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CONTROLLABILITY OF NONLINEAR FRACTIONAL IMPULSIVE EVOLUTION SYSTEMS

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ABSTRACT. In this paper, we consider controllability of nonlinear fractional impulsive evolution systems. We firstly give a mild solution expression for nonlinear fractional impulsive evolution systems. Sufficient conditions for controllability results are obtained by Krasnoselskii's fixed point theorem in the infinite-dimensional spaces.

1. Introduction. Control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. It is well known that controllability plays a significant role in modern control theory and engineering since they are closely related to pole assignment, structural decomposition and quadratic optimal control. Different techniques were developed to investigate the controllability of various systems, such as geometric analysis [**12, 16**], Lie algebraic approach [**17**], functional analysis [**5**] and algebraic method [**15, 20**]. See also [**1–4, 6**].

In recent years, the study of impulsive control systems has received increasing interest, since dynamical systems with impulsive effects have great importance in applied sciences (see [**7, 8, 10–13, 18, 20**]).

In the paper, we study the following controllability of nonlinear fractional impulsive evolution systems by the Krasnoselskii's fixed point theorem.

(1.1)
$$\begin{cases} {}^cD^q_{0+}x(t) = Ax(t) + Bu(t) + f(t, x(t)) \quad t \in J\backslash\{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = I_i(x(t_i)), \\ x(0) = x_0, \end{cases}$$

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where ${}^cD^q$ is the Caputo fractional derivative of order $0 < q \leq 1$ with the lower limit zero, the state $x(\cdot)$ takes values in a Banach space X and the control function $u(\cdot)$ is given in $L^2(J, U)$ with U as a Banach space. A is the infinitesimal generator of a strongly continuous semigroup $T(t), t \geq 0$ in X, B is a bounded linear operator from U into X and $f : J \times X \to X$ is a given X-value function. $I_i : X \to X$ is continuous, $i = 1, 2, \dots, k$, $J = [0, T], 0 = t_0 < t_1 < \cdots < t_i < \cdots < t_k < t_{k+1} = T$, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $x(t_i^+)$ and $x(t_i^-)$ denote the right and the left limits of $x(t)$ at $t = t_i$ $(i = 1, 2, \dots, k)$.

The organization of the paper is as follows. In Section 2, we give some preliminaries and introduce mild solutions of system (1.1). In Section 3, by using Krasnoselskii's fixed point theorem, we obtain the condition of controllability of nonlinear fractional impulsive evolution systems.

2. Preliminaries. Let X be a Banach space and $A : D(A)(\subseteq X) \to X$ the infinitesimal generator of a uniformly bounded strongly continuous semigroup $\{T(t), t \geq 0\}$. Therefore, there exists an $M_1 \geq 1$ such that $\sup_{t\in[0,+\infty)} \|T(t)\| \leq M_1$. Let $L^p(J, R)(1 \leq p < \infty)$ denote the Banach space of all Lebesgue measurable functions from J into R with $\|\varphi\|_{L^p(J,R)} := (\int_J |\varphi(t)|^p dt)^{1/p} < \infty$. And let $L^p(J, X)$ be the Banach space of functions $\varphi : J \to X$ which are Bochner integrable normed by $\|\varphi\|_{L^p(J,X)}$. We introduce the Banach space $C(J, X)$ endowed with supnorm given $\|x\|_{C(J,X)} = \sup_{t\in J} \|x(t)\|_X$, for $x \in C(J, X)$ and $PC(J, X) = \{x : J \to X | x \in C((t_k, t_{k+1}], X)$, $i = 0, 1 \dots, k$, and $x(t_i^+)$ exist$\}$ with the norm $\|x\|_{PC(J,X)} = \sup_{t\in J}\{\|x(t+0)\|, \|x(t-0)\|\}$.

For the convenience of the readers, we first present some useful definitions and fundamental facts of fractional calculus theory, which can be found in [**9, 14**].

