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GLOBAL EXISTENCE AND BLOW-UPS FOR CERTAIN ORDINARY INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We will study ordinary integro-differential equations of second order with nonlinearity given as a convolution, but differently from the widely investigated cases. In addition, the kernel depends on the solution. Such equations play a key role in the theory of glass-forming liquids, and we will establish results on global existence and investigate the long-term behavior. In contrast, we give examples where blow-ups occur.

1. Introduction. In the theory of glass-forming liquids, which is a subject of soft-condensed matter physics, the so called mode-coupling equation appears. It is an integro-differential equation of the form

$$\begin{aligned} &\lambda\ddot{\phi}(t) + \dot{\phi}(t) + \phi(t) + \int_0^t m(\phi(t-s))\dot{\phi}(s)\,\mathrm{d}s = 0, \\ &\phi(0) = \phi_0, \\ &\dot{\phi}(0) = \phi_1, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a constant parameter and m is a matrix-valued function determined by the physical properties of the studied fluid. The function ϕ is a correlation function which represents the microscopic dynamics of the fluid in a statistical mean.

A detailed derivation of this equation by the Zwanzig-Mori formalism is given in [**7**]. The function m is mostly assumed to be a second-order polynomial with positive coefficients in the applications [**6, 11**], but also linear ([**5**]) or higher order cases are of interest ([**3**]).

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For these applications, the initial conditions are $\phi(0) = 1$, $\dot{\phi}(0) = 0$, and the long-term limit has a direct physical meaning; if it is 0 the fluid stays in its phase and in the other case there is a glass-transition.

New results in the modeling lead to an additional dependence of the function m on the parameters t and s and to complex-valued problems **[1, 4]**. The example given in **[1]** reads

$$m(x, t-s) = \begin{pmatrix} m_{||}^s(x_1, x_2, t-s) & 0 \\ 0 & m_\perp^s(x_1, x_2, t-s) \end{pmatrix}$$

with

$$m_{||}^s(x_1, x_2, t-s) = \frac{v_1^s \overline{x}_1 + v_s^2 x_2}{1 - ik_{||}F^{ex}} \Phi(t-s)$$

and

$$m_\perp^s(x_1, x_2, t-s) = \frac{v_1^s x_2 + v_s^2 \Re(x_1)}{1 + (k_\perp F^{ex})^2} \Phi(t-s),$$

where the function Φ is a known correlator function and v_1^s, v_s^2, $k_{||}$, $k_\perp$ and F^{ex} are real constants.

The literature for integro-differential equations (e.g., **[2, 9, 13]**) concentrates on kernels which do not depend on the solution, and thus it cannot be applied directly to our nonlinearity. Also, the methods presented there to deduce the long-term limit cannot be adopted, because they make use of the sign of m' or the decay of the kernel, which both depend again on ϕ in our case.

Some results for the mode-coupling equation with polynomial functions m are established in **[10]** by physical arguments. If m has positive coefficients the solution for $\phi(0) > 0$ and $\dot{\phi}(0) = 0$ will stay positive and is bounded by $\phi(0)$, and thus the global existence follows.

In **[14]**, the local existence of solutions to (1) for locally Lipschitz continuous functions m is shown and, under the additional assumption, that m is bounded by a linear growing function, the global existence is deduced without any restrictions on the data. Furthermore, the damping in (1) is used there to obtain exponentially stable solutions for small data, if $m(x) = \mathcal{O}(x^\alpha)$ holds near zero for $\alpha > 1$.

On this basis, we will investigate a generalized first-order system, where both factors in the convolution are nonlinear,

$$\dot{x}(t) = -Ax(t) + \int_0^t M(x(t-s))G(x(s))\,\mathrm{d}s + F(t), \quad t \geq 0, \tag{1.2}$$
$$x(0) = x_0$$

where $M \in C^0(\mathbb{C}^n, \mathbb{C}^{n\times n})$, $G \in C^0(\mathbb{C}^n, \mathbb{C}^n)$, $F \in C^0([0,\infty), \mathbb{C}^n)$ and $x_0 \in \mathbb{C}^n$. The second order mode-coupling equation (1) can easily be transformed in such a system.

For $\lambda = 0$, the system (1) represents an alternative mode-coupling equation for which first results are shown in [**8**]. The proofs are based on the monotonicity of solutions, and we cannot carry this idea over to our second order systems. Further results can be found in [**12**]; here also the non-monotone case is treated.

In Section 2, we deduce the global well-posedness for any data, if the functions M and G are locally Lipschitz continuous and if each function is at most of linear growth. To extend the results in [**12**] we will carry this over to the first order mode-coupling equation ($\lambda = 0$), which is not of the form (1.2). In Section 3, we give criteria for the monotonicity of solutions for the second order equation; the important assumptions therefore are $m(\phi_0) < -1$ and that m is monotonically decreasing. We then use this monotonicity in Section 4 to show that our result on the global well-posedness for large data is sharp in the sense of polynomials. More precisely, we show that, for any $\epsilon > 0$, there exists a function m with $m(x) = \mathcal{O}(|x|^{1+\varepsilon})$ for $x \to -\infty$ and data $\phi_0, \phi_1 \in \mathbb{R}$ such that the local solution to (1) has a blow-up in finite time. In Section 5, we investigate the case of an exponentially stable linear system and small data. We extend the results of [**14**] where, for $m(x) = \mathcal{O}(x^\alpha)$ near zero with $\alpha > 1$ and $G(x) = x$, the existence of global and exponentially stable solutions to (1) is established, to the system (1.2). By a similar technique, we will also treat the case $\alpha = 1$, which is important in the applications.

This work is based on the author's PhD thesis ([**15**]) written at the University of Konstanz. Further details and results for partial integro-differential equations can also be found there.

2. Well posedness for large Data. To prove the well posedness of solutions we investigate the equation

(2.1)
$$\begin{aligned} \dot{x}(t) &= -Ax(t) + \int_0^t M(x(t-s))G(x(s))\,\mathrm{d}s + F(t), && T \le t, \\ x(t) &= \Phi(t), && 0 \le t \le T \end{aligned}$$

with $T \geq 0$ and given functions $\Phi \in C^0([0,T],\mathbb{C}^n)$ and $F \in C^0([T,\infty),\mathbb{C}^n)$. For $T = 0$ and $\Phi(0) = x_0$, we obtain solutions to (1.2) and, taking $T > 0$, gives us the possibility of showing that a bounded solution to (1.2) can be extended onto some larger interval of existence.

Theorem 2.1. *Let $T \geq 0$, $A \in \mathbb{C}^{n\times n}$ and $M \in C^0(\mathbb{C}^n,\mathbb{C}^{n\times n})$, $G \in C^0(\mathbb{C}^n,\mathbb{C}^n)$ locally Lipschitz continuous.*

Then, for every
$$\Phi \in C^0([0,T],\mathbb{C}^n) \quad \textit{and} \quad F \in C^0([T,\infty),\mathbb{C}^n),$$
there is a $\Delta T > 0$ with $\Delta T = c(1+T)^{-1}$, where $c > 0$ is independent of T, and a unique local solution
$$x \in C^1([T,T+\Delta T],\mathbb{C}^n)$$
to (2.1).

Proof. We will construct a sequence of solutions to a linearized equation and deduce the uniform boundedness and, with this, the convergence to a solution of our nonlinear equation. Let
$$h_E(t) := \begin{cases} \Phi(t), & 0 \le t < T, \\ h(t), & T \le t. \end{cases}$$
For functions h with $h(T) = \Phi(T)$ we have $h_E \in C^0([0,\infty),\mathbb{C}^n)$.

