of (1.1). Furthermore, either $T_{\max} = \infty$ or else $T_{\max} < \infty$ and $\|u\|_{H^{\alpha/2}(\mathbb{R}^N)} \to \infty$ as $t \to T_{\max}$.*

Proof. Cho, et al., [**4,** Propositions 4.1–4.3] have shown, using the Banach fixed-point theorem, that there exists a unique mild solution $u \in \Pi_T := C([0,T], H^{\alpha/2}(\mathbb{R}^N))$ of (1.1). Using the uniqueness of solution, we conclude the existence of a solution on a maximal interval $[0, T_{\max})$, where

$$\begin{aligned} T_{\max} &:= \sup\{T > 0;\ \text{there exists a mild solution } u \in \Pi_T \text{ to (1.1)}\} \\ &\leq +\infty. \end{aligned}$$

Next, we prove that $\|u\|_{H^{\alpha/2}} \to \infty$ as $t \to T_{\max}$. We suppose

$$\liminf_{t \to T_{\max}} \|u\|_{H^{\alpha/2}} < \infty.$$

Then, we can find a sequence $\{t_k\}_{k\in\mathbb{N}} \subset [0, T_{\max})$ and a positive constant $M > 0$ such that

$$\lim_{k\to\infty} t_k = T_{\max} \tag{2.2}$$

and

$$\sup_{k\in\mathbb{N}} \|u(t_k)\|_{H^{\alpha/2}} \le M. \tag{2.3}$$

From (2.3) and the first part of Theorem 2.2, we can construct a solution $u \in C([t_k, t_k + T(M)); H^{\alpha/2}(\mathbb{R}^N))$ of (2.1) for all $k \in \mathbb{N}$ with some $T(M) > 0$. However, by (2.2), we can take t_k satisfying $t_k + T(M) > T_{\max}$, which contradicts the definition of $T_{\max}$. Therefore, we obtain

$$\liminf_{t\to T_{\max}} \|u\|_{H^{\alpha/2}} = \infty. \qquad \square$$

3. Blow-up of solutions. This section is devoted to deriving the blow-up result of (1.1). We define the following.

Definition 3.1 (Weak solution)**.** Let $f \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $T > 0$. We say that u is a weak solution of problem (1.1) if $u \in L^p_{\text{loc}}((0,T) \times \mathbb{R}^N)$ which verifies the following weak formulation:

$$\begin{aligned} & i\int_{\mathbb{R}^N} f(x)\varphi(x,0) + \lambda\int_0^T\int_{\mathbb{R}^N} |u|^p\varphi(x,t) \\ & = -\int_0^T\int_{\mathbb{R}^N} u(x,t)\Lambda^\alpha\varphi(x,t) \\ & \quad - i\int_0^T\int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t), \end{aligned} \tag{3.1}$$

for all compactly supported real-valued functions $\varphi \in C_0^2([0,T] \times \mathbb{R}^N)$ such that $\varphi(\cdot\,,T) = 0$.

Lemma 3.2. *Consider* $f \in H^{\alpha/2}(\mathbb{R}^N)$*, and let* $u \in C([0,T], H^{\alpha/2}(\mathbb{R}^N))$ *be a mild solution of* (1.1)*. Then,* u *is a weak solution of* (1.1)*, for all* $T > 0$*.*

Proof. Let $T > 0$, $f \in H^{\alpha/2}(\mathbb{R}^N)$ and $u \in C([0,T], H^{\alpha/2}(\mathbb{R}^N))$ be a solution of (2.1). Given a real-valued function $\varphi \in C_0^2([0,T] \times \mathbb{R}^N)$ such that suppφ is compact and $\varphi(\cdot\,,T) = 0$. Then, after multiplying (2.1) by φ and integrating over $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) = \int_{\mathbb{R}^N} S(t)f(x)\varphi(x,t) - i\lambda \int_{\mathbb{R}^N} \int_0^t S(t-s)|u(s)|^p \, ds\varphi(x,t).$$

We differentiate to obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t) &= \int_{\mathbb{R}^N} \frac{d}{dt}\left(S(t)f(x)\varphi(x,t)\right) \\ &\quad - i\lambda \int_{\mathbb{R}^N} \frac{d}{dt}\int_0^t S(t-s)|u(s)|^p \, ds\varphi(x,t). \end{aligned} \tag{3.2}$$