Definition 2.1. For $s > 0$, the integral

$$I_{0+}^s f(x) = \frac{1}{\Gamma(s)} \int_0^x \frac{f(t)}{(x-t)^{1-s}}\, dt,$$

is called the Riemann-Liouville fractional integral of order s.

Definition 2.2. For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^s f(x) = \frac{1}{\Gamma(n-s)} \left(\frac{d}{dx}\right)^n \int_0^x \frac{f(t)}{(x-t)^{s-n+1}}\, dt, \quad n = [s]+1$$

is called the Riemann-Liouville fractional derivative of order $s > 0$.

Definition 2.3. The Caputo derivative of order s for a function $f : [0,\infty) \to R$ can be written as

$${}^cD_{0+}^s f(t) = D^s[f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0)], \quad t > 0,\ n-1 < s < n.$$

Remark. (i) If $f^{(n)} \in C[0,\infty)$, then

$${}^cD_{0+}^s f(x) = \frac{1}{\Gamma(n-s)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{s+1-n}}\, dt, \quad t > 0,\ n-1 < s < n.$$

(ii) The Caputo derivative of a constant is equal to zero.

Definition 2.4. For any $u \in L^2(J,U)$, $x \in PC(J,X)$ is called a mild solution of system (1.1) if $x(t)$ satisfies

$$x(t) = \begin{cases} T_q(t)x_0 + \int_0^t (t-s)^{q-1} S_q(t-s) f(s,x(s))\, ds \\ \quad + \int_0^t (t-s)^{q-1} S_q(t-s) Bu(s)\, ds & t \in [0,t_1] \\ T_q(t)x_0 + \int_0^t (t-s)^{q-1} S_q(t-s) f(s,x(s))\, ds \\ \quad + \int_0^t (t-s)^{q-1} S_q(t-s) Bu(s)\, ds \\ \quad + \sum_{j=1}^i T_q(t-t_j) I_j(x(t_j)) \\ & t \in (t_i, t_{i+1}],\ i = 1,2,\dots,k, \end{cases}$$

where $T_q(t) = \int_0^\infty \xi_q(\theta) T(t^q\theta)\, d\theta$, $S_q(t) = q\int_0^\infty \theta\xi_q(\theta) T(t^q\theta)\, d\theta$ and, for $\theta \in (0,\infty)$, $\xi_q(\theta) = (1/q)\theta^{-1-(1/q)}\overline{w}_q(\theta^{-1/q}) \geq 0$, $\overline{w}_q(\theta) = 1/\pi \sum_{n=1}^\infty (-1)^{n-1}\theta^{-qn-1}(\Gamma(nq+1)/n!)\sin(n\pi q)$. Here, ξ_q is a probability density function defined on $(0,\infty)$, that is, $\xi_q(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \xi_q(\theta)\, d\theta = 1$.

Lemma 2.5 [**21**, Lemmas 3.2 and 3.3]. *The operators T_q and S_q have the following properties*:

(i) *For any fixed $t \geq 0$, $T_q(t)$ and $S_q(t)$ are linear and bounded operators, i.e., for any $x \in X$, $\|T_q(t)x\| \leq M_1\|x\|$ and $\|S_q(t)x\| \leq (qM_1)/(\Gamma(1+q))\|x\|$.*

(ii) $\{T_q(t), t \geq 0\}$ *and* $\{S_q(t), t \geq 0\}$ *are strongly continuous.*

Lemma 2.6 (Krasnoselskii's fixed point theorem). *Let X be a Banach space, Ω a bounded closed and convex subset of X and F_1, F_2 maps Ω into X such that $F_1x + F_2y \in \Omega$ for every pair $x, y \in \Omega$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on Ω.*

3. Controllability results.

Definition 3.1. The fractional system (1.1) is said to be completely controllable on $[0, t_f](t_f \in (0, T])$ if, for every $x_0, x_{t_f} \in X$, there exists a control $u \in L^2(J, U)$ such that the solution $x(t)$ of (1.1) satisfies $x(t_f) = x_{t_f}$.

