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DIFFERENTIABILITY OF SOLUTIONS OF ABSTRACT NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we study the differentiability of solutions of abstract neutral integro-differential equations with infinite delay. We consider specially the cases when the underlying space is reflexive or at least has the Radon-Nikodym property.

1. Preliminaries. In this work we are concerned with regularity properties of solutions of abstract neutral integro-differential equations with infinite delay.

Let X be a Banach space endowed with a norm $\|\cdot\|$. In this paper we study the existence of classical solutions for the class of abstract neutral integro-differential equations described in the form

$$\frac{d}{dt}\left(x(t)+\int_0^t N(t-s)x(s)\,ds\right) = Ax(t)+\int_0^t B(t-s)x(s)\,ds+f(t,x_t), \tag{1.1}$$

for $t \in I=[0,a]$, with initial condition

$$x_0=\varphi\in\mathcal{B} \tag{1.2}$$

In this description $x(t)\in X$ and the history $x_t:(-\infty,0]\to X$, given by $x_t(\theta)=x(t+\theta)$ for $\theta\le 0$, belongs to some abstract phase space

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$\mathcal{B}$ defined axiomatically. Moreover, $N(t)$ denotes a bounded linear operator from X into X, $A, B(t)$ are closed linear operators defined in the Banach space X, and $f : I \times \mathcal{B} \to X$ is an appropriate function.

Partial neutral integro-differential equations with unbounded delay arise, for instance, in the theory developed by Gurtin and Pipkin [**20**] and Nunziato [**29**] for the description of heat conduction in materials with fading memory.

We next establish some basic properties of the initial boundary value problem

$$\frac{d}{dt}\left(x(t) + \int_0^t N(t-s)x(s)\,ds\right) = Ax(t) + \int_0^t B(t-s)x(s)\,ds, \quad t \geq 0, \tag{1.3}$$

$$x(0) = z \in X. \tag{1.4}$$

We associate with equation (1.3) a resolvent operator. The subject of the existence and qualitative properties of a resolvent operator for integro-differential equations has been considered in several works. The book of Gripenberg, Londen and Staffans [**19**] contains an overview of the theory for the case the underlying space X has finite dimension. The subject also has been studied by several authors for abstract integro-differential equations. Related to our work, we mention the articles [**5–8, 16–18, 23, 26, 27**]. Furthermore, integro-differential equations with infinite delay have been considered recently by Ezzinbi, Toure and Zabsonre [**13**], and neutral integro-differential equations with infinite delay have been studied by Dos Santos et al. in [**10, 11**].

Throughout this paper, for Banach spaces X, Y we denote by $\mathcal{L}(X, Y)$ the Banach space of bounded linear operators from X into Y endowed with the operator norm, and we abbreviate this notation by $\mathcal{L}(X)$ in the case $X = Y$. Moreover, we denote the dual space of X by X^*. If C is a linear operator defined in a dense subspace $D(C)$, then C^* will represent the adjoint operator of C, and if $x \in X$ and $x^* \in X^*$, we will write indistinctly $\langle x^*, x\rangle$ or $\langle x, x^*\rangle$ for the value $x^*(x)$. We denote by $D(A)$ the domain of operator A, and by $[D(A)]$ the space $D(A)$ endowed with the graph norm

$$\|x\|_1 = \|x\| + \|Ax\|, \quad x \in D(A).$$

In addition, we denote by $\rho(A)$ the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ is the resolvent operator of A. Finally, we designate by $\widehat{f}$ the Laplace transform of an appropriate function $f : [0, \infty) \to X$.

We introduce the following concept of resolvent operator for problem (1.3)–(1.4).

Definition 1.1. A one parameter family of bounded linear operators $(\mathrm{R}\,(t))_{t\geq 0}$ on X is called a resolvent operator for problem (1.3)–(1.4) if the following conditions are verified.

(a) The function $\mathrm{R}\,(\cdot) : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous, exponentially bounded and $\mathrm{R}\,(0) = I$.

(b) For $x \in D(A)$, $\mathrm{R}\,(\cdot)x \in C([0, \infty), [D(A)]) \cap C^1((0, \infty), X)$, and

$$\frac{d}{dt}\left[\mathrm{R}\,(t)x + \int_0^t N(t-s)\mathrm{R}\,(s)x\,ds\right] = A\mathrm{R}\,(t)x + \int_0^t B(t-s)\mathrm{R}\,(s)x\,ds, \tag{1.5}$$

$$\frac{d}{dt}\left[\mathrm{R}\,(t)x + \int_0^t \mathrm{R}\,(t-s)N(s)x\,ds\right] = \mathrm{R}\,(t)Ax + \int_0^t \mathrm{R}\,(t-s)B(s)x\,ds, \tag{1.6}$$

for every $t \geq 0$.

