JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 26, Number 3, Fall 2014

ON THE PROBLEMS OF PERIDYNAMICS WITH SPECIAL CONVOLUTION KERNELS

S.A. ALIMOV, YANZHAO CAO AND O.A. ILHAN

ABSTRACT. The well-posedness and regularity of a peridynamic model with a special kernel is studied. The differential-integral equation describing the model is first converted to an operator valued Volterra integral equation. Then the existence and regularity of the solution of the peridynamics problem are established through the study of the Volterra integral equation. The regularity results improve the previous known results for more general peridynamics models.

1. Introduction. In the classical theory of solid mechanics, the behavior of solids is described by partial differential equations (PDES) through Newton's second law of motion. However, when spontaneous cracks and fractures exist, such PDE models are inadequate to characterize the discontinuities of physical quantities such as the displacement field. Recently, a peridynamic continuum model was proposed which only involves the integration over the differences of the displacement field [**3, 7, 10, 11**]. A linearized peridynamic model can be described by the following integro-differential equation with initial values.

(1.1)
$$\frac{\partial^2 u(x,t)}{\partial t^2} + \int_\Omega K(x,y)[u(x,t)-u(y,t)]\,dy = f(x,t),\ x\in\Omega,\ t>0,$$
$$u(x,0)=\phi(x),\quad x\in\Omega,$$
$$u_t(x,0)=\psi(x),\quad x\in\Omega,$$

Keywords and phrases. Peridynamics, Volterra integral equations, nonlocal modeling.

This research is partially supported by the National Science Foundation under grant number DMS0914554 and US Air Force Office of Scientific Research under grant number FA550-07-1-0154. The project was supported by Guangdong Provincial Government of China through the Computational Science Innovative Research Team program.

Received by the editors on August 24, 2013, and in revised form on January 22, 2014.

DOI:10.1216/JIE-2014-26-3-301

where $\Omega \subset \mathbf{R}^n$. Here $u : \Omega \times [0,T] \to \mathbf{R}^n$ is the unknown function, the $n \times n$ matrix-function K defined on $\Omega \times \Omega$ is the kernel, and $f : \Omega \times [0,T] \to \mathbf{R}^n$ is the external force. We consider all the integrals in terms of tempered distributions (see **[8]**).

A typical kernel for peridynamic models is given by the following function (see **[11]**).

$$K(x,y) = \frac{(x-y) \otimes (x-y)}{\lambda(|x-y|)}, \tag{1.2}$$

where $\otimes$ denotes the dyadic product and λ is a real valued function. This problem has been studied by several authors. In particular, when Ω is bounded, it is shown in **[5, 6]** that if

$$\int_0^1 \frac{r^{n-1}}{\lambda(r)}\, dr < \infty, \tag{1.3}$$

then a unique solution $u \in C^1([0,T], L^2(\Omega)^d)$ of (1.1) exists. When $\Omega = R^n$, Du and Zhou **[14]** show that a unique solution of (1.1) also exists in a certain function space when

$$\int_0^1 \frac{r^{n+3}}{\lambda(r)}\, dr < \infty, \tag{1.4}$$

which is a weaker condition than (1.3).

In this paper, we study the regularities of the solution of (1.1) with kernel K given by

$$K(x,y) = \frac{(x-y) \otimes (x-y)}{|x-y|^{\alpha+2}}, \quad x, y \in \mathbf{R}^n,\ n > 1. \tag{1.5}$$

Equation (1.5) is an important special case of a class of linearized peridynamic models (see **[4,** page 50**]**). To study such a special case, we first rewrite K defined above as a differential operator applied to an integral operator with kernel $|x-y|^{2-\alpha}$. Then we convert (1.1) into an operator valued Volterra integral equation of the second kind with a differential-integral operator kernel. Using pseudo-differential operator theory, we demonstrate the well-posedness and regularity of (1.1) for $0 < \alpha \leq n$ through well-posedness of the Volterra integral equation. For $0 < \alpha < n$, our result improves the regularity result of **[5]**; for $\alpha = n$, our result improves a result of Du and Zhou **[14]**.

The paper is organized as follows. In the next section, we convert (1.1) into an operator valued Volterra integral equation of the second kind and study the existence and regularity of its solution for $0 < \alpha < n$. In the last section, we consider the $\alpha = n$ case.

2. Solution regularity for $\alpha < n$.

2.1. Conversion to operator valued Volterra integral equations. Let operator A and matrix Λ be defined as

$$Av(x) = \int_\Omega K(x,y)v(y)\,dy, \quad x \in \Omega, \tag{2.1}$$

and

$$\Lambda(x) = \int_\Omega K(x,y)\,dy, \quad x \in \Omega, \tag{2.2}$$

where K is given by (1.5). Then we may rewrite (1.1) as

$$\frac{\partial^2 u(x,t)}{\partial t^2} - [A - \Lambda(x)]u(x,t) = f(x,t), \quad x \in \Omega. \tag{2.3}$$

Integrating (2.3) twice with respect to t, we obtain the Volterra type operator valued integral equation

$$u(x,t) - [A - \Lambda(x)]\int_0^t (t-s)u(x,s)\,ds = F(x,t), \quad x \in \Omega,\ t > 0. \tag{2.4}$$

where

$$F(x,t) = \phi(x) + t\psi(x) + \int_0^t (t-s)f(x,s)\,ds, \qquad x \in \Omega, \quad t > 0. \tag{2.5}$$

We refer the readers to **[1, 2]** for studies of operator valued integral equations.