Now we make the following assumptions:

(H_1) The operator A generates a uniformly bounded strongly continuous semigroup $\{T(t), t \geq 0\}$ in X. So there exists a constant M_1 such that $\sup_{t\in[0,+\infty)} \|T(t)\| \leq M_1$.

(H_2) The linear operator $B : L^2(J, U) \to L^1(J, X)$ is bounded, $W : L^2(J, U) \to X$ defined by

$$Wu = \int_0^{t_f} (t_f - s)^{q-1} S_q(t_f - s)Bu(s)\, ds$$

has an inverse operator W^{-1} which takes value in $L^2(J, U)/\ker W$, and there exist two positive constants $M_2, M_3 > 0$ such that

$$\|B\| \leq M_2, \qquad \|W^{-1}\| \leq M_3.$$

(H_3) $f : J \times X \to X$ is continuous and there exists a constant $q_1 \in (0, q)$ and $p(\cdot) \in L^{1/q_1}(J, R^+)$ such that $\|f(t, x_1) - f(t, x_2)\| \leq p(t)\|x_1 - x_2\|$, $x_i \in X$, $i = 1, 2$.

(H_4) $I_i : X \to X$, $i = 1, 2, \ldots, k$, are continuous and there exist constants L_i such that $\|I_i(x_1) - I_i(x_2)\| \leq L_i\|x_1 - x_2\|$,

$$\sum_{i=1}^{k} L_i = L.$$

(H_5) For $t > 0$, $S_q(t)$ is continuous in the sense of uniformly operator topology.

(H_6) For all bounded subset Ω, the set

$$\Pi_{h,\delta}(t) = \{q \int_0^{t-h} \int_\delta^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s,x(s))\,d\theta\,ds, x \in \Omega\}$$

is relatively compact in X for arbitrary $h \in (0,t)$ and $\delta > 0$.

For convenience, we denote $n_1 = [(1 - q_1/q - q_1)T^{(q-q_1)/(1-q_1)}]^{1-q_1}$, $n_2 = \max_{t\in J}\|f(t,0)\|$, $n_3 = \sum_{j=1}^{k}\|I_i(0)\|$.

Theorem 3.1. *Assume that* (H_1)–(H_5) *hold. In addition, assume that*

$$b_2[1 + \frac{M_1M_2M_3T^q}{\Gamma(1+q)}] < 1, \tag{3.1}$$

holds, where $b_2 = [(M_4M_1q)/(\Gamma(1+q)) + M_1L]$, $M_4 = n_1\|p\|_{L^{1/q_1}(J,R^+)}$. *Then system* (1.1) *is completely controllable on* $[0, t_f]$ *for some* $t_f \in (0, T]$.

Proof. By (H_2), for any $x \in PC(J,X)$, we define the control $u_x(t)$ by

$$u_x(t) = \begin{cases} W^{-1}[x_{t_f} - T_q(t_f)x_0 - \int_0^{t_f}(t_f-s)^{q-1}S_q(t_f-s)f(s,x(s))\,ds](t), \\ \qquad\qquad t \in [0,t_1] \\ W^{-1}[x_{t_f} - T_q(t_f)x_0 - \int_0^{t_f}(t_f-s)^{q-1}S_q(t_f-s)f(s,x(s))\,ds \\ -\sum_{j=1}^{i} T_q(t_f-t_j)I_j(x(t_j))](t), \qquad t \in (t_i, t_{i+1}],\ i = 1, 2, \ldots, k. \end{cases}$$

According to the control, we shall prove the operator $Q : PC(J,X) \to PC(J,X)$ defined by

$$(Qx)(t)=\begin{cases} T_q(t)x_0 + \int_0^t (t-s)^{q-1}S_q(t-s)f(s,x(s))\,ds \\ \quad + \int_0^t (t-s)^{q-1}S_q(t-s)Bu_x(s)\,ds, & t\in[0,t_1] \\ T_q(t)x_0 + \int_0^t (t-s)^{q-1}S_q(t-s)f(s,x(s))\,ds \\ \quad + \int_0^t (t-s)^{q-1}S_q(t-s)Bu_x(s)\,ds \\ + \sum_{j=1}^{i} T_q(t-t_j)I_j(x(t_j)), & t\in(t_i,t_{i+1}],\ i=1,2,\ldots,k, \end{cases}$$

has a fixed point x, which is a mild solution of system (1.1).