Now, using that A is a skew-adjoint operator and a property of the group $S(t)$ [**3,** Chapter 3], we have:

$$\begin{aligned} \int_{\mathbb{R}^N} \frac{d}{dt}\left(S(t)f(x)\varphi(x,t)\right)\, dx &= \int_{\mathbb{R}^N} A\left(S(t)f(x)\right)\varphi(x,t)\, dx \\ &\quad + \int_{\mathbb{R}^N} S(t)f(x)\varphi_t(x,t)\, dx \\ &= \int_{\mathbb{R}^N} S(t)f(x)A\varphi(x,t)\, dx \\ &\quad + \int_{\mathbb{R}^N} S(t)f(x)\varphi_t(x,t)\, dx, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} i\lambda \int_{\mathbb{R}^N} \frac{d}{dt}\int_0^t S(t-s)|u(s)|^p \, ds\varphi(x,t)\, dx &= i\lambda \int_{\mathbb{R}^N} |u(t)|^p\varphi(x,t)\, dx \\ &\quad + i\lambda \int_{\mathbb{R}^N}\int_0^t A\left(S(t-s)|u(s)|^p\right)\, ds\varphi(x,t) \\ &\quad + i\lambda \int_{\mathbb{R}^N}\int_0^t S(t-s)|u(s)|^p\, ds\varphi_t(x,t)\, dx \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
&= i\lambda \int_{\mathbb{R}^N} |u(t)|^p \varphi(x,t)\,dx \\
&\quad + i\lambda \int_{\mathbb{R}^N} \int_0^t S(t-s)|u(s)|^p\,ds A\varphi(x,t) \\
&\quad + i\lambda \int_{\mathbb{R}^N} \int_0^t S(t-s)|u(s)|^p\,ds \varphi_t(x,t)\,dx.
\end{aligned}
$$

Thus, using (2.1), (3.3) and (3.4), we conclude that (3.2) implies

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N} u(x,t)\varphi(x,t)\,dx = &\int_{\mathbb{R}^N} u(x,t)A\varphi(x,t)\,dx \\
&+ \int_{\mathbb{R}^N} u(x,t)\varphi_t(x,t)\,dx \\
&- i\lambda \int_{\mathbb{R}^N} |u(s)|^p \varphi(x,t)\,dx.
\end{aligned}
$$

The result follows by integrating in time over $[0,T]$ and using the fact that $\varphi(\cdot\,,T)=0$. □

In order to state our result, we set $\lambda = \lambda_1 + i\lambda_2$ and $f = f_1 + if_2$. We introduce the following assumption on the data:

$$
f_1 \in L^1(\mathbb{R}^N), \qquad \lambda_2 \int_{\mathbb{R}^N} f_1\,dx > 0, \tag{3.5}
$$

or

$$
f_2 \in L^1(\mathbb{R}^N), \qquad \lambda_1 \int_{\mathbb{R}^N} f_2\,dx < 0. \tag{3.6}
$$

Theorem 3.3. *Under the same conditions as Theorem* 2.2, *if f satisfies* (3.5) *or* (3.6) *and if*

$$
1 < p \leq 1 + \frac{\alpha}{N},
$$

then the mild solution of (1.1) *blows-up in finite time.*

Proof. We argue by contradiction, supposing that u is a global mild solution of (1.1). Using Lemma 3.2, we have $u \in L^p((0,R^\alpha), L^p(B_{2\rho}))$, for all $\rho > 0$ and that it satisfies (3.1), where $B_{2\rho}$ stands for the closed ball of center 0 and radius 2ρ. We define the function $\varphi(x,t) := \varphi_1(x/BR)(\varphi_2(t))^\ell$, where

$$\ell = \frac{2p-1}{p-1},$$

$R, B > 0$ and $0 \leq \varphi_1 \in D(\Delta_D^{\alpha/2})$ is the first eigenfunction of the fractional Laplacian operator $\Delta_D^{\alpha/2}$ in B_2, with the homogeneous Dirichlet boundary condition, associated to the first eigenvalue κ, and

$$\varphi_2(t) = \psi\left(\frac{t}{R^\alpha}\right),$$

where ψ is a smooth non-increasing function on $[0, \infty)$ such that

$$\psi(r) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1, \\ 0 & \text{if } r \geq 2. \end{cases}$$

The constant $B > 0$ in the definition of φ_1 is fixed and will be chosen later. In fact, it plays some role only in the critical case $p = 1 + \alpha/N$; in the subcritical case $p < 1 + \alpha/N$ we simply take $B = 1$.