2.2. Solution regularity for $0 < \alpha < n$. Assume that the kernel function K is given by (1.5). Set

$$\nabla \otimes \nabla = \begin{pmatrix} \frac{\partial^2}{\partial x_1^2} & \frac{\partial^2}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} \\ \frac{\partial^2}{\partial x_2 \partial x_1} & \frac{\partial^2}{\partial x_2^2} & \cdots & \frac{\partial^2}{\partial x_2 \partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2}{\partial x_n \partial x_1} & \frac{\partial^2}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2}{\partial x_n^2} \end{pmatrix}$$

and

$$\nabla^2 = \Delta I = \begin{pmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \Delta \end{pmatrix}$$

where Δ is the Laplace operator and I is the identity matrix.

Proposition 2.1. *For $\alpha \neq 2$ and $\alpha < n$, the following holds*:

$$\left[\nabla \otimes \nabla - \frac{1}{n-\alpha}\nabla^2\right] \frac{1}{|x-y|^{\alpha-2}} = \alpha(\alpha-2)\frac{(x-y)\otimes(x-y)}{|x-y|^{\alpha+2}}. \tag{2.6}$$

Proof. Notice that

$$\frac{\partial}{\partial x_k} \frac{1}{|x-y|^{\alpha-2}} = -(\alpha-2)\frac{x_k - y_k}{|x-y|^{\alpha}},$$

and, for $j \neq k$,

$$\frac{\partial^2}{\partial x_j \partial x_k} \frac{1}{|x-y|^{\alpha-2}} = \alpha(\alpha-2)\frac{(x_k-y_k)(x_j-y_j)}{|x-y|^{\alpha+2}}.$$

Also notice that

$$\frac{\partial^2}{\partial x_k^2} \frac{1}{|x-y|^{\alpha-2}} = \alpha(\alpha-2)\ \frac{(x_k-y_k)^2}{|x-y|^{\alpha+2}} - (\alpha-2)\frac{1}{|x-y|^{\alpha}}.$$

Hence,

$$(\nabla \otimes \nabla)\frac{1}{|x-y|^{\alpha-2}} = \alpha(\alpha-2)\ \frac{(x-y)\otimes(x-y)}{|x-y|^{\alpha+2}} - \frac{\alpha-2}{|x-y|^{\alpha}}I.$$

Let $r = |x|$. Then

$$\Delta \frac{1}{|x-y|^{\alpha-2}} = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} \frac{\partial r^{2-\alpha}}{\partial r} \right) = \frac{2-\alpha}{r^{n-1}} \frac{\partial r^{n-\alpha}}{\partial r}$$
$$= -\frac{(n-\alpha)(\alpha-2)}{|x-y|^{\alpha}}.$$

Thus,

$$\frac{1}{n-\alpha} \Delta \frac{1}{|x-y|^{\alpha-2}} = -\frac{\alpha-2}{|x-y|^{\alpha}},$$
$$\frac{1}{n-\alpha} \nabla^2 \frac{1}{|x-y|^{\alpha-2}} = -\frac{\alpha-2}{|x-y|^{\alpha}} I,$$

and (2.6) follows. □

Set

$$L_\alpha(\nabla) = \frac{1}{\alpha(\alpha-2)} \left[(\nabla \otimes \nabla) - \frac{1}{n-\alpha} \nabla^2 \right].$$

In the rest of this section we assume that Ω is a bounded domain with piecewise smooth boundary. In view of Proposition 2.1, we may rewrite the operator A defined by (2.1) as

$$Au(x) = L_\alpha(\nabla) \int_\Omega \frac{u(y)}{|x-y|^{\alpha-2}}\, dy \quad x \in \Omega.$$

For any $p > 1$ and $\beta > 0$ we define the Sobolev space $L_p^\beta(\mathbf{R}^n)$ as the Banach space of the functions $f \in L_p(\mathbf{R}^n)$ with the finite norm defined by (see [**9,** page 154])

$$\|f\|_{L_p^\beta} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\beta/2} \cdot \mathcal{F}f(\xi)]\|_{L_p(\mathbf{R}^n)}.$$

Here

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx$$

is the Fourier transform in terms of distributions.

Remark 2.2. For the integer $\beta > 0$, the space $L_p^\beta(\mathbf{R}^n)$ coincides with the usual Sobolev space $W_p^\beta(\mathbf{R}^n)$.

Theorem 2.3. *Let $0 < \alpha < n$, $\alpha \neq 2$ and $1 < p < \infty$. Then A is a bounded operator from $L_p(\Omega)$ to Sobolev space $L_p^{n-\alpha}(\mathbf{R}^n)$.*

Proof. For an arbitrary $R > 0$, we introduce $\chi \in C^\infty(\mathbf{R})$ such that $0 \leq \chi(r) \leq 1$ for $r \in \mathbf{R}$ and

$$\chi(r) = \begin{cases} 1 & 0 \leq |r| \leq R, \\ 0 & |r| \geq 2R. \end{cases} \tag{2.7}$$

Set

$$g_\alpha(x) = \frac{\chi(|x|)}{|x|^{\alpha-2}}, \quad x \in \mathbf{R}^n,$$

and consider the linear operator G_α defined as

$$\begin{aligned} G_\alpha u(x) &= \int_{\mathbf{R}^n} g_\alpha(x-y)\, u(y)\, dy \\ &= \int_{\mathbf{R}^n} \frac{u(y)}{|x-y|^{\alpha-2}} \chi(|x-y|)\, dy, \quad x \in \Omega, \end{aligned}$$

for $u \in L_p(\mathbf{R}^n)$ such that $u(x) = 0$ for $x \notin \Omega$. It is clear that for $R > \operatorname{diam}(\Omega)$

$$A = L_\alpha(\nabla) G_\alpha.$$

Note that the Fourier transform $\widehat{g_\alpha}$ of g_α is a $C^\infty(\mathbf{R}^n)$ function. To study the behavior of $\widehat{g_\alpha} = \widehat{g_\alpha}(\xi)$ as $|\xi| \to \infty$, we write

$$\begin{aligned} \widehat{g_\alpha}(\xi) &= \int_{\mathbf{R}^n} \frac{e^{-ix\cdot\xi}}{|x|^{\alpha-2}}\, dx - \int_{\mathbf{R}^n} \frac{e^{-ix\cdot\xi}}{|x|^{\alpha-2}} [1 - \chi(|x|)]\, dx \\ &= \frac{c_{n,\alpha}}{|\xi|^{n+2-\alpha}} - \int_{|x|\geq R} \frac{e^{-ix\cdot\xi}}{|x|^{\alpha-2}} [1 - \chi(|x|)]\, dx. \end{aligned}$$