The existence of a resolvent operator for problem (1.3)–(1.4) has been considered in the Appendix. In what follows, we use the terminology and notations introduced in the Appendix. In particular, we always assume that conditions (P1)–(P4) are verified, and consequently that there exists a resolvent operator for problem (1.3)–(1.4).

We complete the preliminaries with a brief description of the paper. This paper has five sections. In Section 2 we study the existence of classical solutions for the non-homogeneous problem

$$\frac{d}{dt}\left(x(t) + \int_0^t N(t-s)x(s)\,ds\right) = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t), \quad t \in I, \tag{1.7}$$

with initial condition (1.4), where $f : [0, a] \to X$ is a continuous function. Moreover, we include in this section an application to the existence of classical solutions for a semi-linear equation of type (1.7). In Section 3 we discuss the existence of classical solutions for an abstract neutral integro-differential equation with infinite delay. In Section 4 we apply our results to an equation that arises in the study of heat conduction in material with fading memory. Finally, in Section 5 we have met the basic properties of resolvent operators which will be used throughout the text.

2. Classical solutions for the semi-linear equation. We begin this section by characterizing the elements $x \in X$ for which the function $\mathrm{R}\,(\cdot)x$ is differentiable. Initially we generalize a well-known result for strongly continuous semi-groups of linear operators [**4**].

Theorem 2.1. *Assume that X is reflexive and that $b(\cdot)$ is locally bounded. If*

$$\liminf_{t\to 0^+} \left\| \frac{\mathrm{R}\,(t)x - x}{t} \right\| < \infty, \tag{2.1}$$

then $x \in D(A)$.

Proof. Under condition (2.1), it follows that there is a sequence $(\tau_n)_n$ convergent to 0 such that the set

$$\left\{ \frac{\mathrm{R}\,(\tau_n)x - x}{\tau_n} : n \in \mathbf{N} \right\}$$

is bounded. Since X is reflexive, by passing to a subsequence, we may assume that $\frac{\mathrm{R}\,(\tau_n)x - x}{\tau_n}$ converges to some element $y \in X$ as $n \to \infty$ in the weak topology. Using Lemma 5.8, we have that

$$\frac{1}{\tau_n} V(\tau_n)x = \frac{1}{\tau_n} \int_0^{\tau_n} \mathrm{R}\,(s)x \, ds \in D(A)$$

and

$$A\frac{1}{\tau_n}V(\tau_n)x = \frac{\mathrm{R}\,(\tau_n)x - x}{\tau_n} + \frac{1}{\tau_n}\int_0^{\tau_n} N(\tau_n - s)\mathrm{R}\,(s)x\,ds - \frac{1}{\tau_n}\int_0^{\tau_n} B(\tau_n - s)V(s)x\,ds. \tag{2.2}$$

It is clear that

$$\frac{1}{\tau_n}\int_0^{\tau_n} N(\tau_n - s)\mathrm{R}\,(s)x\,ds \to N(0)x, \quad n \to \infty.$$

Moreover,

$$\left\|\frac{1}{\tau_n}\int_0^{\tau_n} B(\tau_n - s)V(s)x\,ds\right\| \le \frac{1}{\tau_n}\int_0^{\tau_n} b(\tau_n - s)\|V(s)x\|_1\,ds.$$

Since $V(\cdot)x$ is $\|\cdot\|_1$-continuous and $b(\cdot)$ is locally bounded, it follows that

$$\frac{1}{\tau_n}\int_0^{\tau_n} B(\tau_n - s)V(s)x\,ds \longrightarrow 0, \quad n \to \infty.$$

Using (2.2), we obtain that $A(\frac{1}{\tau_n}V(\tau_n)x)$ converges as $n \to \infty$ in the weak topology. Since $1/\tau_n V(\tau_n)x \to x$, $n \to \infty$, and A is a closed operator, we conclude that $x \in D(A)$. □

Employing the Eberlein-Smulian theorem and arguing as above we can substitute the reflexivity of X by a compactness condition. For the sake of brevity we omit the proof.

Proposition 2.2. *If the set* $\{1/t(\mathrm{R}\,(t)x - x) : 0 < t \le 1\}$ *is relatively weakly compact and* $b(\cdot)$ *is locally bounded, then* $x \in D(A)$.

We now consider the differentiability of solutions of the inhomogeneous equation (1.7) with initial condition (1.4). In this part we always assume that f is a continuous function. We refer the reader to Definition 5.7 for the concept of the mild solution of problem (1.4)–(1.7).