We set $x^{(0)}(t) := \Phi(T)$ for $t \geq T$ and define $(x^{(l)})_l \subset C^1([T,\infty),\mathbb{C}^n)$ by

(2.2)
$$\begin{aligned} \dot{x}^{(l)}(t) &= -Ax^{(l)}(t) + \int_0^t M\left(x_E^{(l-1)}(t-s)\right) G\left(x_E^{(l-1)}(s)\right) \mathrm{d}s + F(t), \ t \geq T, \\ x^{(l)}(T) &= \Phi(T). \end{aligned}$$

It is easy to see that $(x^{(l)})_l$ is well defined, and we now show by induction

$$\sup_{t\in[T,T+\Delta T]} \left|x^{(l)}(t)\right|^2 \leq 2 \sup_{t\in[0,T]} |\Phi(t)|^2 + 2(1+|F(T)|)^2 =: b \tag{2.3}$$

for small ΔT. If the inequality holds for $x^{(l-1)}$, $\left|x_E^{(l-1)}(t)\right|^2 \leq b$ also follows, and thus

$$\left|\int_0^t M\left(x_E^{(l-1)}(t-s)\right) G\left(x_E^{(l-1)}(s)\right) \mathrm{d}s\right| \leq t \sup_{|x|^2,|y|^2\leq b} |M(x)|\,|G(y)| =: tc_{MG}.$$

We choose $\Delta T > 0$ small enough to guarantee

$$\sup_{t\in[T+\Delta T]} |F(t)| \leq 2(1+|F(T)|).$$

A multiplication of (2.2) by $x^{(l)}(t)$ leads to

$$\left|x^{(l)}(t)\right|^2 \leq |\Phi(T)|^2 + (2c_A + 1 + (T+\Delta T)) \int_T^t \left|x^{(l)}(s)\right|^2 \mathrm{d}s + \left((T+\Delta T)c_{MG}^2 + 4(1+|F(T)|)^2\right)\Delta T,$$

where $c_A := \sup_{|x|=1} |Ax|$, and Gronwall's inequality yields

$$\left|x^{(l)}(t)\right|^2 \leq \left(|\Phi(T)|^2 + \left((T+\Delta T)c_{MG}^2 + 4(1+|F(T)|)^2\right)\Delta T\right) e^{(2c_A+1+(T+\Delta T))\Delta T}.$$

We now assume $\Delta T = \delta(1+T)^{-1}$ with a constant $\delta > 0$ independent of T.

On the one hand, we have $\Delta T \leq \delta$, and on the other hand,

$$(T+\Delta T)\Delta T = \left(T + \frac{\delta}{1+T}\right)\frac{\delta}{1+T} \leq (T+\delta)\frac{\delta}{1+T} = \delta\left(\frac{T}{1+T} + \frac{\delta}{1+T}\right) \leq (1+\delta)\delta.$$

So,

$$\left|x^{(l)}(t)\right|^2 \leq \left(|\Phi(T)|^2 + \left((1+\delta)c_{MG}^2 + 4\,|F(T)|^2\right)\delta\right) e^{(2c_A+1)(1+\delta)\delta}$$

follows and, for δ sufficiently small, (2.3) holds.

To prove the convergence, we set $w^{(l)} := x^{(l)} - x^{(l-1)}$. For $l \geq 2$, these functions fulfill

$$\begin{aligned} \dot{w}^{(l)}(t) &= -Aw^{(l)}(t) + \int_0^t M\left(x_E^{(l-1)}(t-s)\right) G\left(x_E^{(l-1)}(s)\right) \\ &\qquad - M\left(x_E^{(l-2)}(t-s)\right) G\left(x_E^{(l-2)}(s)\right)\,\mathrm{d}s, \\ w^{(l)}(T) &= 0, \end{aligned}$$

and we carry out the multiplier method as above. There is a Lipschitz constant L for M and G, depending only on the bound b of the sequence $(x^{(l)})_l$, with

$$\left|w^{(l)}(t)\right|^2 \leq (2Lc_{MG}\Delta T)^2 \Delta T e^{(2c_A+1)(t-T)} \sup_{t\in[T,T+\Delta T]} \left|w^{(l-1)}(s)\right|^2.$$

This yields

$$\sup_{t\in[T,T+\Delta T]} \left|w^{(l)}(t)\right|^2 \leq \left((2Lc_{MG}\delta)^2\delta e^{(2c_A+1)\delta}\right)^l \sup_{t\in[T,T+\Delta T]} \left|w^{(1)}(s)\right|^2,$$

and, taking δ additionally small enough to have

$$(2Lc_{MG}\delta)^2\delta e^{(2c_A+1)\delta} \leq \frac{1}{2},$$

we conclude

$$\begin{aligned} \sup_{t\in[T,T+\Delta T]} \left|x^{(i)}(t) - x^{(j)}(t)\right| &\leq \sum_{l=j+1}^{i} \sup_{t\in[T,T+\Delta T]} \left|x^{(l)}(t) - x^{(l-1)}(t)\right| \\ &\leq \sum_{l=j+1}^{i} \frac{1}{2^l} \sup_{t\in[T,T+\Delta T]} \left|w^{(1)}(s)\right|^2 \\ &\longrightarrow 0 \quad (i,j\to\infty). \end{aligned}$$

So the sequence converges to some $x \in C^0([T,T+\Delta T],\mathbb{C}^n)$, which implies the convergence of the convolution-term and hence, by the equation, $\dot{x}^{(l)}$ also converges and the limit is a solution. The uniqueness easily follows by the local Lipschitz continuity of M and G. $\square$

With this local existence theorem we can show that a uniform a priori bound for $x(t)$ is sufficient to extend the solution.

Lemma 2.2. *Under the assumptions of Theorem* 2.1 *the local solution of* (2.1) *exists as long as*

$$|x(t)| \leq C$$

holds for some C independent of t.

The following theorem states that, for functions M and G, which are at most of linear growth, there is a global solution. But, stability of that solution cannot be expected because, by taking $G(x) = x$ and $M = -2$, we obtain exponentially growing solutions for certain initial values.

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ and $M \in C^0(\mathbb{C}^n, \mathbb{C}^{n\times n})$, $G \in C^0(\mathbb{C}^n, \mathbb{C}^n)$ be locally Lipschitz continuous with*

$$|M(x)G(y)| \leq c(1 + |x| + |y| + |x|\,|y|)$$

for some $c > 0$ and all $x, y \in \mathbb{C}^n$.

Then, for any $x_0 \in \mathbb{C}^n$ and $F \in C^0([0,\infty), \mathbb{C}^n)$, there is a global unique solution $x \in C^1([0,\infty), \mathbb{C}^n)$ to (1.2).

Proof. For the local solution $x \in C^1([0,T], \mathbb{C}^n)$, let $c_T := \sup\limits_{t\in[0,T]} |x(t)|$. Multiplication of (1.2) by $\overline{x}(t)$ gives with $c_A := \sup_{|x|=1} |Ax|$ for $T \leq t \leq 2T$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,|x(t)|^2 \leq \left(c_A + \frac{1}{2}\right)|x(t)|^2 \\ + c\,|x(t)| \int_0^t 1 + 2\,|x(s)| + |x(t-s)||x(s)| \ \mathrm{d}s + \frac{1}{2}\,|F(t)|^2 \,.$$

We separate the integral into two parts and obtain for $T \leq t \leq 2T$

$$\begin{aligned}\int_0^t |x(t-s)|\,|x(s)|\ \mathrm{d}s &= \int_0^T |x(t-s)|\,|x(s)|\ \mathrm{d}s \\ &\quad + \int_T^t |x(t-s)|\,|x(s)|\ \mathrm{d}s \\ &\leq c_T \int_0^T |x(t-s)|\ \mathrm{d}s + c_T \int_T^t |x(s)|\ \mathrm{d}s \\ &\leq 2c_T \int_0^t |x(s)|\ \mathrm{d}s.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{1}{2}\,|x(t)|^2 \leq \frac{1}{2}\,|x_0|^2 + \left(c_A + \frac{1}{2} + 5cT + 4c_T cT\right) \int_0^t |x(s)|^2\ \mathrm{d}s \\ + \frac{1}{2} \int_0^{2T} |F(s)|^2\ \mathrm{d}s + cT^2,\end{aligned}$$

and thus we have a uniform bound in $[T, 2T]$. Iteratively, we get a bound in $[T, 2^n T]$ for any $n \in \mathbb{N}$, and so there is a global solution. □

Example 2.4 (Linear functions)**.**
For

$$m(x) = \sum_{i=1}^{d} C_i x_i,$$

with matrices $C_i \in \mathbb{C}^{d \times d}$, there is, for any $\phi_0, \phi_1 \in \mathbb{C}^d$, a global solution to (1).

Example 2.5 (First order equation)**.** We can also treat the first order equation investigated in **[12]** with our results. It reads

$$\begin{aligned}&\dot{\phi}(t) + \phi(t) + \int_0^t m(\phi(t-s))\dot{\phi}(s)\,\mathrm{d}s = f(t), \quad t \geq 0, \\ &\phi(0) = \phi_0.\end{aligned} \tag{2.4}$$

Assuming that m and f are C^1-functions we obtain

$$\begin{aligned}
&\ddot{\phi}(t) + (\mathrm{id} + m(\phi_0))\dot{\phi}(t) \\
&\quad + \int_0^t m'(\phi(t-s))\dot{\phi}(t-s)\dot{\phi}(s)\,\mathrm{d}s = \dot{f}(t), \quad t \geq 0, \\
&\qquad \phi(0) = \phi_0, \\
&\qquad \dot{\phi}(0) = -\phi(0) + f(0),
\end{aligned}$$

by differentiating the equation. For $x = (\phi, \dot{\phi})$, we get the first order system

$$\begin{aligned}
\dot{x}(t) = &- \begin{pmatrix} 0 & -\mathrm{id} \\ 0 & \mathrm{id} + m(\phi_0) \end{pmatrix} x(t) \\
&- \int_0^t \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & m'(x_1(t-s))x_2(t-s) \end{pmatrix}}_{\widehat{=} -M(x(t-s))} \underbrace{\begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix}}_{\widehat{=} G(x(s))} \mathrm{d}s + \begin{pmatrix} 0 \\ \dot{f}(t) \end{pmatrix} \\
x(0) = &\begin{pmatrix} \phi_0 \\ -\phi_0 + f(0) \end{pmatrix}.
\end{aligned}$$

In contrast to (2.4), there is no derivative in the convolution and the system has the form (1.2).