Note that

$$e^{-ix\cdot\xi} = -\frac{1}{|\xi|^2} \Delta_x e^{-ix\cdot\xi}.$$

Hence, using Green's formula, we have that

$$\widehat{g_\alpha}(\xi) = \frac{c_{n,\alpha}}{|\xi|^{n+2-\alpha}} - \frac{(-1)}{|\xi|^2} \int_{|x|\geq R} e^{-ix\cdot\xi} \Delta_x \frac{1 - \chi(|x|)}{|x|^{\alpha-2}}\, dx.$$

Repeating this process m times we obtain

$$\widehat{g_\alpha}(\xi) = \frac{c_{n,\alpha}}{|\xi|^{n+2-\alpha}} - \frac{(-1)^m}{|\xi|^{2m}} \int_{|x|\geq R} e^{-ix\cdot\xi} \Delta_x^m \frac{1-\chi(|x|)}{|x|^{\alpha-2}}\, dx.$$

Note that, for $|x| > 2R$, all derivatives of $\chi = \chi(x)$ vanish. Thus,

$$\widehat{g_\alpha}(\xi) = \frac{c_{n,\alpha}}{|\xi|^{n+2-\alpha}} - \frac{(-1)^m}{|\xi|^{2m}} \int_{R\leq|x|\leq 2R} e^{-ix\cdot\xi} \Delta_x^m \frac{1-\chi(|x|)}{|x|^{\alpha-2}}\, dx$$
$$- \frac{b_{m,\alpha}}{|\xi|^{2m}} \int_{|x|\geq 2R} e^{-ix\cdot\xi} \frac{1}{|x|^{\alpha-2+2m}}\, dx.$$

Now it is clear that, for $|\xi| \geq 1$ and for any multiindex μ with $|\mu| \leq n$, we have

$$D_\xi^\mu \widehat{g_\alpha}(\xi) = \frac{O(1)}{|\xi|^{n+2-\alpha+|\mu|}}.$$

Taking into account the fact that $\widehat{g_\alpha}$ and all its derivatives are bounded in $\mathbf{R}^n$, we get

$$|D_\xi^\mu \widehat{g_\alpha}(\xi)| \leq \frac{C}{(1+|\xi|)^{n+2-\alpha+|\mu|}}, \quad \xi \in \mathbf{R}^n,\ |\mu| \leq n.$$

Set

$$m_\alpha(\xi) = \widehat{g_\alpha}(\xi) \cdot (1+|\xi|^2)^{(n+2-\alpha)/2}.$$

It is easy to see that

$$|D_\xi^\mu m_\alpha(\xi)| \leq \text{const}\,(1+|\xi|)^{-|\mu|}.$$

Hence, m_α is a multiplier from $L_p(\mathbf{R}^n)$ to $L_p(\mathbf{R}^n)$ for $p > 1$ (see **[9]**). Therefore,

$$\widehat{g_\alpha}(\xi) = \frac{m_\alpha(\xi)}{(1+|\xi|^2)^{(n+2-\alpha)/2}}$$

is a multiplier from $L_p(\mathbf{R}^n)$ to Sobolev space $L_p^{n+2-\alpha}(\mathbf{R}^n)$. This means that G_α is a continuous operator from $L_p(\Omega)$ to $L_p^{n+2-\alpha}(\mathbf{R}^n)$.

Since $L_\alpha(\nabla)$ is a second order differential operator, we conclude that $A = L_\alpha(\nabla)G_\alpha$ is a continuous operator from $L_p(\Omega)$ to $L_p^{n-\alpha}(\mathbf{R}^n)$. □

Corollary 2.4. *Let $0 < \alpha < n$ and $\alpha \neq 2$. Then for any $p > 1$, A is bounded from $L_p(\Omega)$ to $L_p(\Omega)$.*

Define operator B by

$$Bu(x,t) = \int_0^t (t-s)[A - \Lambda(x)]u(x,s)\, ds, \quad x \in \Omega,\ t > 0, \tag{2.8}$$

and set

$$M = \|(A - \Lambda)\|_{L_p(\Omega)\to L_p(\Omega)}.$$

We say that $u \in C[\overline{R_+} \to L_p(\Omega)]$ if $u(\cdot,t)$ continuously depends on t for $t \geq 0$ in the $L_p(\Omega)$ norm and $u \in C[\overline{R_+} \to L_p^\beta(\Omega)]$ if $u(\cdot,t)$ continuously depends on t for $t \geq 0$ in the $L_p^\beta(\Omega)$ norm. For $u \in C[\overline{R_+} \to L_p(\Omega)]$, define

$$\|u\|_t = \sup_{0\leq s\leq t} \|u(x,s)\|_{L_p(\Omega)}. \tag{2.9}$$

Proposition 2.5. *Let $u \in C[\overline{R_+} \to L_p(\Omega)]$. Then, for any $m \geq 1$, the following holds*:

$$\|B^m u\|_t \leq \frac{M^m t^{2m}}{(2m-1)!!} \|u\|_t, \tag{2.10}$$

where $(2m-1)!! = \Pi_{k=1}^m (2k-1)$.