In the following, we denote by Ω_1 and Ω_2 the supports of φ_1 and φ_2, respectively:

$$\Omega_1 = \left\{x \in \mathbb{R}^N : |x| \leq 2BR\right\},$$
$$\Omega_2 = \{t \in [0, \infty) : t \leq 2R^\alpha\}.$$

Since u is a weak solution, we have

$$\begin{aligned} &\lambda \int_{\Omega_2} \int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \\ &\quad + i \int_{\Omega_1} f(x)\varphi(x,0)\,dx \\ &= -i \int_{\Omega_2} \int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dx\,dt \\ &\quad + \int_{\Omega_2} \int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dx\,dt. \end{aligned} \tag{3.7}$$

In order to obtain non-negativity on the left hand side of (3.7) (for $R, B \gg 1$), we consider four cases:

Case I. If $\lambda_1 > 0$, then

$$\int_{\mathbb{R}^N} f_2\,dx < 0;$$

therefore, by taking the real part (Re) on the both sides of (3.7), we get:

$$\begin{aligned} 0 &\leq \lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt - \int_{\Omega_1} f_2(x)\varphi(x,0)\,dx \\ &= \mathrm{Re}\Big[-i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dx\,dt\Big] \\ &\quad + \mathrm{Re}\Big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dx\,dt\Big]. \end{aligned}$$

Case II. If $\lambda_1 < 0$, then $\int_{\mathbb{R}^N} f_2\,dx > 0$; therefore, by taking $(-\mathrm{Re})$ on both sides of (3.7), we get:

$$\begin{aligned} 0 &\leq -\lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt + \int_{\Omega_1} f_2(x)\varphi(x,0)\,dx \\ &= \mathrm{Re}\Big[i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dx\,dt\Big] \\ &\quad + (-\mathrm{Re})\Big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dx\,dt\Big]. \end{aligned}$$

Case III. If $\lambda_2 > 0$, then

$$\int_{\mathbb{R}^N} f_1\,dx > 0;$$

therefore, by taking the imaginary part (Im) on both sides of (3.7), we get:

$$\begin{aligned} 0 &\leq \lambda_2 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt + \int_{\Omega_1} f_1(x)\varphi(x,0)\,dx \\ &= \mathrm{Im}\Big[-i\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^\ell(t)\,dx\,dt\Big] \\ &\quad + \mathrm{Im}\Big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^\ell(t)\Lambda^\alpha(\varphi_1(x/BR))\,dx\,dt\Big]. \end{aligned}$$

Case IV. If $\lambda_2 < 0$, then $\int_{\mathbb{R}^N} f_1\, dx < 0$; therefore, by taking $(-\mathrm{Im})$ on both sides of (3.7), we get:

$$\begin{aligned}
0 &\le -\lambda_2 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\, dx\, dt - \int_{\Omega_1} f_1(x)\varphi(x,0)\, dx \\
&= \mathrm{Im}\Big[i \int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_1(x/BR)\partial_t\varphi_2^{\ell}(t)\, dx\, dt\Big] \\
&\quad + \mathrm{Im}\Big[\int_{\Omega_2}\int_{\Omega_1} u(x,t)\varphi_2^{\ell}(t)\Lambda^{\alpha}(\varphi_1(x/BR))\, dx\, dt\Big].
\end{aligned}$$

We only consider Case I since the others may be treated identically. In this case, we assume $f_2 \in L^1$ and

$$\int_{\mathbb{R}^N} f_2\, dx < 0. \tag{3.8}$$

Thus, we have:

$$\begin{aligned}
&\lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\, dx\, dt \\
&\le \kappa B^{-\alpha} \int_{\Omega_2}\int_{\Omega_1} |u|(x,t)\varphi_2^{\ell}(t)R^{-\alpha}\varphi_1(x/BR)\, dx\, dt \\
&\quad + \ell \int_{\Omega_2}\int_{\Omega_1} |u|(x,t)\varphi_1(x/BR)\varphi_2^{\ell-1}(t)\partial_t\varphi_2(t)\, dx\, dt \\
&:= I_2 + I_1,
\end{aligned} \tag{3.9}$$