It follows, that for $m \in C^1(\mathbb{C}^n, \mathbb{C}^n)$ with a locally Lipschitz continuous derivative m', there is for any $\phi_0 \in \mathbb{C}^d$ and $f \in C^1([0, \infty), \mathbb{C}^d)$ a unique local solution $\phi \in C^2([0, T], \mathbb{C}^d)$ to (2.4).

The local solutions can be extended to a global one, if additionally m' is bounded, which is equivalent to m being at most of linear growth. This is an extension of the results in **[12]**, where the boundedness of m itself is assumed to obtain global solutions.

Choosing in **[12,** Theorem 7] the function $F(x) = x + x\,|x|^{\varepsilon} - 2$ for an arbitrary $\varepsilon > 0$ we obtain, that for $\phi_0 = 1$, $f = 0$ and $m(x) = F(x)$ there is a blow-up in finite time for the local solution to (2.4).

By a different method than in **[12]** we will derive a similar result in Section 4 for the second order equation. So we found the weakest polynomial growth condition, under which for any data there is a global solution.

Remark 2.6. Our results also hold for functions M and G, which additionally depend continuously on the parameters t and s.

Example 2.7. The functions $m^s_{||}, m^s_{\perp} : \mathbb{C}^2 \times [0,\infty) \to \mathbb{C}$ given in **[5]** fulfill the necessary estimate; they read

$$m^s_{||}(x_1, x_2, t-s) = \frac{v^s_1 \overline{x}_1 + v^2_s x_2}{1 - ik_{||}F^{ex}} \Phi(t-s)$$

and

$$m^s_{\perp}(x_1, x_2, t-s) = \frac{v^s_1 x_2 + v^2_s \Re(x_1)}{1 + (k_{\perp}F^{ex})^2} \Phi(t-s)$$

with a known function Φ and constants $v^s_1, v^2_s, F^{ex}, k_{||}, k_{\perp}$. In our notation, we have

$$M(x, t-s) = \begin{pmatrix} m^s_{||}(x_1, x_2, t-s) & 0 \\ 0 & m^s_{\perp}(x_1, x_2, t-s) \end{pmatrix}$$

and $G(x) = x$.

3. Monotonicity of solutions. For $m \in C^1(\mathbb{R}^d, \mathbb{R})$ and $\phi_0, \phi_1 \in \mathbb{R}^d$, we treat here the system

$$\begin{aligned} \lambda\ddot{\phi}_k(t) + \dot{\phi}_k(t) + \phi_k(t) + \int_0^t m(\phi(t-s))\dot{\phi}_k(s)\,\mathrm{d}s = 0 \quad (1 \le k \le d), \\ \phi(0) = \phi_0, \\ \dot{\phi}(0) = \phi_1. \end{aligned} \tag{3.1}$$

The equations for the components of ϕ are coupled by m; this is a special case of (1), which is in wide use in the physical application.

Integration by parts leads to

$$\begin{aligned} \lambda\ddot{\phi}_k(t) + \dot{\phi}_k(t) + \phi_k(t) + m_0\phi_k(t) - m(\phi(t))\phi_{0k} \\ + \int_0^t \phi_k(s)(\nabla m)(\phi(t-s))\dot{\phi}(t-s)\,\mathrm{d}s = 0, \end{aligned}$$

where $m_0 := m(\phi(0))$. Since

$$(\nabla m)(\phi(t-s))\dot{\phi}(t-s) = \sum_{l=1}^{d} \partial_l m(\phi(t-s))\dot{\phi}_l(t-s)$$

and

$$m(\phi(t))\phi_{0k} = m_0\phi_{0k} + \phi_{0k}\int_0^t \sum_{l=1}^d \partial_l m(\phi(t-s))\dot{\phi}_l(t-s)\,\mathrm{d}s,$$

we have the equivalent system

$$\begin{aligned} &\lambda\ddot{\phi}_k(t) + \dot{\phi}_k(t) + \phi_k(t) + m_0(\phi_k(t) - \phi_{0k}) \\ &\qquad + \sum_{l=1}^d \int_0^t (\phi_k(s) - \phi_{0k})\partial_l m(\phi(t-s))\dot{\phi}_l(t-s)\,\mathrm{d}s = 0, \\ &\phi(0) = \phi_0, \\ &\dot{\phi}(0) = \phi_1. \end{aligned} \tag{3.2}$$

The form of this system allows us to prove the monotonicity of solutions under certain assumptions.

Theorem 3.1. *Let $m \in C^1(\mathbb{R}^d, \mathbb{R})$, $\phi_0, \phi_1 \in \mathbb{R}^d$ and ϕ be the local solution to* (3) *for this data. If*

$$\phi_{0k} \geq 0, \qquad \phi_{1k} \leq 0 \quad (1 \leq k \leq d)$$

and

$$m(\phi_0) < -1 \quad \textit{and} \quad \partial_l m(x) \geq 0 \textit{ for } x_k \leq \phi_{0k} \quad (1 \leq k, l \leq d)$$

holds, then ϕ_k $(1 \leq k \leq d)$ is monotonically decreasing in its interval of existence.

Proof. First let $1 \leq k \leq d$ be arbitrary. In the case $\phi_{0k} = \phi_{1k} = 0$, the conjecture is obvious.

If now $\phi_{1k} = \dot{\phi}_k(0) < 0$ or $\dot{\phi}_k(0) = 0$ and $\lambda\ddot{\phi}_k(0) = -\phi_k(0) < 0$ holds, there is a $t_0^k > 0$ with $\dot{\phi}_k(t) < 0$ for $t \in (0, t_0^k)$.

Assume that ϕ is not monotonically decreasing in every component. Then there is a smallest $t_1 > 0$ with $\dot{\phi}_l(t) \leq 0$ $(0 \leq t \leq t_1,\ 1 \leq l \leq d)$ and a component ϕ_k for which we have $\dot{\phi}_k(t_1) = 0$ and $\ddot{\phi}_k(t_1) \geq 0$.

Equation (3.2) yields

$$\begin{aligned}\dot\phi_k(t_1) &= -\lambda\ddot\phi_k(t_1) - (1+m_0)\phi_k(t_1) + m_0\phi_{0k}\\ &\quad - \sum_{l=1}^{d}\int_0^{t_1}(\phi_k(s)-\phi_{0k})\partial_l m(\phi(t_1-s))\dot\phi_l(t_1-s)\,\mathrm{d}s\\ &\le -(1+m_0)\phi_k(t_1) + m_0\phi_{0k}\\ &\quad - \sum_{l=1}^{d}\int_0^{t_1}(\phi_k(s)-\phi_{0k})\partial_l m(\phi(t_1-s))\dot\phi_l(t_1-s)\,\mathrm{d}s.\end{aligned} \tag{3.3}$$

In $(0,t_1)$, the function $\phi_k(t)$ is monotonically decreasing and not constant because of $(0,t_0^k)\subset(0,t_1)$. This implies $\phi_k(t_1)-\phi_{0k}<0$ for the case $\phi_k(t_1)\ge 0$, and hence

$$-(1+m_0)\phi_k(t_1)+m_0\phi_{0k} = -\phi_k(t_1) - m_0(\phi_k(t_1)-\phi_{0k}) < 0.$$

If $\phi_k(t_1)<0$, then $1+m_0<0$ also leads to

$$-(1+m_0)\phi_k(t_1)+m_0\phi_{0k}<0.$$

Since $\partial_l m(\phi(t_1-s))\ge 0$, we have

$$\sum_{l=1}^{d}\int_0^{t_1}(\phi_k(s)-\phi_{0k})\partial_l m(\phi(t_1-s))\dot\phi_l(t_1-s)\,\mathrm{d}s \ge 0,$$

and, by inserting this into (3.3), we conclude

$$\dot\phi_k(t_1)<0,$$

which is a contradiction to $\dot\phi_k(t_1)=0$, thus ϕ is in every component monotonically decreasing. □

By investigating the derivative of equation (3),

$$\lambda\dddot\phi_k(t)+\ddot\phi_k(t)+(1+m_0)\dot\phi_k(t) + \int_0^t\sum_{l=1}^{d}\partial_l m(\phi(t-s))\dot\phi_l(t-s)\dot\phi_k(s)\,\mathrm{d}s = 0,$$

we can carry over this result to the derivatives $\dot\phi_k$, if also $\phi_0+\phi_1>0$ holds.