Proof. The proof follows from mathematical induction:

$$\begin{aligned} \|B^{m+1}u(x,t)\| &\leq M \int_0^t (t-s)\|B^m u(x,s)\|\, ds \\ &\leq\ Mt \int_0^t \frac{M^m s^{2m}}{(2m-1)!!} \|u\|_s\, ds \\ &\leq Mt\|u\|_t \frac{M^m t^{2m+1}}{(2m+1)!!} = \frac{M^{m+1} t^{2(m+1)}}{(2m+1)!!} \|u\|_t. \qquad \square \end{aligned}$$

Now consider the equation

$$(I - B)u = F. \tag{2.11}$$

Proposition 2.6. *For any $F \in C[\overline{R_+} \to L_p(\Omega)]$, equation* (2.11) *has a solution $u \in C[\overline{R_+} \to L_p(\Omega)]$. Moreover,*

$$\|u - F\|_t \leq \left(e^{Mt^2} - 1\right) \|F\|_t. \tag{2.12}$$

Proof. It follows from Proposition 2.5 that the Neumann series

$$u(x,t) = F(x,t) + \sum_{k=1}^{\infty} B^k F(x,t)$$

converges in $L_p(\Omega)$ uniformly with respect to t for any compact set of t. It is clear that u is the solution to (2.11):

$$(I - B)u = \left(F + \sum_{k=1}^{\infty} B^k F \right) - \left(BF + B \sum_{k=1}^{\infty} B^k F \right) = F.$$

The required estimate (2.12) follows from (2.10):

$$\begin{aligned} \|u - F\|_t &\le \sum_{k=1}^{\infty} \|B^k F\|_t \le \sum_{k=1}^{\infty} \frac{M^k t^{2k}}{(2k-1)!!} \|F\|_t \\ &\le \|F\|_t \sum_{k=1}^{\infty} \frac{M^k t^{2k}}{k!} = \left(e^{Mt^2} - 1 \right) \|F\|_t. \end{aligned}$$ □

Proposition 2.7. *The solution of* (2.11) *is unique.*

Proof. Indeed, if $(I - B)v = 0$, then, according to (2.10),

$$v = Bv = B^2 v = \cdots = B^m v \longrightarrow 0, \quad m \to \infty,$$

and consequently $v = 0$. □

Now we are ready to prove the existence, uniqueness and regularity of the solution to problem (1.1). We say that $f \in L_p^\beta(\Omega)$ if there exists a function $f_* \in L_p^\beta(R^n)$ such that $f(x) = f_*(x)$ for $x \in \Omega$. The corresponding norm is defined as

$$\|f\|_{L_p^\beta(\Omega)} = \inf_{f_*} \|f_*\|_{L_p^\beta(\mathbf{R}^n)}.$$

Also, we say that $u \in C^2[\overline{R_+} \to L_p(\Omega)]$ if $u''_{tt} \in C[\overline{R_+} \to L_p(\Omega)]$.

Theorem 2.8. *Let* $p > 1$, $0 < \alpha < n$, $\alpha \ne 2$ *and* $0 \le \beta \le n - \alpha$. *Assume that* $\phi \in L_p^\beta(\Omega)$, $\psi \in L_p^\beta(\Omega)$ *and* $f \in C[\overline{R_+} \to L_p^\beta(\Omega)]$. *Then problem* (1.1) *has a unique solution in* $C^2[\overline{R_+} \to L_p^\beta(\Omega)]$.

Proof. It follows from Propositions 2.6 and 2.7 that the solution u of (2.3) exists, is unique and belongs to $C[\overline{R_+} \to L_p(\Omega)]$. Directly from this equation, one can see that $u \in C^2[\overline{R_+} \to L_p(\Omega)]$ and u is the unique solution to problem (1.1).

To prove that $u \in C[\overline{R_+} \to L_p^\beta(\Omega)]$, we use the following representation of the solution of (1.1) (see Appendix A):

$$u(x,t) = \int_0^t \frac{1}{\sqrt{\Lambda(x)}} \sin(\sqrt{\Lambda(x)}(t-s))\, Au(x,s)\, ds + F_1(x,t) \tag{2.13}$$

where

$$F_1(x,t) = \cos\sqrt{\Lambda(x)}t\,\phi(x) + \frac{\sin\sqrt{\Lambda(x)}t}{\sqrt{\Lambda(x)}}\,\psi(x) + \frac{1}{\sqrt{\Lambda(x)}}\int_0^t \sin\left(\sqrt{\Lambda(x)}(t-s)\right) f(x,s)\, ds. \tag{2.14}$$

According to Theorem 2.3, $\Lambda \in L_p^{n-\alpha}(\Omega)$. Furthermore, if $u \in C[\overline{R_+} \to L_p(\Omega)]$, then, according to the same theorem, $Au \in C[\overline{R_+} \to L_p^{n-\alpha}(\Omega)]$. Therefore, if $F_1 \in C[\overline{R_+} \to L_p^\beta(\Omega)]$ for $\beta \le n-\alpha$, then it follows from (2.13) that $u \in C[\overline{R_+} \to L_p^\beta(\Omega)]$. Using again the representation (2.13), we conclude that $u \in C^2[\overline{R_+} \to L_p^\beta(\Omega)]$. $\square$

Remark 2.9. The representation (2.13) can also be used to study other properties of the solution to (1.1), such as smoothness and oscillatory behaviors.