For any $r > 0$, let $\Omega_r = \{x \in PC(J,X) : \|x\|_{PC(J,X)} \leq r\}$. It is obvious that Ω_r is a bounded, closed and convex set of $PC(J,X)$. Now we divide the proof into the following several steps.

Step 1. We prove that $Q(\Omega_r) \subseteq \Omega_r$. Suppose that $Q(\Omega_r) \subseteq \Omega_r$ is not true. Then there exists a function $\widehat{x} \in \Omega_r$, $\|(Q\widehat{x})(t)\| > r$ for some $t \in J$.

By (H_3) and the Holder inequality, we get that:

$$\int_0^{t_f} (t_f-s)^{q-1}p(s)\,ds \leq \left[\left(\frac{1-q_1}{q-q_1}T^{(q-q_1)/(1-q_1)}\right)\right]^{1-q_1} \|p\|_{L^{1/q_1}}(J,R^+),$$

$$\int_0^{t_f} (t_f-s)^{q-1}\|f(s,0)\|\,ds \leq \frac{n_2T^q}{q}.$$

Denote $M_4 = n_1\|p\|_{L^{1/q_1}(J,R^+)}$, $M_5 = (n_2T^q)/q$. Thus, for all $(t_i,t_{i+1}]$, $i = 0,1,\ldots,k$, we have

$$\begin{aligned}\|u_x(t)\| &\leq \|W^{-1}\|\{\|x_{t_f}\| + \|T_q(t_f)x_0\| + \sum_{j=1}^{i}\|T_q(t_f-t_j)I_j(x(t_j))\| \\ &\quad + \int_0^{t_f}(t_f-s)^{q-1}\|S_q(t_f-s)f(s,x(s))\|\,ds \\ &\leq M_3\|x_{t_f}\| + M_3M_1\|x_0\| + M_3M_1\sum_{j=1}^{i}\|I_j(x(t_j))\|\end{aligned}$$

$$
\begin{aligned}
&+ \frac{M_1 M_3 q}{\Gamma(1+q)} \int_0^{t_f} (t_f - s)^{q-1} \|f(s, x(s))\| \, ds \\
&\le M_3 \|x_{t_f}\| + M_3 M_1 \|x_0\| + M_3 M_1 \sum_{j=1}^{i} [L_j \|x(t_j)\| + \|I_j(0)\|] \\
&+ \frac{M_1 M_3 q}{\Gamma(1+q)} \int_0^{t_f} (t_f - s)^{q-1} [p(s)\|x(s)\| + \|f(s,0)\|] \, ds \\
&\le M_3 \|x_{t_f}\| + M_3 M_1 \|x_0\| \\
&+ \frac{M_4 M_3 M_1 q}{\Gamma(1+q)} \|x\|_{PC(J,X)} + \frac{M_5 M_3 M_1 q}{\Gamma(1+q)} \\
&+ M_1 M_3 L \|x\|_{PC} + M_1 M_3 n_3 \\
&= M_3 \|x_{t_f}\| + M_3 b_1 + M_3 b_2 \|x\|_{PC(J,X)},
\end{aligned}
$$

where $b_1 = [M_1\|x_0\| + (M_5 M_1 q)/(\Gamma(1+q)) + M_1 n_3]$, $b_2 = [(M_4 M_1 q)/(\Gamma(1+q)) + M_1 L]$. By (H_2) and (H_3), we have