Lemma 3.2. *Assume additionally to the preliminaries of Theorem* 3.1*, that*

$$\phi_{0k} + \phi_{1k} > 0 \quad (1 \le k \le d).$$

Then also $\dot{\phi}_k$ *is monotonically decreasing for* $1 \le k \le d$*.*

Remark 3.3. Under the assumptions of Lemma 3.2 $\dot{\phi}_k$ is monotonically decreasing and $\phi_{0k} + \phi_{1k} > 0$ excludes the case $\dot{\phi}_k(t) = 0$ for all t. So we can find some $t_0^k \ge 0$ with $\dot{\phi}_k(t_0^k) < 0$, and this implies

$$\begin{aligned}\phi_k(t) &= \phi_k(t_0^k) + \int_{t_0^k}^t \dot{\phi}_k(s)\,\mathrm{d}s \\ &\le \phi_k(t_0^k) + (t - t_0^k)\dot{\phi}_k(t_0^k) \to -\infty \quad (t \to T^*)\end{aligned}$$

for some $T^* \in (0, \infty]$. Thus, ϕ is not bounded from below.

We can combine our global existence Theorem 2.3 with the monotonicity result.

Corollary 3.4. *Let* $m \in C^1(\mathbb{R}^d, \mathbb{R})$ *and* $\phi_0, \phi_1 \in \mathbb{R}^d$*. If*

$$\phi_{0k} \ge 0, \qquad \phi_{1k} \le 0 \quad (1 \le k \le d)$$

as well as

$$\begin{gathered} m(\phi(0)) < -1, \qquad \partial_l m(x) \ge 0 \quad \text{and} \quad |m(x)| \le c(1 + |x|) \\ (x_k \le \phi_{0k},\ 1 \le k, l \le d) \end{gathered}$$

hold for some $c > 0$*, then there is a global unique solution to* (3)*, which is monotonically decreasing.*

4. Blow up in finite time. In this section, we show that, without the estimate $|m(x)| \le c(1 + |x|)$, global solvability for arbitrary data cannot be expected.

We will investigate for $m \in C^1(\mathbb{R}^d, \mathbb{R})$, $\phi_0, \phi_1 \in \mathbb{R}^d$ the system of coupled integro-differential equations

$$\lambda\ddot{\phi}_k(t) + \dot{\phi}_k(t) + \phi_k(t) + \int_0^t m(\phi(t-s))\dot{\phi}_k(s)\,\mathrm{d}s = 0 \quad (1 \leq k \leq d),$$
$$\phi(0) = \phi_0, \qquad \dot{\phi}(0) = \phi_1.$$

Under the assumptions of Lemma 3.2 we get an estimate for the local solution, which we can use to show a blow-up in finite time.

Lemma 4.1. *Let $m \in C^1(\mathbb{R}^d, \mathbb{R})$, $\phi_0, \phi_1 \in \mathbb{R}^d$ and ϕ be the local solution of* (3) *to this data. If*

$$\phi_{0k} \geq 0, \phi_{1k} \leq 0, \phi_{0k} + \phi_{1k} > 0 \quad (1 \leq k \leq d),$$
$$m(\phi_0) < -1$$

and

$$\partial_l m(x) \geq 0 \quad (x_k \leq \phi_{0k},\ 1 \leq k, l \leq d),$$

hold, then there is, for any $c > 0$, a $t_0 > 0$ with $\phi_k(t_0) \leq -c$ $(1 \leq k \leq d)$ and

$$(t - t_0 + \lambda)\,|\phi_k(t)| \geq \lambda c + |\phi_{1k}| \int_{t_0}^t \int_0^s \int_0^r |1 + m(\phi(v))|\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \tag{4.1}$$

for $t \geq t_0$.

Proof. ϕ_k and $\dot{\phi}_k$ are monotonically decreasing and, as seen in Remark 3.3, we can find for any $c > 0$ a t_0 with

$$\phi_k(t_0) \leq -c - \frac{1}{2\lambda}\phi_{0k}. \tag{4.2}$$

By using $\phi_k(t) = \phi_{0k} + \int_0^t \dot{\phi}_k(s)\,\mathrm{d}s$, we obtain

$$\lambda\ddot{\phi}_k(t) + \dot{\phi}_k(t) + \phi_{0k} + \int_0^t (1 + m(\phi(t-s)))\dot{\phi}_k(s)\,\mathrm{d}s = 0.$$

Integration leads to

$$\lambda\dot{\phi}_k(t) - \lambda\phi_{1k} + \phi_k(t) - \phi_{0k} + t\phi_{0k} + \int_0^t \int_0^s (1 + m(\phi(s-r)))\dot{\phi}_k(r)\,\mathrm{d}r\,\mathrm{d}s = 0,$$

and because of $\phi_{1k} \le 0$, there follows

$$\begin{aligned}\lambda\dot{\phi}_k(t) + \phi_k(t) &= \lambda\phi_{1k} + \phi_{0k}(1-t) \\ &\quad - \int_0^t \int_0^s (1 + m(\phi(s-r)))\dot{\phi}_k(r)\,\mathrm{d}r\,\mathrm{d}s \\ &\le \phi_{0k}(1-t) - \int_0^t \int_0^s (1 + m(\phi(s-r)))\dot{\phi}_k(r)\,\mathrm{d}r\,\mathrm{d}s.\end{aligned}$$

A second integration gives

$$\begin{aligned}\lambda\phi_k(t) + \int_{t_0}^t \phi_k(s)\,\mathrm{d}s &\le \phi_{0k}\int_{t_0}^t (1-s)\,\mathrm{d}s + \lambda\phi_k(t_0) \\ &\quad - \int_{t_0}^t \int_0^s \int_0^r (1 + m(\phi(r-v)))\dot{\phi}_k(v)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \\ &\le -\lambda c - \int_{t_0}^t \int_0^s \int_0^r (1 + m(\phi(r-v)))\dot{\phi}_k(v)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s,\end{aligned}$$

where we used (4.2).

The monotonicity of ϕ_k and $\dot{\phi}_k$ gives, on the one hand, $\dot{\phi}_k(t) \le \phi_{1k} < 0$ for all $t > 0$ and, on the other hand, $\phi_k(t) < \phi_k(t_0) < 0$ for all $t > t_0$.

So $m(\phi(t))$ is also monotonically decreasing and, especially, $1 + m(\phi(t)) \le 1 + m(\phi(0)) < 0$, which implies

$$\lambda\phi_k(t) + \int_{t_0}^t \phi_k(s)\,\mathrm{d}s \le -\lambda c - \phi_{1k}\int_{t_0}^t \int_0^s \int_0^r (1 + m(\phi(r-v)))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s.$$

Since $\int_{t_0}^t \phi_k(s)\,\mathrm{d}s \ge (t-t_0)\phi_k(t)$, we now see

$$(t - t_0 + \lambda)\phi_k(t) \le -\lambda c - \phi_{1k}\int_{t_0}^t \int_0^s \int_0^r (1 + m(\phi(v)))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s,$$

or equivalently,

$$(t - t_0 + \lambda)\,|\phi_k(t)| \ge \lambda c + |\phi_{1k}|\int_{t_0}^t \int_0^s \int_0^r |1 + m(\phi(v))|\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s. \quad \square$$

The easy proof of the following Gronwall-type inequality is omitted here.

Lemma 4.2. *Let $k \in C^0([0,\infty),\mathbb{R})$ be monotonically increasing, $c_1, c_2, c_3 > 0$, $t_0 > t_1 \geq 0$ and $u, w : [0,T) \mapsto [0,\infty)$ piecewise continuous with*

$$u(t) \geq \frac{1}{c_2 + t - t_0} c_1 + \frac{1}{c_2 + t - t_0} c_3 \int_{t_0}^{t} \int_{t_1}^{s} \int_{t_1}^{r} k(u(v))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s$$

and

$$w(t) < \frac{1}{c_2 + t - t_0} c_1 + \frac{1}{c_2 + t - t_0} c_3 \int_{t_0}^{t} \int_{t_1}^{s} \int_{t_1}^{r} k(w(v))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s$$

for $t \geq t_0$ and $u(t) \geq w(t)$ for $t < t_0$. Then

$$u(t) > w(t)$$

holds in $[t_0, T)$.