To abbreviate our next statements, we introduce some additional notation. In the sequel we denote by M_0 and M_1 a pair of positive constants such that $\|\mathrm{R}(t)\| \leq M_0$ and $\|AV(t)\| \leq M_1$ for $0 \leq t \leq a$, and we represent by $\upsilon(f)$ the variation of f on $[0,a]$, and by $\upsilon(t,f)$ the variation of f on the interval $[0,t]$.

For a function $h:[0,a]\to X$, we denote by h^τ the function given by

$$h^\tau(s) = \begin{cases} h(s+\tau) & 0 \leq s+\tau \leq a, \\ h(0) & s+\tau \leq 0, \\ h(a) & a \leq s+\tau. \end{cases}$$

We consider the following condition for functions of bounded variation:

(T) $\upsilon(h^\tau - h) \to 0$, $\tau \to 0^+$.

Remark 2.3. In this remark we collect some properties relative to the condition (T).

(i) If h satisfies assumption (T), then h is continuous. In fact, for $0 \leq t < a$, we have that

$$\|h(t+\tau) - h(t))\| = \|h^\tau(a) - h(a) - (h^\tau(t) - h(t))\| \leq \upsilon(h^\tau - h)$$

which implies that h is right-continuous at t. Similarly, we can show that h is left-continuous on $(0,a]$.

(ii) If h is continuously differentiable, then h satisfies condition (T). In fact, it is well known [**3**] that

$$\upsilon(h^\tau - h) = \int_0^a \|h'(t+\tau) - h'(t)\|\,dt \to 0, \quad \tau \to 0.$$

(iii) There are continuous functions of bounded variation that do not satisfy condition (T). We next illustrate this assertion with an example. Let $h_n(t) = 1/n \cos n\pi t$ for $0 \leq t \leq 1$ and $n \in \mathbf{N}$. Since $h_n'(t) = -\pi \sin n\pi t$, we deduce that $\{h_n : n \in \mathbf{N}\}$ is equicontinuous on $[0,1]$ and

$$\upsilon(h_n) = \int_0^1 |h_n'(t)|\,dt = \pi \int_0^1 |\sin n\pi t|\,dt = 2.$$

We define $h : [0,1] \to \ell^\infty$ by $h(t) = (h_n(t))_n$. Using the equicontinuity of functions h_n we get that h is continuous. On the other hand, let $d = \{\xi_0, \xi_1, \dots, \xi_k\}$, where $0 = \xi_0 < \xi_1 < \cdots < \xi_k = 1$, be a division of the interval $[0,1]$. For each $\varepsilon > 0$, we can choose $n_i \in \mathbf{N}$ such that

$$\begin{aligned} \upsilon_d(h) &= \sum_{i=1}^{k} \|h(\xi_i) - h(\xi_{i-1})\|_\infty \\ &\le \sum_{i=1}^{k} |h_{n_i}(\xi_i) - h_{n_i}(\xi_{i-1})| + \varepsilon \\ &\le \pi \sum_{i=1}^{k} |\xi_i - \xi_{i-1}| + \varepsilon \\ &\le \pi + \varepsilon. \end{aligned}$$

This implies that h is a function of bounded variation with $\upsilon(h) \le \pi$. To complete this remark, we will estimate $\upsilon(h^\tau - h)$. It is clear that $\upsilon(h^\tau - h) \ge \upsilon(h_n^\tau - h_n)$ for every $n \in \mathbf{N}$. Again, using [**3**], we can write

$$\begin{aligned} \upsilon(h_n^\tau - h_n) &= \int_0^1 \left| \frac{d}{dt}(h_n^\tau(t) - h_n(t)) \right| dt \\ &\ge \int_0^{1-\tau} |-\pi \sin n\pi(t+\tau) + \pi \sin n\pi t|\, dt \\ &= \pi \int_0^{1-\tau} |(1 - \cos n\pi\tau) \sin n\pi t - \sin n\pi\tau \cos n\pi t|\, dt. \end{aligned}$$

Hence, for $\tau = 1/n$, we get that

$$\begin{aligned} \upsilon(h_n^{1/n} - h_n) &\ge \pi \int_0^{1-(1/n)} |2 \sin n\pi t|\, dt \\ &= 2\pi \int_0^1 |\sin n\pi t|\, dt \\ &\qquad - 2\pi \int_{1-(1/n)}^1 |\sin n\pi t|\, dt \\ &\ge 4 - \frac{2\pi}{n}. \end{aligned}$$

Consequently, $\upsilon(h^\tau - h)$ does not converge to zero as $\tau \to 0$.