3. Existence and regularity for $\alpha = n$. In this section, we assume that $n \ge 3$. The case when $\alpha = n$ is an important one since it represents a more suitable nonlocal continuum model for the corresponding material under consideration (see [**14,** page 1761]). However, this case is much more complicated since it is difficult to find a Hilbert space H such that operator $A : H \to H$ is bounded. To overcome this difficulty, we shall modify the operator A and introduce a sequence of Hilbert spaces in which A is a bounded operator. We start with the following proposition which can be proved through simple computations.

Proposition 3.1. *For $n \geq 3$ and $x \neq y$, the following holds:*

$$(\nabla \otimes \nabla) \frac{1}{|x-y|^{n-2}} - \nabla^2 \frac{\ln |x-y|}{|x-y|^{n-2}} = n(n-2) \frac{(x-y) \otimes (x-y)}{|x-y|^{n+2}}. \tag{3.1}$$

The following lemma is needed to prove Propositions 3.3 and 3.4.

Lemma 3.2. *For $\xi \in \mathbf{R}^n$, the following holds:*

$$\int_{|x|=r} e^{-i\xi x} ds = (2\pi)^{n/2} r^{n-1} \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}},$$

where ds denotes the surface integral on $\{x; |x| = r\} \subset \mathbf{R}^n$.

Proof. Let $\psi(x) = e^{-i\xi x}$, $x, \xi \in \mathbf{R}^n$. Notice that ψ satisfies the following partial differential equation

$$\Delta\psi(x) + |\xi|^2\psi(x) = 0, \quad x \in R^n.$$

By (18.2.1) (with $q = 0$ and $\lambda_n = |\xi|^2$) and (18.3.4) (with $\mathbf{u}$=0) of **[12]** we have that (after a change from area integral to angular integral)

$$\begin{aligned}\int_{|x|=r} e^{-ix\xi}\, ds &= (2\pi)^{n/2} r^{n-1} \psi(0) \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}} \\ &= (2\pi)^{n/2} r^{n-1} \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}}.\end{aligned}$$

□

The next proposition shows the difficulty of studying the integral equation (2.11) in its current form.

Proposition 3.3. *Operator $A_0 : L_2(\mathbf{R}^n) \to L_2(\mathbf{R}^n)$ defined by*

$$A_0 u(x) = \int_{\mathbf{R}^n} \frac{u(y)}{|x-y|^n}\, dy$$

is not bounded in the domain $D(A_0) = S(\mathbf{R}^n)$.

Proof. 1) First, we consider the case $n > 3$. Define a kernel function K_* as

$$K_*(x) = \frac{\ln|x|}{|x|^{n-2}}, \quad x \in \mathbf{R}^n,$$

and assume that $\widehat{K_*}$ is its Fourier transform. By Lemma 3.2, we have that

$$\begin{aligned}
\widehat{K_*}(\xi) &= \int_{R^n} K_*(x) e^{-ix\cdot\xi}\, dx \\
&= \int_{R^n} \frac{\ln|x|}{|x|^{n-2}}\, e^{-ix\cdot\xi}\, dx \\
&= (2\pi)^{n/2} \int_0^\infty \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}}\, \frac{\ln r}{r^{n-2}}\, r^{n-1}\, dr \\
&= (2\pi)^{n/2}|\xi|^{1-n/2} \int_0^\infty J_{n/2-1}(r|\xi|) \ln r\, r^{2-n/2}\, dr \\
&= \frac{(2\pi)^n}{|\xi|^2} \int_0^\infty J_{n/2-1}(t) \ln t\, t^{2-n/2}\, dt \\
&\quad - \frac{(2\pi)^{n/2}}{|\xi|^2} \ln|\xi| \int_0^\infty J_{n/2-1}(t)\, t^{2-n/2}\, dt.
\end{aligned}$$

Hence,

$$\widehat{K_*}(\xi) = \frac{a_1}{|\xi|^2} - a_2 \frac{\ln|\xi|}{|\xi|^2},$$

where

$$\begin{aligned}
a_1 &= \int_0^\infty J_{n/2-1}(t) \ln t\, t^{2-n/2} dt, \\
a_2 &= (2\pi)^n \int_0^\infty J_{n/2-1}(t)\, t^{2-n/2} dt
\end{aligned}$$

are constants (see [**13,** 13.24(1)]). This implies that

$$\widehat{\Delta K_*}(\xi) = a_2 \ln|\xi| - a_1.$$

Obviously $\widehat{\Delta K_*}$ is not a multiplier since it is unbounded. According to the equation

$$\Delta K_*(x-y) = -\frac{n-2}{|x-y|^n},$$

the operator A_0 is also unbounded.

2) We now consider the case $n = 3$. Write

$$\frac{\ln|x|}{|x|} = \frac{\ln|x|}{|x|}\,\chi(|x|) + \frac{\ln|x|}{|x|}\,[1-\chi(|x|)] \\ := K_1(x) + K_2(x),$$

where $\chi \in C_0^\infty(\mathbf{R})$ such that

$$\chi(r) = \begin{cases} 1 & \text{for } r \le 1, \\ 0 & \text{for } r \ge 2, \end{cases}$$

and $0 \le \chi(r) \le 1$ for $r \in \mathbf{R}$.

It is clear that the function $\Delta[\Delta K_2(x)]$ is integrable over $\mathbf{R}^3$. Hence, its Fourier transform is bounded, and therefore

$$|\xi|^2 |\widehat{\Delta K_2}(\xi)| \le \text{const},$$

which implies that $\widehat{\Delta K_2}(\xi)$ is bounded for $|\xi| \ge 1$.