Now we can give explicit examples for blow-ups in finite time in the case $d = 1$.

Conclusion 4.3. Let $\varepsilon > 0$ be arbitrary. Set

$$\phi_1 = -(\lambda+1)\left(\frac{3}{\varepsilon}+2\right)\left(\frac{3}{\varepsilon}+1\right)\frac{3}{\varepsilon} \quad \text{and} \quad \phi_0 > |\phi_1| \text{ arbitrary}$$

and

$$m(x) = -2 - \phi_0^{1+\varepsilon} + x\,|x|^{\varepsilon}.$$

Then the local solution ϕ to (3) has a finite time of existence.

Proof. The assumptions of Lemma 4.1 are fulfilled for the initial values ($\phi_0 > 0$, $\phi_1 < 0$ and $\phi_0 + \phi_1 > |\phi_1| + \phi_1 = 0$) and, by definition, we have $m(\phi_0) = -2$ and $m'(x) = (1+\varepsilon)\,|x|^{\varepsilon} \geq 0$. So ϕ is monotonically decreasing, and corresponding to

$$c_\varepsilon := \frac{\lambda+2}{\lambda}\left(1 + \frac{3}{\varepsilon} + \frac{1}{2}\frac{3}{\varepsilon}\left(\frac{3}{\varepsilon}+1\right)\right),$$

there is a t_0 with $\phi(t_0) \leq -c_\varepsilon$ and

$$(t - t_0 + \lambda)\,|\phi(t)| \geq \lambda c_\varepsilon + |\phi_1| \int_{t_0}^{t} \int_{0}^{s} \int_{0}^{r} 1 + \phi_0^{1+\varepsilon} - \phi(v)\,|\phi(v)|^{\varepsilon}\,\mathrm{d}v\mathrm{d}r\mathrm{d}s$$

for $t \geq t_0$.

We simplify the integral by using the fact that ϕ has exactly one root t_1. $\phi_0^{1+\varepsilon} - \phi(v)\,|\phi(v)|^{\varepsilon} > 0$ holds for $0 < v \leq t_1$, and it follows that

$$(t - t_0 + \lambda)\,|\phi(t)| \geq \lambda c_\varepsilon + |\phi_1| \int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} k(|\phi(v)|)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s,$$

with $k(x) := \phi_0^{1+\varepsilon} + x^{1+\varepsilon}$.

We now set $T := t_0 + 1$ and define $w : [0, T) \mapsto \mathbb{R}$ by

$$w(t) = \begin{cases} 0 & t < t_0, \\ (T-t)^{-3/\varepsilon} & t \geq t_0. \end{cases}$$

In $[t_0, T)$, we have

$$\begin{aligned} &\int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} k(w(v))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \\ &\geq \int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} w(v)^{1+\varepsilon}\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \\ &\geq -\frac{1}{2}\alpha_2^{-1} - (\alpha_1\alpha_2)^{-1} - (\alpha_0\alpha_1\alpha_2)^{-1} + (\alpha_0\alpha_1\alpha_2)^{-1}(T-t)^{-\alpha_0}, \end{aligned}$$

with $\alpha_0 := 3/\varepsilon$, $\alpha_1 := 3/\varepsilon + 1$, $\alpha_2 := 3/\varepsilon + 2$ and $\alpha_3 := 3/\varepsilon + 3$ (remember that $T - t_0 = 1$).

Because of $|\phi_1| = (\lambda + 1)\alpha_0\alpha_1\alpha_2$, we conclude that

$$\begin{aligned} \lambda c_\varepsilon + |\phi_1| &\int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} k(w(v))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \\ &\geq (\lambda+1)w(t) + \lambda c_\varepsilon - (\lambda+1)\Big(1 + \alpha_0 + \frac{1}{2}\alpha_0\alpha_1\Big), \end{aligned}$$

and, by the choice of c_ε,

$$\begin{aligned} \lambda c_\varepsilon - (\lambda+1)\Big(1 + \alpha_0 + \frac{1}{2}\alpha_0\alpha_1\Big) &= (\lambda+2)\Big(1 + \alpha_0 + \frac{1}{2}\alpha_0\alpha_1\Big) \\ &\quad - (\lambda+1)\Big(1 + \alpha_0 + \frac{1}{2}\alpha_0\alpha_1\Big) > 0. \end{aligned}$$

It follows that

$$\lambda c_\varepsilon + |\phi_1| \int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} k(w(v))\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s \\ > (\lambda+1)w(t) \geq (\lambda + t - t_0)w(t).$$

and, with ϕ fulfilling the estimate,

$$(t - t_0 + \lambda)\,|\phi(t)| \geq \lambda c_\varepsilon + |\phi_1| \int_{t_0}^{t}\int_{t_1}^{s}\int_{t_1}^{r} k(|\phi(v)|)\,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}s,$$

we can apply Lemma 4.2, so

$$|\phi(t)| > w(t)$$

holds in $[t_0, T)$ and ϕ exists at most in $[0, T)$. □

5. Long term behavior for small data. In this section, we show that the integro-differential equation (1.2) with $G(x) = x$ and $F = 0$

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + \int_0^t M(x(t-s))x(s)\,\mathrm{d}s, \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{5.1}$$

has global solutions for small data, if the linear system is stable.

Our main assumption on M will be $M(x) = \mathcal{O}(|x|^\alpha)$ for some $\alpha \geq 1$. We obtain exponentially decaying solutions if $\alpha > 1$, $\alpha = 1$ leads to a polynomial decay.

A function G decaying faster than linear for $x \to 0$ does not change the qualitative behavior; it would just change certain constants. We also drop the right hand side F for simplicity.

5.1. Exponential stability.

Theorem 5.1. *Let $x_0 \in \mathbb{C}^n$ and $M \in C^0(\mathbb{C}^n, \mathbb{C}^{n\times n})$ be locally Lipschitz continuous. Assume that the real parts of the eigenvalues of $A \in \mathbb{C}^{n\times n}$ are strictly positive and denote by $\mu_A > 0$, $c_A \geq 1$ as constants with $\left|e^{-At}x\right| \leq c_A e^{-\mu_A t}\,|x|$ $(x \in \mathbb{C}^n,\ t \in [0,\infty))$.*

Suppose, for some constants $c_M > 0$, $\alpha > 1$ and an arbitrary $c_e > c_A$,

$$|M(z)x| \le c_M |z|^\alpha |x| \quad (x, z \in \mathbb{C}^n, |z| \le |x_0| c_e).$$

If the smallness condition

$$c_M |x_0|^\alpha \le \frac{c_e/c_A - 1}{c_e^{\alpha+1}} (\alpha - 1)\mu(\mu_A - \mu)$$

holds for some $\mu \in (0, \mu_A)$, then there is a unique global solution $x \in C^1([0,\infty), \mathbb{C}^n)$ to (5.1) with

$$|x(t)| \le |x_0| c_e e^{-\mu t}.$$

Proof. We show that the operator $K : X \subset C^0([0,\infty), \mathbb{C}^n) \to C^0([0,\infty), \mathbb{C}^n)$ defined as

$$(K(x))(t) = e^{-At}x_0 + \int_0^t e^{-A(t-s)} \int_0^s M(x(s-r))x(r)\,\mathrm{d}r\,\mathrm{d}s$$

has a fixed-point in the set

$$X := \left\{ x \in C^0([0,\infty), \mathbb{C}^n) \;\middle|\; |x(t)| \le |x_0| c_e e^{-\mu t} \right\}.$$

A direct calculation yields, with $\mu_A - \mu > 0$ and $\alpha - 1 > 0$,

$$\begin{aligned}|K(x)(t)| &\le |x_0| c_A e^{-\mu_A t} \\ &\qquad + c_A e^{-\mu_A t} \int_0^t e^{\mu_A s} \int_0^s c_M c_e^{\alpha+1} |x_0|^{\alpha+1} e^{-\alpha\mu s} e^{\mu(\alpha-1)r}\,\mathrm{d}r\,\mathrm{d}s \\ &\le |x_0| c_A \left(1 + \frac{c_M c_e^{\alpha+1} |x_0|^\alpha}{\mu(\alpha-1)(\mu_A - \mu)}\right) e^{-\mu t}\end{aligned}$$

for $x \in X$, and our smallness condition gives $|K(x)(t)| \le |x_0| c_e e^{-\mu t}$, which means $K(X) \subset X$.

By the Lipschitz continuity of M, we deduce the continuity of K on X and, for $x \in X$ follows, that $|\frac{\mathrm{d}}{\mathrm{d}t}(K(x))(t)| \le c$ for some $c > 0$ independent of x, so the set $K(X)$ is bounded in $C^1([0,\infty), \mathbb{C}^n)$.