$$
\begin{aligned}
r < \|(Q\widehat{x})(t)\| &\le \|T_q(t) x_0\| + \sum_{j=1}^{i} \|T_q(t - t_j) I_j(\widehat{x}(t_j))\| \\
&+ \int_0^t (t-s)^{q-1} \|S_q(t-s) f(s, \widehat{x}(s))\| \, ds \\
&+ \int_0^t (t-s)^{q-1} \|S_q(t-s) B u_x(s)\| \, ds \\
&\le M_1 \|x_0\| + M_1 L \|\widehat{x}\|_{PC(J.X)} \\
&+ M_1 n_3 + \frac{M_4 M_1 q}{\Gamma(1+q)} \|\widehat{x}\|_{PC(J.X)} \\
&+ \frac{M_5 M_1 q}{\Gamma(1+q)} + \frac{M_1 M_2 T^q}{\Gamma(1+q)} \max_{s \in [0, t_f]} \|u_x(s)\| \\
&= b_1 \left[1 + \frac{M_1 M_2 M_3 T^q}{\Gamma(1+q)}\right] + \frac{M_1 M_2 M_3 T^q}{\Gamma(1+q)} \|x_{t_f}\| \\
&+ b_2 \left[1 + \frac{M_1 M_2 M_3 T^q}{\Gamma(1+q)}\right] r.
\end{aligned}
$$

Taking the limit as $r \to \infty$, we get

$$
b_2 \left[1 + \frac{M_1 M_2 M_3 T^q}{\Gamma(1+q)}\right] \ge 1,
$$

which contradicts (3.1). Hence, $Q(\Omega_r) \subseteq \Omega_r$.

Now, we define the following operators Q_1 and Q_2 on Ω_r as

$$(Q_1x)(t) = \int_0^t (t-s)^{q-1} S_q(t-s) Bu_x(s)\,ds + \sum_{j=1}^{i} T_q(t-t_j) I_j(x(t_j)),$$

$$(Q_2x)(t) = T_q(t)x_0 + \int_0^t (t-s)^{q-1} S_q(t-s) f(s, x(s))\,ds$$

for $t \in J$, respectively. Obviously, $Q = Q_1 + Q_2$.

Next we show that Q_1 is a contraction and Q_2 is completely continuous.

Step 2. Q_1 is a contraction. Let $x, y \in \Omega_r$. By (H_2)–(H_3), we get

$$\begin{aligned}\|(Q_1x)(t) - (Q_2y)(t)\| &\le \int_0^t (t-s)^{q-1} B\|S_q(t-s)(u_x - u_y)\|\,ds \\ &\quad + \sum_{j=1}^{i} \|T_q(t-t_j)[I_j(x(t_j)) - I_j(y(t_j))]\| \\ &\le \frac{M_2M_1q}{\Gamma(1+q)} \int_0^t (t-s)^{q-1}\,ds \max_{[0,t_f]} \|u_x - u_y\| \\ &\quad + M_1 \sum_{j=1}^{i} \|I_j(x(t_j)) - I_j(y(t_j))\| \\ &\le \frac{M_2M_1T^q}{\Gamma(1+q)} b_2M_3\|x-y\|_{PC(J,X)} \\ &\quad + M_1L\|x-y\|_{PC(J,X)} \\ &\le \left[\frac{b_2M_1M_2M_3T^q}{\Gamma(1+q)} + M_1L\right]\|x-y\|_{PC(J,X)}.\end{aligned}$$

In view of (3.1), we derive that $(b_2M_1M_2M_3T^q)/(\Gamma(1+q)) + M_1L < 1$, which implies Q_1 is a contraction.

Step 3. Q_2 is completely continuous.

i) Q_2 is continuous on Ω_r.

It is easily proved that Q_2 is continuous on Ω_r.

ii) We prove that $Q_2(\Omega_r) \subset PC(J, X)$ is equicontinuous.