where we have used the fact that $\Delta_D^{\alpha/2}\varphi_1(x/BR) = R^{-\alpha}B^{-\alpha}\kappa\varphi_1(x/R)$. Hence, by the ε-Young inequality $ab \le \varepsilon a^p + C(\varepsilon)b^{\ell-1}$ (note that $1/p + 1/(\ell-1) = 1$) with $\varepsilon > 0$, we deduce:

$$\begin{aligned}
I_1 &= \ell \int_{\Omega_2}\int_{\Omega_1} |u|(x,t)\varphi^{1/p}\varphi^{-1/p}\varphi_1(x/BR)\varphi_2^{\ell-1}(t)\partial_t\varphi_2(t)\, dx\, dt \\
&\le \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\, dx\, dt \\
&\quad + C\int_{\Omega_2}\int_{\Omega_1} \varphi^{-(\ell-1)/p}\varphi_1^{(\ell-1)}(x/BR)\varphi_2^{(\ell-1)^2}(t)|\partial_t\varphi_2(t)|^{\ell-1}\, dx\, dt
\end{aligned}$$

$$\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt$$
$$+ C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2(t)|\partial_t\varphi_2(t)|^{\ell-1}\,dx\,dt,$$

and

$$\begin{aligned} I_2 &= \kappa B^{-\alpha}\int_{\Omega_2}\int_{\Omega_1} |u|(x,t)\varphi^{1/p}\varphi^{-1/p}\varphi_2^{\ell}(t)R^{-\alpha}\varphi_1(x/BR)\,dx\,dt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \\ &\quad + C\int_{\Omega_2}\int_{\Omega_1} \varphi^{-(\ell-1)/p}\varphi_2^{\ell(\ell-1)}(t)B^{-\alpha(\ell-1)}R^{-\alpha(\ell-1)}\varphi_1^{\ell-1}(x/BR)\,dx\,dt \\ &\leq \frac{\lambda_1}{4}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \\ &\quad + C\int_{\Omega_2}\int_{\Omega_1} \varphi_2^{\ell}(t)B^{-\alpha(\ell-1)}R^{-\alpha(\ell-1)}\varphi_1(x/BR)\,dx\,dt. \end{aligned}$$

Hence, from (3.9), we have:

$$\begin{aligned} &\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \\ &\quad \leq C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2(t)|\partial_t\varphi_2(t)|^{\ell-1}\,dx\,dt \\ &\qquad + C\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2^{\ell}(t)B^{-\alpha(\ell-1)}R^{-\alpha(\ell-1)}\,dx\,dt. \end{aligned}$$

Note that $N+\alpha-\alpha(\ell-1)\leq 0$ if and only if $p\leq 1+\alpha/N$. Therefore, we consider two cases.

• If $p<1+\alpha/N$, we suppose that $B=1$. Thus, by taking the change of variables $\xi=R^{-1}x$ and $\tau=R^{-\alpha}t$, we have

$$\begin{aligned} &\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \\ &\quad \leq C\int_0^2\int_{|\xi|\leq 2} \varphi_1(\xi)\varphi_2(R^\alpha\tau)R^{-\alpha(\ell-1)}|\partial_\tau\varphi_2(R^\alpha\tau)|^{\ell-1}R^N R^\alpha\,d\xi\,d\tau \\ &\qquad + C\int_0^2\int_{|\xi|\leq 2} \varphi_1(\xi)\varphi_2^{\ell}(R^\alpha\tau)R^{-\alpha(\ell-1)}R^N R^\alpha\,d\xi\,d\tau. \end{aligned}$$

Therefore, we easily obtain

$$\int_{\Omega_2}\int_{\Omega_1} |u|^p(x,t)\varphi(x,t)\,dx\,dt \le CR^{N+\alpha-\alpha(\ell-1)}, \tag{3.10}$$

where the constant C on the right hand side of (3.10) is independent of R. Hence, computing the limit $R \to \infty$ and using the Lebesgue dominated convergence theorem yields

$$\int_0^\infty \int_{\mathbb{R}^N} |u|^p(x,t)\varphi_1(0)\,dx\,dt = 0.$$

Then, $u(x,t) = 0$ for all t and almost every x. Hence, we obtain a contradiction with (3.8).