Therefore, for $(x_n)_n \subset X$, the sequences $(y_n)_n := (K(x_n))_n$ and $(\dot{y}_n)_n$ are bounded, and thus there is, for every $t \ge 0$, a pointwise limit $y(t)$ of a subsequence. The convergence is uniform on any compact interval, which implies $y \in X$. Hence, y decays exponentially, and this yields the uniform convergence of a subsequence on $[0, \infty)$.

We now have that K is compact, and so we can apply Schauder's fixed-point theorem. This fixed-point is a solution to (5.1). The uniqueness follows as in Section 2. □

Applying these results to the special case (1) we obtain stable solutions for the mode-coupling equation.

Corollary 5.2. *Let* $\lambda > 0$, id *the unit matrix in* $\mathbb{C}^{d\times d}$ *and*

$$A := \begin{pmatrix} 0 & -\operatorname{id} \\ \frac{1}{\lambda}\operatorname{id} & \frac{1}{\lambda}\operatorname{id} \end{pmatrix},$$

where $c_A \geq 1$, $\mu_A > 0$ *are constants with* $\left|e^{-At}x\right| \leq c_A e^{-\mu_A t}\left|x\right|$ $(x \in \mathbb{C}^{2d},\ t \geq 0)$. *Let* $\phi_0, \phi_1 \in \mathbb{C}^d$ *be given, and suppose that* $m \in C^0(\mathbb{C}^d, \mathbb{C}^{d\times d})$ *is locally Lipschitz continuous with*

$$\left|m(z)x\right| \leq c_m \left|z\right|^\alpha \left|x\right| \quad (x, z \in \mathbb{C}^d, \left|z\right| \leq \left|(\phi_0, \phi_1)\right| c_e)$$

for constants $c_m > 0$, $\alpha > 1$ *and an arbitrary* $c_e > c_A$.

If

$$c_m \left|(\phi_0, \phi_1)\right|^\alpha \leq \lambda \frac{c_e/c_A - 1}{c_e^{\alpha+1}}(\alpha - 1)\mu(\mu_A - \mu) \tag{5.2}$$

holds for some $\mu \in (0, \mu_A)$, *then there is a unique solution* $\phi \in C^2([0,\infty), \mathbb{C}^d)$ *to* (1) *with*

$$\left|(\dot{\phi}(t), \phi(t))\right| \leq \left|(\phi_0, \phi_1)\right| c_e e^{-\mu t}.$$

Remark 5.3 (Possible extensions)**.**

(i) We dropped for simplicity the right hand sides f, but it is easy to see that we can carry over the proofs for $f \in C_b^0([0,\infty), \mathbb{C}^d)$, when $c_f := (1/\lambda) \sup\limits_{t\in[0,\infty)} \int_0^t e^{\mu_A s}\left|f(s)\right| \mathrm{d}s < \infty$. This changes the smallness condition to

$$c_m(\left|(\phi_0, \phi_1)\right| + c_f)^\alpha \leq \lambda \frac{c_e/c_A - 1}{c_e^{\alpha+1}}(\alpha - 1)\mu(\mu_A - \mu).$$

(ii) For functions m, which additionally depend continuously on the parameter s and t, we need an estimate of the form

$$|m(t,s,x)| \leq c_m h(t,s)\,|x|^{\alpha}$$

with a bounded function h, to adopt the proofs. For $h \in C_b^0(\mathbb{R}^2,\mathbb{R})$ only the constant $\|h\|_\infty$ appears in the smallness condition, while an exponentially decaying h leads to better decay rates for the solution, limited by the decay rate of the linear system.

Remark 5.4 (Choice of constants)**.**

(i) Taking $\mu := \mu_A/2$ gives the greatest value for $\mu(\mu_A - \mu)$, but $\mu > \mu_A/2$ leads to better decay rates.

(ii) If the estimate $|m(z)x| \leq c_m\,|z|^{\alpha}\,|x|$ holds on $\mathbb{C}^d$, the choice $c_e := c_A(\alpha + 1/\alpha)$ is optimal.

Together with $\mu = \mu_A/2$, the weakest assumption on the data then reads

$$c_m(|(\phi_0,\phi_1)| + c_f)^{\alpha} \leq \frac{\lambda}{4}\frac{\alpha^{\alpha}(\alpha-1)}{(\alpha+1)^{\alpha+1}}c_A^{-(\alpha+1)}\mu_A^2.$$

Example 5.5 ($d = 1$)**.** For $d = 1$, the eigenvalues of A are

$$\mu_{\pm} = \frac{1}{2\lambda}(1 \pm \sqrt{1-4\lambda}).$$

To estimate $e^{-At}x$ we have to distinguish two cases:

$\lambda \neq \frac{1}{4}$: Here we have

$$|e^{-At}x| \leq |\mu_+|\,\sqrt{2}\left(1 + \frac{1}{\lambda\,|\mu_+ - \mu_-|} + \frac{2}{|\mu_+ - \mu_-|}\right)e^{-t\mu_-}.$$

$\lambda = \frac{1}{4}$: In this case, A has only one eigenvalue $\mu = 2$ and is not diagonalizable. This means that we cannot choose $\mu_A = \mu_- = 2$, but for $\mu_A \in (0,2)$ arbitrary, $e^{-At}x$ can be estimated by

$$\left|e^{-At}x\right| \leq \frac{2e\sqrt{2}}{2-\mu_A}e^{-\mu_A t}\,|x|\,.$$

Example 5.6 (Quadratic kernels)**.** For $d = 1$, $\lambda \neq 1/4$ and $m(z) = c_m z^2$,

$$c_m(|\phi_0|^2 + |\phi_1|^2) \leq \frac{1}{27c_A^3}\frac{(\Re 1 - \sqrt{1-4\lambda})^2}{4\lambda}$$

is the weakest condition to obtain solutions ϕ by Corollary 5.2. We then have

$$|\phi(t)|^2 + \left|\dot{\phi}(t)\right|^2 \leq \frac{9}{4}c_A^2(|\phi_0|^2 + |\phi_1|^2)e^{-(\Re 1-\sqrt{1-4\lambda})/\lambda t}.$$

In **[3]**, there are several examples given for physically relevant kernels, which are quadratic. Also, other polynomials without a linear part are considered there.

Example 5.7. Once again, we take $d = 1$, but now we investigate a kernel with different powers in z,

$$m(z) = az^\alpha + bz^\beta,$$

where $a, b \in \mathbb{C}\backslash\{0\}$ and $1 < \alpha < \beta$. Here, c_m depends on the choice of c_e, because the dominating term is bz^β for large z and az^α for small z.

For $z \in \mathbb{C}^d$, $|z| \leq c_e\,|(\phi_0, \phi_1)|$ it holds that

$$|m(z)| \leq |a|\,|z|^\alpha + |b|\,|z|^\beta \leq (|a| + |b|\,(c_e\,|(\phi_0, \phi_1)|)^{\beta-\alpha})\,|z|^\alpha =: c_m\,|z|^\alpha\,.$$

The smallness condition now reads:

$$(|a| + |b|\,(c_e\,|(\phi_0, \phi_1)|)^{\beta-\alpha})\,|(\phi_0, \phi_1)|^\alpha \leq \frac{\lambda}{4}\frac{c_e/c_A - 1}{c_e^{\alpha+1}}\mu_A^2,$$

and so the optimal choice of c_e depends not only on α and β, we also have to take $|(\phi_0, \phi_1)|$ as well as a and b into account.