Next, we consider the Fourier transform of K_1 given by

$$\begin{aligned} \widehat{K_1}(\xi) &= \int_{R^3} \frac{\ln|x|}{|x|}\,\chi(|x|)\,e^{-ix\cdot\xi}\,dx \\ &= (2\pi)^3 \int_0^\infty \frac{J_{1/2}(r|\xi|)}{(r|\xi|)^{1/2}}\,\frac{\ln r}{r}\,\chi(r)\,r^2\,dr \\ &= \frac{(2\pi)^3}{|\xi|^2} \int_0^\infty \sqrt{t} J_{1/2}(t) \ln t\, \chi\Big(\frac{t}{|\xi|}\Big)\,dt \\ &\quad - \frac{(2\pi)^3}{|\xi|^2} \ln|\xi| \int_0^\infty \sqrt{t} J_{1/2}(t)\,\chi\Big(\frac{t}{|\xi|}\Big)\,dt. \end{aligned}$$

Set

$$I(\xi) = \int_0^\infty \sqrt{t} J_{1/2}(t)\,\chi\Big(\frac{t}{|\xi|}\Big)\,dt = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin t \cdot \chi\Big(\frac{t}{|\xi|}\Big)\,dt.$$

Then

$$\sqrt{\frac{\pi}{2}}\, I(\xi) = -\cos t \cdot \chi\Big(\frac{t}{|\xi|}\Big)\Big|_{t=0}^{t=\infty} \\ + \frac{1}{|\xi|} \int_0^\infty \cos t \cdot \chi'\Big(\frac{t}{|\xi|}\Big)\,dt$$

$$= 1 - \frac{1}{|\xi|^2}\int_0^\infty \sin t \cdot \chi''\left(\frac{t}{|\xi|}\right) dt$$
$$= 1 + \frac{O(1)}{|\xi|}.$$

Hence, for $|\xi| \to \infty$,

$$\widehat{K_1}(\xi) = c_0 \frac{\ln|\xi|}{|\xi|^2} + \frac{O(1)}{|\xi|^2}, \quad c_0 > 0.$$

Consequently,

$$\widehat{K_*}(\xi) = \widehat{\Delta K_1}(\xi) + \widehat{\Delta K_2}(\xi) = c_0 \ln|\xi| + O(1),$$
$$|\xi| \to \infty,$$

which show that $\widehat{K_*}$ is not a multiplier from $L_2(\mathbf{R}^3)$ to $L_2(\mathbf{R}^3)$. □

Remark. Note a the similar singular operator A^j defined as

$$A^{(j)}u(x) = \int_{\mathbf{R}^n} \frac{u(y)}{|x-y|^n} \cos\phi_j(x,y)\; dy, \quad x \in \mathbf{R}^n,$$

where $\cos\phi_j(x,y) = (x_j - y_j)/|x-y|$, according to the Calderon-Zigmund theorem, is bounded and may be considered as a continuous operator from $L_2(\mathbf{R}^n)$ to $L_2(\mathbf{R}^n)$.

Now we consider the modified kernel

$$K_{**}(x) = \frac{\ln|x|}{|x|^{n-2}}\chi(|x|). \tag{3.2}$$

Proposition 3.4. *The Fourier transform of* K_{**} *satisfies the following estimate*:

$$|\widehat{K_{**}}(\xi)| \le C_1 \frac{\ln(e+|\xi|)}{|\xi|^2}, \tag{3.3}$$

where C_1 *is a constant.*

Proof. Again, using Lemma 3.2, we have that

$$\begin{aligned}\widehat{K_{**}}(\xi) &= \int_{R^n} K_{**}(x)e^{-ix\cdot\xi}\,dx\\ &= \int_{R^n} \frac{\ln|x|}{|x|^{n-2}}\,\chi(|x|)\,e^{-ix\cdot\xi}\,dx\\ &= (2\pi)^{n/2}\int_0^2 \frac{J_{n/2-1}(r|\xi|)}{(r|\xi|)^{n/2-1}}\,\frac{\ln r}{r^{n-2}}\,\chi(r)\;r^{n-1}\,dr.\end{aligned}$$

It is clear that this integral is bounded for $|\xi|\le 1$. Hence, it suffices to prove estimate (3.3) for $|\xi|>1$. To this end, we notice that

$$\begin{aligned}\widehat{K_{**}}(\xi) &= (2\pi)^{n/2}|\xi|^{1-n/2}\int_0^2 J_{n/2-1}(r|\xi|)(\ln r)\,\chi(r)\,r^{2-n/2}\,dr\\ &= \frac{(2\pi)^{n/2}}{|\xi|^2}\int_0^{2|\xi|} J_{n/2-1}(t)(\ln t)\,\chi\!\left(\frac{t}{|\xi|}\right)t^{2-n/2}\,dt\\ &\quad-\frac{(2\pi)^{n/2}}{|\xi|^2}\ln|\xi|\int_0^{2|\xi|} J_{n/2-1}(t)\,\chi\!\left(\frac{t}{|\xi|}\right)t^{2-n/2}\,dt.\end{aligned}$$

The required estimate (3.3) then follows from the fact that both integrals on the right hand side of the above are bounded with respect to $|\xi|$. □

Set

$$K_n(x) = \frac{(x-y)\otimes(x-y)}{|x-y|^{n+2}}\,\chi(|x-y|). \tag{3.4}$$

The following proposition follows directly from Propositions 3.1 and 3.3.