For any $x \in \Omega_r$, let $0 \le \tau_1 \le \tau_2 \le T$. We get

$$\begin{aligned}
&\|(Q_1x)(\tau_2) - (Q_2x)(\tau_1)\| \\
&\quad \le \|T_q(\tau_2)x_0 - T_q(\tau_1)x_0\| \\
&\qquad + \| \int_0^{\tau_2} (\tau_2 - s)^{q-1} S_q(\tau_2 - s) f(s, x(s))\, ds \\
&\qquad - \int_0^{\tau_1} (\tau_1 - s)^{q-1} S_q(\tau_1 - s) f(s, x(s))\, ds\| \\
&\quad \le \|T_q(\tau_2)x_0 - T_q(\tau_1)x_0\| \\
&\qquad + \left\| \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] S_q(\tau_2 - s) f(s, x(s))\, ds \right\| \\
&\qquad + \left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} S_q(\tau_2 - s) f(s, x(s))\, ds \right\| \\
&\qquad + \left\| \int_0^{\tau_1} (\tau_1 - s)^{q-1} [S_q(\tau_2 - s) - S_q(\tau_1 - s)] f(s, x(s))\, ds \right\|.
\end{aligned}$$

We denote

$$\begin{aligned}
J_1 &= T_q(\tau_2)x_0 - T_q(\tau_1)x_0, \\
J_2 &= \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] S_q(\tau_2 - s) f(s, x(s))\, ds, \\
J_3 &= \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{q-1} S_q(\tau_2 - s) f(s, x(s))\, ds, \\
J_4 &= \int_0^{\tau_1} (\tau_1 - s)^{q-1} [S_q(\tau_2 - s) - S_q(\tau_1 - s)] f(s, x(s))\, ds, \\
\alpha &= \frac{q - q_1}{1 - q_1}.
\end{aligned}$$

Obviously, $\|(Q_1x)(\tau_2) - (Q_2x)(\tau_1)\| \le \|J_1\| + \|J_2\| + \|J_3\| + \|J_4\|$. Now, we need to check that J_1, J_2, J_3, J_4 tend to 0 uniformly for all $x \in \Omega_r$ when $\tau_2 \to \tau_1$. In fact, by Lemma 2.5 (ii), we can deduce that $\lim_{\tau_2 \to \tau_1} J_1 = 0$. Using (H_3), we have

$$\begin{aligned}
J_2 \le &\int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] \|S_q(\tau_2 - s) p(s) x(s)\|\, ds \\
&+ \int_0^{\tau_1} [(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}] \|S_q(\tau_2 - s) f(s, 0)\|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{rM_1q}{\Gamma(1+q)}\int_0^{\tau_1}[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}]p(s)\,ds\\
&\quad+\frac{n_2M_1q}{\Gamma(1+q)}\int_0^{\tau_1}\left[(\tau_2-s)^{q-1}-(\tau_1-s)^{q-1}\right]ds\\
&\leq \frac{M_1rq\alpha^{q_1-1}}{\Gamma(1+q)}[\tau_2^{\alpha}-\tau_1^{\alpha}-(\tau_2-\tau_1)^{\alpha}]^{1-q_1}\|p\|_{L^{1/q_1}(J,R^+)}\\
&\quad+\frac{n_2M_1}{\Gamma(1+q)}[\tau_2^{q}-\tau_1^{q}-(\tau_2-\tau_1)^{q}],
\end{aligned}
$$

$$
\begin{aligned}
J_3 &\leq \int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}\|S_q(\tau_2-s)f(s,x(s))\|\,ds\\
&\leq \frac{rM_1q}{\Gamma(1+q)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}p(s)\,ds\\
&\quad+\frac{n_2M_1q}{\Gamma(1+q)}\int_{\tau_1}^{\tau_2}(\tau_2-s)^{q-1}\,ds\\
&\leq \frac{M_1rq\alpha^{q_1-1}}{\Gamma(1+q)}(\tau_2-\tau_1)^{q-q_1}\|p\|_{L^{\frac{1}{q_1}}(J,R^+)}+\frac{n_2M_1}{\Gamma(1+q)}(\tau_2-\tau_1)^{q}.
\end{aligned}
$$