5.2. Polynomial stability.

Theorem 5.8. *Let $x_0 \in \mathbb{C}^n$ and $M \in C^0(\mathbb{C}^n, \mathbb{C}^{n\times n})$ be locally Lipschitz continuous. Assume that the real parts of the eigenvalues of $A \in \mathbb{C}^{n\times n}$ are strictly positive, and denote by $\mu_A > 0$, $c_A \geq 1$ constants with $\left|e^{-At}x\right| \leq c_A e^{-\mu_A t}\,|x|$ $(x \in \mathbb{C}^n,\ t \in [0, \infty))$.*

Take $p > 1$ arbitrary, and set

$$k(p) := \begin{cases} 1 & p < \mu_A, \\ e^{\mu_A - p} \left(\frac{p}{\mu_A}\right)^p & p \geq \mu_A. \end{cases}$$

Suppose, for $c_M > 0$, $\alpha \geq 1$ and an arbitrary $c_p > k(p) c_A$,

$$|M(z)x| \leq c_M |z|^\alpha |x| \quad (x, z \in \mathbb{C}^n, |z| \leq |x_0| c_p).$$

Choose $t_0 \geq 0$ with $t_0 > (p/\mu_A) - 1$, and set

$$k_{t_0} := \int_0^{t_0} e^{\mu_A s} \frac{1}{(1+s)^p} \, \mathrm{d}s.$$

If

$$c_m |(\phi_0, \phi_1)|^\alpha \leq \lambda \left(\frac{c_p}{c_A} - k(p)\right) c_p^{-\alpha-1} \frac{p-1}{2^p} \left(k_{t_0} k(p) + \frac{1+t_0}{\mu_A(1+t_0) - p}\right)^{-1}$$

holds, then there is a unique solution $x \in C^1([0,\infty), \mathbb{C}^n)$ to (5.1) with

$$|x(t)| \leq |x_0| (1+t)^{-p}.$$

Proof. As in the proof of Theorem 5.1 we investigate the operator $K : X \to C^0([0,\infty), \mathbb{C}^n)$, defined as

$$(K(x))(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} \int_0^s M(x(s-r)) x(r) \, \mathrm{d}r \, \mathrm{d}s \, \mathrm{d}s,$$

now in the set

$$X := \left\{ x \in C^0([0,\infty), \mathbb{C}^n) \, \middle| \, |x(t)| \leq |x_0| c_p (1+t)^{-p} \right\}.$$

For $x \in X$, it follows that

$$|M(x(s-r))x(r)| \leq c_M c_p^{\alpha+1} |x_0|^{\alpha+1} (1+s-r)^{-p} (1+r)^{-p}.$$

Using

$$(1+s-r)^{-p}(1+r)^{-p} = (2+s)^{-p} \left(\frac{1}{1+s-r} + \frac{1}{1+r}\right)^p,$$

we obtain

$$\begin{aligned}
&\int_0^s (1+s-r)^{-p}(1+r)^{-p}\,\mathrm{d}r \\
&\leq (2+s)^{-p}2^{p-1}\int_0^s \frac{1}{(1+s-r)^p}+\frac{1}{(1+r)^p}\,\mathrm{d}r \leq \frac{2^p}{p-1}(1+s)^{-p},
\end{aligned}$$

and thus,

$$\int_0^s |M(x(s-r))x(r)|\,\mathrm{d}r \leq c_M c_p^{\alpha+1}\,|x_0|^{\alpha+1}\,\frac{2^p}{p-1}(1+s)^{-p}.$$

It follows that

$$\begin{aligned}
|(K(x))(t)| &\leq c_A\,|x_0|\,e^{-\mu_A t} \\
&\quad + c_A c_M c_p^{\alpha+1}\,|x_0|^{\alpha+1}\,\frac{2^p}{p-1}\int_0^t e^{-\mu_A(t-s)}(1+s)^{-p}\,\mathrm{d}s.
\end{aligned}$$

To handle $\int_0^t e^{-\mu_A(t-s)}(1+s)^{-p}\,\mathrm{d}s$, we choose a $t_0 \geq 0$ with $t_0 > p/\mu_A - 1$.

For $t \geq t_0$, we separate the integral and integrate by parts to conclude:

$$\begin{aligned}
\int_0^t \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s &= \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s + \int_{t_0}^t \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s \\
&= \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s + \frac{1}{\mu_A}\frac{e^{\mu_A t}}{(1+t)^p} - \frac{1}{\mu_A}\frac{e^{\mu_A t_0}}{(1+t_0)^p} \\
&\quad + \frac{p}{\mu_A}\int_{t_0}^t \frac{e^{\mu_A s}}{(1+s)^{p+1}}\,\mathrm{d}s \\
&\leq \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s + \frac{1}{\mu_A}\frac{e^{\mu_A t}}{(1+t)^p} \\
&\quad + \frac{p}{\mu_A}\frac{1}{1+t_0}\int_{t_0}^t \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s \\
&= \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s + \frac{1}{\mu_A}\frac{e^{\mu_A t}}{(1+t)^p} \\
&\quad + \frac{p}{\mu_A}\frac{1}{1+t_0}\int_0^t \frac{e^{\mu_A s}}{(1+s)^p}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&- \frac{p}{\mu_A} \frac{1}{1+t_0} \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s \\
&= \frac{\mu_A(1+t_0) - p}{\mu_A(1+t_0)} \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s \\
&\quad + \frac{1}{\mu_A} \frac{e^{\mu_A t}}{(1+t)^p} + \frac{p}{\mu_A} \frac{1}{1+t_0} \int_0^{t} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s.
\end{aligned}
$$

$$
\Longrightarrow \int_0^t \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s \le \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s + \frac{1+t_0}{\mu_A(1+t_0) - p} \cdot \frac{e^{\mu_A t}}{(1+t)^p}.
$$

For $t < t_0$, obviously

$$
\int_0^t \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s \le \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s + \frac{1+t_0}{\mu_A(1+t_0) - p} \cdot \frac{e^{\mu_A t}}{(1+t)^p}
$$

holds. These estimates lead, along with $k_{t_0} := \int_0^{t_0} e^{\mu_A s} \frac{1}{(1+s)^p} \, \mathrm{d}s$, to

$$
\int_0^t e^{-\mu_A(t-s)} (1+s)^{-p} \, \mathrm{d}s \le k_{t_0} e^{-\mu_A t} + \frac{1+t_0}{\mu_A(1+t_0) - p} \cdot \frac{1}{(1+t)^p},
$$

and hence,

$$
\begin{aligned}
|(K(x))(t)| \le c_A \left| x_0 \right| e^{-\mu_A t} &+ c_A c_M c_p^{\alpha+1} \left| x_0 \right|^{\alpha+1} \frac{2^p}{p-1} \\
&\left(k_{t_0} e^{-\mu_A t} + \frac{1+t_0}{\mu_A(1+t_0) - p} \cdot \frac{1}{(1+t)^p} \right).
\end{aligned}
$$

For $\mu_A \le p$, we have

$$
e^{-\mu_A t} \le \left(\frac{p}{\mu_A} \right)^p e^{\mu_A - p} (1+t)^{-p}
$$

for all $t \in [0, \infty)$, and it follows that

$$
\begin{aligned}
|(K(x))(t)| \le c_A &\left(\frac{p}{\mu_A} \right)^p e^{\mu_A - p} \frac{|x_0|}{(1+t)^p} \\
&+ c_A c_M c_p^{\alpha+1} \left| x_0 \right|^{\alpha} \frac{2^p}{p-1} \\
&\quad \left(k_{t_0} \left(\frac{p}{\mu_A} \right)^p e^{\mu_A - p} + \frac{1+t_0}{\mu_A(1+t_0) - p} \right) \frac{|x_0|}{(1+t)^p}.
\end{aligned}
$$

$c_p > e^{\mu_A - p}(p/\mu_A)^p c_A$ for $p \geq \mu_A$ implies

$$\delta_{\mu_A \leq p} := \left(\frac{c_p}{c_A} - e^{\mu_A - p}\left(\frac{p}{\mu_A}\right)^p\right) c_p^{-\alpha-1}\frac{p-1}{2^p} \left(k_{t_0}\left(\frac{p}{\mu_A}\right)^p e^{\mu_A - p} + \frac{1+t_0}{\mu_A(1+t_0)-p}\right)^{-1} > 0,$$

and if

$$c_M \left|x_0\right|^\alpha \leq \delta_{\mu_A \leq p}$$

holds, it is

$$c_A\left(\frac{p}{\mu_A}\right)^p e^{\mu_A - p} + c_A c_M c_p^{\alpha+1}\left|x_0\right|^\alpha \frac{2^p}{p-1} \left(k_{t_0}\left(\frac{p}{\mu_A}\right)^p e^{\mu_A - p} + \frac{1+t_0}{\mu_A(1+t_0)-p}\right) \leq c_p,$$

and thus $K(x) \in X$.

In the case $\mu_A > p$, we have

$$e^{-\mu_A t} \leq (1+t)^{-p},$$

and thus,

$$\begin{aligned} &|(K(x))(t)| \\ &\leq c_A\frac{|x_0|}{(1+t)^p} + c_A c_M c_p^{\alpha+1}\left|x_0\right|^\alpha \frac{2^p}{p-1}\left(k_{t_0} + \frac{1+t_0}{\mu_A(1+t_0)-p}\right)\frac{|x_0|}{(1+t)^p}. \end{aligned}$$

Because of $c_p > c_A$, it follows that

$$\delta_{\mu_A > p} := \left(\frac{c_p}{c_A} - 1\right) c_p^{-\alpha-1}\frac{p-1}{2^p}\left(k_{t_0} + \frac{1+t_0}{\mu_A(1+t_0)-p}\right)^{-1} > 0,$$

and if

$$c_M \left|x_0\right|^\alpha \leq \delta_{\mu_A > p}$$

holds, we also have

$$c_A + c_A c_M c_p^{\alpha+1}\left|x_0\right|^\alpha \frac{2^p}{p-1}\left(k_{t_0} + \frac{1+t_0}{\mu_A(1+t_0)-p}\right) \leq c_p.$$

This gives $K(x) \in X$. The rest of the proof follows as before. $\square$

We again carry over the result for the system (5.1) to the special case (1).