Proposition 3.5. *The Fourier transform of* K_n *satisfies the following estimate*:

$$|\widehat{K_n}(\xi)| \;\le\; C_1\ln(e+|\xi|). \tag{3.5}$$

Next we introduce Banach spaces H_m, for $m=1,2,\ldots$, defined by

$$H_m = \{v;\,(2\pi)^{-n}\int_{\mathbf{R}^n}|\widehat{v}(\xi)|^2\;\ln^{2m}\sqrt{e^2+|\xi|^2}\,d\xi<\infty\}$$

with norm

$$\|v\|_m = \left((2\pi)^{-n} \int_{\mathbf{R}^n} |\widehat{v}(\xi)|^2 \, \ln^{2m} \sqrt{e^2 + |\xi|^2 d\xi} \right)^{1/2}.$$

Clearly, $H_m \subset H_{m+1}$, $m = 1, 2, \ldots$, and

$$\|v\|_m \le \|v\|_{m+1}, \quad \text{for all } v \in H_m.$$

Proposition 3.6. *Let $W_2^1(\mathbf{R}^n)$ be the usual Sobolev space and $f \in W_2^1(\mathbf{R}^n)$. Then $f \in H_k$ for $k = 1, 2, \ldots$. Moreover,*

$$\|f\|_k \ \le \ k! \, \|f\|_{W_2^1}, \quad k = 1, 2 \ldots .$$

Proof. Note that, for $a \ge 1$

$$\ln a \le k \frac{a^{1/k}}{e}.$$

Therefore,

$$\ln^{2k} a \le \left(\frac{k}{e}\right)^{2k} a^2.$$

Thus,

$$\begin{aligned} \|f\|_k^2 &\le \left(\frac{k}{e}\right)^{2k} (2\pi)^{-n} \int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 (e^2 + |\xi|^2) \, d\xi \\ &\le e^2 \left(\frac{k}{e}\right)^{2k} (2\pi)^{-n} \int_{\mathbf{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2) \, d\xi \\ &= e^2 \left(\frac{k}{e}\right)^{2k} \|f\|_{W_2^1}^2, \end{aligned}$$

which implies that

$$\|f\|_k \ \le \ e \left(\frac{k}{e}\right)^k \|f\|_{W_2^1}. \tag{3.6}$$

Note that

$$\left(\frac{k}{e}\right)^k \le \frac{k!}{e}. \tag{3.7}$$

Indeed, set

$$c_k = \frac{1}{k!}\left(\frac{k}{e}\right)^k.$$

Then

$$\frac{c_{k+1}}{c_k} = \frac{1}{e}\left(1+\frac{1}{k}\right)^k < 1.$$

Hence,

$$c_{k+1} < c_k \leq c_1 = \frac{1}{e}.$$

The desired estimate then follows from (3.6) and (3.7). □

For $f : \overline{\mathbf{R}_+} \to H_m$, we set

$$\|f\|_{m,t} = \sup_{0\leq s\leq t} \|f(s)\|_m.$$

Define the integral operator A_n by

$$A_n v(x) = \int_{R^n} K_n(x,y)v(y)\,dy, \quad x \in \mathbf{R}^n, \tag{3.8}$$

where K_n is defined by (3.4). It is easy to see from (3.5) that A_n is a bounded operator from H_{m+1} to H_m, that is,

$$\|A_n v\|_m \leq M\|v\|_{m+1}, \quad \text{for all } v \in H_m, \quad m = 0,1,\ldots, \tag{3.9}$$

where M is a constant.

Next we introduce operator L defined by

$$Lv(t) = \int_0^t (t-s)A_n v(s)\,ds \tag{3.10}$$

and set

$$v_{k+1}(t) = Lv_k(t) = \int_0^t (t-s)A_n v_k(s)\,ds, \quad v_0(t) = f(t).$$

It is clear that $v_k(t) = L^k f(t)$.

Proposition 3.7. *For $f \in H_m$, $m = 1,2,\ldots$, the following holds*:

$$\|v_k\|_{m,t} \leq M^k \|f\|_{k+m,t}\,\frac{t^{2k}}{(2k-1)!!}. \tag{3.11}$$

In particular,

$$\|v_k(t)\| \le M^k \|f\|_{k,t} \frac{t^{2k}}{(2k-1)!!}. \tag{3.12}$$

Proof. Using (3.9) recursively, we have

$$\begin{aligned} \|v_{k+1}\|_{m,t} &\le \int_0^t (t-s)\|Av_k(s)\|_m \, ds \\ &\le M \int_0^t (t-s)\|v_k(s)\|_{m+1} \, ds \\ &\le \cdots \le M^{k+1}\|f\|_{k+m+1,t} \int_0^t (t-s) \frac{s^{2k}}{(2k-1)!!} \, ds \\ &\le M^{k+1}\|f\|_{k+m+1,t} t \int_0^t \frac{s^{2k}}{(2k-1)!!} \, ds \\ &= M^{k+1}\|f\|_{k+m+1,t} \frac{t^{2k+2}}{(2k+1)!!}. \end{aligned}$$

□

The following proposition is an immediate consequence of Propositions 3.6 and 3.7.

Proposition 3.8. *Assume that $f \in W_2^1$. Then, for $k = 0, 1, 2, \ldots$, the estimate*

$$\|L^k f(t)\|_m \le M^k t^{2k} \|f(t)\|_{W_2^1}, \quad t > 0. \tag{3.13}$$

is valid.

We are ready to prove the existence theorem for $\alpha = n$.

Theorem 3.9. *Let A_n be the integral operator defined in (3.8). There exists $T > 0$ such that, for every $f \in C([0,T] \to W_2^1(\mathbf{R}^n))$, the solution $u \in C([0,T] \to L_2(\mathbf{R}^n))$ of the equation*

$$u(x,t) = f(x,t) + \int_0^t (t-s) A_n u(x,s) \, ds. \tag{3.14}$$

exists and is unique.