• In the critical case $p = 1+\alpha/N$, we choose $1 \le B < R$ large enough such that, when $R \to \infty$, we do not simultaneously have $B \to \infty$. We estimate the first term on the right hand side of inequality (3.9) by the ε-Young inequality and the second term by the Hölder inequality (with $\overline{p} = p/(p-1) = \ell - 1$), as follows:

$$\begin{aligned} &\lambda_1 \int_{\Omega_2}\int_{\Omega_1} |u|^p\varphi(x,t)\,dx\,dt \\ &\quad\le \frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1} |u|^p\varphi(x,t)\,dx\,dt \\ &\qquad + C\int_{\Omega_2}\int_{\Omega_1} \varphi^{-\bar{p}/p}\varphi_2^{\ell\bar{p}}(t)\varphi_1^{\bar{p}}(x/BR)(RB)^{-\alpha\bar{p}}\,dx\,dt \\ &\qquad + \ell\biggl(\int_{\Omega_3}\int_{\Omega_1} |u|^p\varphi(x,t)\,dx\,dt\biggr)^{1/p} \\ &\quad\times\biggl(\int_{\Omega_2}\int_{\Omega_1} \varphi_1(x/BR)\varphi_2(t)\,|\partial_t\varphi_2(t)|^{\bar{p}}\,dx\,dt\biggr)^{1/\bar{p}}. \end{aligned} \tag{3.11}$$

Here, $\Omega_3 = \{t \in [0,\infty) : R^\alpha \le t \le 2R^\alpha\} \subset \Omega_2$ is the support of $\partial_t\varphi_2$.

Note that

$$
\begin{aligned}
&\lim_{R\to\infty}\int_{\Omega_3}\int_{\Omega_1}|u|^p\varphi(x,t)\,dx\,dt\\
&\quad=\lim_{R\to\infty}\int_{|t|\leq 2R^\alpha}\int_{\Omega_1}|u|^p\varphi(x,t)\,dt\,dx\\
&\qquad-\lim_{R\to\infty}\int_{|t|\leq R^\alpha}\int_{\Omega_1}|u|^p\varphi(x,t)\,dt\,dx \qquad (3.12)\\
&\quad=\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\varphi_1(0)\,dx\,dt\\
&\qquad-\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\varphi_1(0)\,dx\,dt=0,
\end{aligned}
$$

where we have used the Lebesgue dominated convergence theorem and the fact that $u\in L^p(\mathbb{R}^N\times(0,\infty))$, cf., (3.10). Now, introducing the new variables $\xi=(BR)^{-1}x$, $\tau=R^{-\alpha}t$ and recalling that $p=1+\alpha/N$, we rewrite (3.11) as:

$$
\begin{aligned}
&\frac{\lambda_1}{2}\int_{\Omega_2}\int_{\Omega_1}|u|^p\varphi(x,t)\,dx\,dt\\
&\leq C\int_0^2\int_{|\xi|\leq 2}\psi^\ell(\tau)\varphi_1(\xi)B^{-\alpha}\,d\xi\,d\tau\\
&\quad+\ell\Big(\int_{\Omega_3}\int_{\Omega_1}|u|^p\varphi(x,t)\,dx\,dt\Big)^{1/p} \qquad (3.13)\\
&\times\Big(\int_0^2\int_{|\xi|\leq 2}\psi(\tau)\varphi_1(\xi)B^N\left|\partial_\tau\psi(\tau)\right|^{\bar p}\,d\xi\,d\tau\Big)^{1/\bar p}\\
&\leq CB^{-\alpha}+CB^{N/\bar p}\Big(\int_{\Omega_3}\int_{\Omega_1}|u|^p\varphi(x,t)\,dx\,dt\Big)^{1/p},
\end{aligned}
$$

where the constant C is independent of R and B. Passing in (3.13) to the limit as $R\to+\infty$ and using (3.12) and the Lebesgue dominated convergence theorem, we obtain

$$
\int_0^\infty\int_{\mathbb{R}^N}|u|^p(x,t)\varphi_1(0)\,dx\,dt\leq CB^{-\alpha}. \qquad (3.14)
$$

Finally, computing the limit $B\to\infty$ in (3.14), we infer that $u(x,t)=0$ for all t and almost every x. A contradiction with (3.8) is again obtained. □

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