Corollary 5.9. *Let* $\lambda > 0$, id *be the unit matrix in* $\mathbb{C}^{d\times d}$ *and*

$$A := \begin{pmatrix} 0 & -\mathrm{id} \\ \frac{1}{\lambda}\,\mathrm{id} & \frac{1}{\lambda}\,\mathrm{id} \end{pmatrix},$$

where $c_A \geq 1$ *and* $\mu_A > 0$ *are constants with* $\left|e^{-At}x\right| \leq c_A e^{-\mu_A t}\left|x\right|$ $(x \in \mathbb{C}^{2d},\ t \geq 0)$. *Let* $\phi_0, \phi_1 \in \mathbb{C}^d$ *be given.*

Take $p > 1$ *arbitrary, and set*

$$k(p) := \begin{cases} 1 & p < \mu_A, \\ e^{\mu_A - p}\left(\frac{p}{\mu_A}\right)^p & p \geq \mu_A. \end{cases}$$

Suppose $m \in C^0(\mathbb{C}^d, \mathbb{C}^{d\times d})$ *is locally Lipschitz continuous, and*

$$|m(z)x| \leq c_m\,|z|^\alpha\,|x| \quad (x, z \in \mathbb{C}^d, |z| \leq |(\phi_0, \phi_1)|\,c_p)$$

for constants $c_m > 0$ *and* $\alpha \geq 1$ *and some arbitrary* $c_p > k(p)c_A$. *Choose* $t_0 \geq 0$ *with* $t_0 > (p/\mu_A) - 1$, *and set*

$$k_{t_0} := \int_0^{t_0} e^{\mu_A s}\frac{1}{(1+s)^p}\,\mathrm{d}s.$$

If

$$c_m\,|(\phi_0, \phi_1)|^\alpha \leq \lambda\left(\frac{c_p}{c_A} - k(p)\right)c_p^{-\alpha-1}\frac{p-1}{2^p}\left(k_{t_0}k(p) + \frac{1+t_0}{\mu_A(1+t_0) - p}\right)^{-1}$$

holds, then there is a unique solution $\phi \in C^2([0,\infty), \mathbb{C}^d)$ *to* (1) *with*

$$\left|(\dot{\phi}(t), \phi(t))\right| \leq |(\phi_0, \phi_1)|\,(1+t)^{-p}.$$

Remark 5.10 (Choice of constants)**.**

(i) If $d = 1$, we have by Remark 5.5 $\mu_A \leq 1$ for $\lambda \geq 1/2$, and thus only $p \geq \mu_A$ is possible. For $\lambda \in (0, \frac{1}{2})$, we have $\mu_A \in (1, 2)$, and so we can choose $p \in (1, \mu_A)$.

(ii) t_0 can be chosen independent of μ_A and p in such a way that we get a smallness condition as weak as possible. Therefore, we have to find a minimizer of the function

$$g(t_0) := k(p) \int_0^{t_0} \frac{e^{\mu_A s}}{(1+s)^p} \, \mathrm{d}s + \frac{1+t_0}{\mu_A(1+t_0)-p}$$

in $[0,\infty) \cap (p/\mu_A - 1, \infty)$. The derivative g' is negative in a neighborhood of the lower interval boundary and g is not bounded for $t_0 \to \infty$. So there is some minimizer $t_{\min} > t_1$, which means that, for $p \leq \mu_A$, the choice $t_0 = 0$ is not optimal.

(iii) Here we have that $c_p = c_A(\alpha + 1/\alpha)k(p)$ gives the weakest condition for functions m, which can globally be estimated by a monomial.

Remark 5.11. As in Remark 5.3, we can handle the right hand sides of f and an additional time dependence of the function m. For m, we then need an estimate of the form

$$|m(t,s,x)| \leq c_m h(t,s) \, |x|^\alpha$$

with some $h \in C_b^0(\mathbb{R}^2, \mathbb{R})$. An example for such an h is $h(t) = 1/(1+(\dot{\gamma}t)^2)$ where $\dot{\gamma} > 0$, which is introduced in **[4]** and in a more general setting in **[1]** to describe special physical systems.

The next example shows that, even if the result is stronger, the smallness condition to obtain exponential stability can be weaker than the one for polynomial stability.

Example 5.12 (Quadratic kernel)**.** We once again investigate $d = 1$ and kernels with

$$m(z) = c_m z^2.$$

In Example 5.6 it was shown that, for $\lambda = 1/3$, and thus $\mu_A = 3/2$, the smallness condition

$$c_m\left(|\phi_0|^2 + |\phi_1|^2\right) \leq \frac{1}{36 c_A^3}$$

implies the existence of an exponentially stable solution. Our result for polynomial stability leads to

$$c_m(|\phi_0|^2 + |\phi_1|^2) \leq \frac{4}{81c_A^3} k(p)^{-2} \frac{p-1}{2^p} \left(k_{t_0} k(p) + \frac{1+t_0}{\mu_A(1+t_0)-p} \right)^{-1}.$$

It holds that $k(p) \geq 1$,

$$k_{t_0} k(p) + \frac{1+t_0}{\mu_A(1+t_0)-p} \geq \frac{1+t_0}{\mu_A(1+t_0)-p} > \frac{1}{\mu_A} = \frac{2}{3},$$

and the maximum of $(p-1)/2^p$ is $1/(2\ln(2)e)$. This implies the rough estimate

$$c_m(|\phi_0|^2 + |\phi_1|^2) \leq \frac{1}{27e\ln(2)c_A^3}.$$

Example 5.13. For

$$m(z) = az^2 + bz,$$

we have $\alpha = 1$, and only the result for polynomial stability can be applied. These functions play an important role in the mode-coupling theory; they are linked to the so-called F_{12}-model (**[1, 3, 7, 4, 11]**). Here, again, we have to restrict m to the set $|z| \leq c_p |(\phi_0, \phi_1)|$ to obtain the needed estimate. It then reads

$$|m(z)| \leq \left|az^2\right| + |bz| \leq (|a|\, c_p\, |(\phi_0,\phi_1)| + |b|)\, |z| =: c_m\, |z|\,.$$

REFERENCES

1. J.M. Brader, T. Voigtmann, M. Fuchs, R.G. Larson and M.E. Cates, *Glass rheology: From mode-coupling theory to a dynamical yield criterion*, Proc. Nat. Acad. Sci. **106** (2009), 15186–15197.

2. T.A. Burton, *Volterra integral and differential equations*, Academic Press, Inc., New York, 1983.

3. S.P. Das, *Mode-coupling theory and the glass transition in supercooled liquids*, Rev. Mod. Phys. **76** (2004), 785–851.

4. M. Fuchs and M.E. Cates, *Schematic models for dynamic yielding of sheared colloidal glasses*, Faraday Discuss. **123** (2003), 267–286.

5. M.V. Gnann, I. Gazuz, A.M. Puertas, M. Fuchs and T. Voigtmann, *Schematic models for active nonlinear microrheology*, Soft Matter **7** (2011), 1390–1396.

6. W. Götze, *The essentials of the mode-coupling theory for glassy dynamics*, Cond. Matter Phys. **1** (1998), 873–904.

7. W. Götze, *Complex dynamics of glass-forming liquids*, Oxford University Press, Oxford, 2009.

8. W. Götze and L. Sjögren, *General properties of certain non-linear integro-differential equations*, J. Math. Anal. Appl. **195** (1995), 230–250.

9. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, Cambridge, 1990.

10. R. Haussmann, *Some properties of mode coupling equations*, Z. Phys.–Condensed Matter **79** (1990), 143–148.

11. W. Kob, *The mode-coupling theory of the glass transition*, Supercooled Liquids **4** (1997), 28–44

12. P. Kurth, *On a new class of integro-differential equations*, Konst. Schrift. Math. **326**, 2014.

13. J. Prüß, *Evolutionary integral equations and applications*, Birkhäuser, New York, 2012.

14. M. Saal, *Well-posedness and asymptotics of some nonlinear integro-differential equations*, J. Integral Equations Appl. **25** (2013), 103–141.

15. ———, *Nichtlineare Integro-Differentialgleichungen zweiter Ordnung*, Ph.D. thesis, University of Konstanz, Germany, 2014.

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