By (H_5), we obtain that

$$
\begin{aligned}
J_4 &\leq \int_0^{\tau_1}(\tau_1-s)^{q-1}\|[S_q(\tau_2-s)-S_q(\tau_1-s)]f(s,x(s))\|\,ds\\
&\leq r\sup_{s\in[0,\tau_1]}\|S_q(\tau_2-s)-S_q(\tau_1-s)\|\int_0^{\tau_1}(\tau_1-s)^{q-1}p(s)\,ds\\
&\quad+n_2\sup_{s\in[0,\tau_1]}\|S_q(\tau_2-s)-S_q(\tau_1-s)\|\int_0^{\tau_1}(\tau_1-s)^{q-1}\,ds\\
&\leq r\sup_{s\in[0,\tau_1]}\|S_q(\tau_2-s)-S_q(\tau_1-s)\|\alpha^{q_1-1}\tau_1^{q-q_1}\|p\|_{L^{1/q_1}(J,R^+)}\\
&\quad+\frac{n_2}{q}\sup_{s\in[0,\tau_1]}\|S_q(\tau_2-s)-S_q(\tau_1-s)\|\tau_1^{q}.
\end{aligned}
$$

Consequently, we conclude that $\|(Q_2x)(\tau_2)-(Q_2x)(\tau_1)\|\to 0$, as $\tau_2\to\tau_1$, for all $x\in\Omega_r$. Therefore, $Q_2(\Omega_r)\subset PC(J,X)$ is equicontinuous.

iii) $Q_2(\Omega_r)(t)\subset X$ is relatively compact in X. Define $\Pi=Q_2(\Omega_r)$ and $\Pi(t)=\{Q_2(x_r)(t):x\in\Omega_r\}$ for $t\in J$.

Set $\Pi_{h,\delta}(t) = \{Q_{2,h,\delta}(x_r)(t) : x \in \Omega_r\}$, where

$$\begin{aligned} &Q_{2,h,\delta}(x_r)(t) \\ &= T(h^q\delta)\int_\delta^\infty \xi_q(\theta)T(t^q\theta - h^q\delta)x_0\,d\theta \\ &\quad + T(h^q\delta)\int_0^{t-h}(t-s)^{q-1}\bigg(q\int_\delta^\infty \theta\xi_q(\theta)T((t-s)^q\theta - h^q\delta)x_0\,d\theta\bigg)f(s,x(s))ds \\ &= \int_\delta^\infty \xi_q(\theta)T(t^q\theta)x_0\,d\theta \\ &\quad + q\int_0^{t-h}\int_0^\infty \theta(t-s)^{q-1}\xi_q(\theta)T((t-s)^q\theta)f(s,x(s))\,d\theta\,ds. \end{aligned}$$

Clearly, $\Pi(0) = \{(Q_2x)(0)|x \in \Omega_r\} = \{x_0\}$ is compact. By Lemma 2.5 (ii), (H_6) and (H_7), we easily show that $\Pi(t) = \{Q_2(x_r)(t) : x \in \Omega_r\}$ is relatively compact in X for all $t \in J$ from Theorem 3.1 of [**19**]. Hence, $Q_2(\Omega_r)(t) \subset X$ is relatively compact in X. As a result, Q_2 is a completely continuous operator. According to the Krasnoselskii's fixed point theorem, Q has a fixed point x on Ω_r. Thus, if $t_f \in [0, t_1]$, it is easy to check that $x(t_f) = x_{t_f}$. Similarly if $t_f \in (t_i, t_{i+1}]$, $i = 1, 2, \dots, k$, $x(t_f) = x_{t_f}$. Consequently, system (1.1) is completely controllable on $[0, t_f]$.

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