Proof. Using the definition (3.10), we may rewrite (3.14) in the form

$$u(t) = f(t) + Lu(t).$$

Consider the Neumann series

$$u(t) = \sum_{k=0}^{\infty} L^k f(t).$$

By Proposition 3.8, if T is chosen such that $T < 1/\sqrt{M}$ where M is a constant that appeared in (3.9), then this series converges uniformly in $C([0,T] \to L_2(\mathbf{R}^n))$ with respect to $t \in [0,T]$, which gives the unique solution to equation (3.14). □

Remark 3.10. Similar results may also be obtained for an arbitrary domain Ω with piecewise smooth boundary. In this case, it is necessary to add some boundary conditions.

Acknowledgments. The authors are grateful to anonymous referees for valuable remarks which helped the authors improve the content and presentation of the paper.

APPENDIX

A. For any $n \times n$ matrix Λ, we define the matrices

$$\cos\sqrt{\Lambda}\, t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}\Lambda^k}{(2k)!} \tag{3.15}$$

and

$$\frac{\sin\sqrt{\Lambda}\, t}{\sqrt{\Lambda}} = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}\Lambda^k}{(2k+1)!}. \tag{3.16}$$

It is clear that

$$\begin{aligned} \frac{d}{dt} \cos\sqrt{\Lambda}\, t &= \sum_{k=1}^{\infty} (-1)^k 2k \frac{t^{2k-1}\,\Lambda^k}{(2k)!} \\ &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{t^{2k+1}\,\Lambda^{k+1}}{(2k+1)!} \end{aligned}$$

$$= -\Lambda \sum_{k=0}^{\infty} (-1)^k \, \frac{t^{2k+1}\Lambda^k}{(2k+1)!}$$
$$= -\Lambda \cdot \frac{\sin\sqrt{\Lambda}\, t}{\sqrt{\Lambda}}.$$

Analogously,

$$\frac{d}{dt}\, \frac{\sin\sqrt{\Lambda}\, t}{\sqrt{\Lambda}} = \cos\sqrt{\Lambda}\, t.$$

Now it easy to check that the vector-function

$$u(t) = \cos\sqrt{\Lambda}\, t\, \phi + \frac{\sin\sqrt{\Lambda}\, t}{\sqrt{\Lambda}}\, \psi + \int_0^t \frac{\sin\sqrt{\Lambda}\,(t-s)}{\sqrt{\Lambda}}\, F(s)\, ds \tag{3.17}$$

satisfies the differential equation of the second order

$$\frac{d^2u}{dt^2} + \Lambda u(t) = F(t), \tag{3.18}$$

and initial conditions

$$u(0) = \phi, \quad u'(0) = \psi. \tag{3.19}$$

Indeed,

$$u'(t) = -\Lambda \frac{\sin\sqrt{\Lambda}\, t}{\sqrt{\Lambda}}\, \phi + \cos\sqrt{\Lambda}\, t\, \psi + \int_0^t \cos\sqrt{\Lambda}\,(t-s)\, F(s)\, ds$$

and

$$u''(t) = -\Lambda \cos\sqrt{\Lambda} t \phi - \Lambda \frac{\sin\sqrt{\Lambda} t}{\sqrt{\Lambda}} \psi - \Lambda \int_0^t \frac{\sin\sqrt{\Lambda}(t-s)}{\sqrt{\Lambda}} F(s)\, ds + F(t),$$

which coincides with (3.18).

REFERENCES

1. S. Albeverio and S.A. Alimov, *On some integral equations in Hilbert space with an application to the theory of elasticity*, Int. Equat. Oper. Theor. **55** (2006), 153–168.

2. S.A. Alimov and O.A. Ilhan, *Solvability of some partial integral equations in Banach space*, Abstract of the 7th AIMS International Conference on Dynamical Systems, Differential Equations and Applications, May, 2008.

3. Qiang Du, James R. Kamm, R.B. Lehoucq and Michael L. Parks, *A new approach for a nonlocal, nonlinear conservation law*, SIAM J. Appl. Math. **72** (2012), 464–487.

4. E. Emmrich, R. Lehoucq and D. Puhst, *Peridynamics: A nonlocal continuum theory*, Lect. Notes Comp. Sci. Eng. **89** (2013), 45–65.

5. E. Emmrich and O. Weckner, *Convergence of peridynamics to classical elasticity theory*, in *Solids, structures and coupled problems in engineering*, C.A. Mota Soares et al., eds., Springer, Lisbon, 2006.

6. ———, *On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity*, Comm. Math. Sci. **5** (2007), 851–864.

7. M. Gunzburger and R. Lehoucq, *A nonlocal vector calculus with application to nonlocal boundary value problems*, Multiscale Model. Simul. **8** (2010), 1581–1598.

8. L. Hörmander, *The analysis of linear partial differential operators*, 1, *Distribution theory and Fourier analysis*, Springer-Verlag, Berlin, 1983.

9. S.M. Nikol'skiy, *Approximation of functions of several variables and imbedding theorems*, Grund. Math. Wiss. **205**, Springer-Verlag, New York, 1975.

10. P. Seleson, Michael L. Parks, M. Gunzburger and R.B. Lehoucq, *Peridynamics as an upscaling of molecular dynamics*, Multiscale Model. Simul. **8** (2009), 204–227.

11. S.A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids **48** (2000), 175–209.

12. E.C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations*, Vol. 2, Oxford, at the Clarendon Press, 1958.

13. G.N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, England, 1944.

14. K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions*, SIAM J. Numer. Anal. **48** (2010), 1759–1780.

Faculty of Applied Mathematics, Lomonosov Moscow State University, Tashkent Branch, Uzbekistan
Email address: sh_alimov@yahoo.com

School of Mathematics and Computational Science, Sun Yat-sun University, China, and Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849
Email address: yzc0009@auburn.edu

Faculty of Education, Erciyes University, 38039 Melikgazi Kayseri, Turkey
Email address: nuralpilhan@icloud.com