We next use the notation

$$u(t) = \int_0^t \mathrm{R}\,(t-s)f(s)\,ds. \tag{2.3}$$

We begin by establishing some preliminary results.

Lemma 2.4. *Assume that X is a reflexive space, and let $h : [0, a] \to X$ be a function of bounded variation. Then the Riemann-Stieltjes integral*

$$\int_0^a AV(s)\,d_s h$$

exists in the weak topology and

$$\left\| \int_0^a AV(s)\,d_s h \right\| \le M_1 \upsilon(h).$$

Furthermore, if $0 < b < a$, then

$$\int_0^a AV(s)\,d_s h = \int_0^b AV(s)\,d_s h + \int_b^a AV(s)\,d_s h. \tag{2.4}$$

Proof. Let $\Lambda : D(A^*) \subseteq X^* \to \mathbf{C}$ be defined by

$$\Lambda(x^*) = \int_0^a \langle V(s)^* A^* x^*,\, d_s h\rangle.$$

The Riemann-Stieltjes integral in the above expression exists because $s \mapsto V(s)^* A^* x^*$ is a continuous function and h has bounded variation ([**25**]). The first assertion is a consequence of the fact that the map $V(\cdot)$, and hence also $V(\cdot)^*$, is continuous for the norm of operators. Moreover, Λ is linear and

$$|\Lambda(x^*)| \le M_1 \upsilon(h) \|x^*\|.$$

Since $D(A^*)$ is dense in X^* ([**12**, Section I.5.14]) the functional Λ has a continuous extension, still denoted by Λ, to X^*. Consequently, $\Lambda \in X^{**}$ and, in view of the fact that X is a reflexive space, we infer the existence of $x \in X$ with $\|x\| \leq M_1 \upsilon(h)$ such that $\Lambda(x^*) = \langle x^*, x\rangle$, for all $x^* \in X^*$. We set $x = \int_0^a AV(s)\, d_s h$.

Proceeding in a similar way, we can establish the existence of $\int_b^a AV(s)\, d_s h$ when $b < a$ and the equality (2.4) is immediate. □

Lemma 2.5. *Assume that X is a reflexive space, and let $h : [0, a] \to X$ be a function of bounded variation that satisfies assumption (T). Then the Riemann-Stieltjes integral*

$$g(t) = \int_0^t AV(t-s)\, d_s h = -\int_0^t AV(s)\, d_s h(t-s)$$

exists in the weak topology and defines a continuous function $g : [0, a] \to X$.

Proof. The existence of $g(t)$ is an immediate consequence of Lemma 2.4. On the other hand, using (2.4), we have

$$\begin{aligned} g(t+\tau) - g(t) &= -\int_0^{t+\tau} AV(s)\, d_s h(t+\tau-s) \\ &\quad + \int_0^t AV(s)\, d_s h(t-s) \\ &= -\int_0^t AV(s)\, d_s [h(t+\tau-s) - h(t-s)] \\ &\quad - \int_t^{t+\tau} AV(s)\, d_s h(t+\tau-s). \end{aligned}$$

We next estimate the two terms on the right hand side of the above expression separately. Since h is continuous, it follows from [**25**, Proposition I.2.9] that

$$\left\| \int_t^{t+\tau} AV(s)\, d_s h(t+\tau-s) \right\| \leq M_1 \upsilon(\tau, h) \longrightarrow 0, \quad \tau \to 0. \tag{2.5}$$

On the other hand,

$$\left\| \int_0^t AV(s)\, d_s [h(t+\tau-s) - h(t-s)] \right\| \leq M_1 \upsilon(h^\tau - h) \longrightarrow 0, \quad \tau \to 0.$$

Collecting this property with (2.5), we conclude that $g(t+\tau)-g(t) \to 0$ as $\tau \to 0$ which shows that $g(\cdot)$ is right continuous at t. Similarly, one can prove that g is left continuous at t, which completes the proof. □

Theorem 2.6. *Assume that X is a reflexive space, and let f be a function of bounded variation on $[0,a]$ that satisfies assumption* (T). *If $z \in D(A)$, then the mild solution $x(\cdot)$ of problem* (1.4)–(1.7) *is a classical solution on $(0,a]$. If, further, $z \in E$, then $x(\cdot)$ is a classical solution on $[0,a]$.*

Proof. We consider a sequence $(f_n)_n$ of step functions, where each f_n is the function given by $f(t_1)\chi_{[t_0,t_1]} + \sum_{i=2}^n f(t_i)\chi_{(t_{i-1},t_i]}$. In this expression, we have chosen the points $t_i = (a/n)i$, $i = 0, 1, \dots, n$, and χ_J represents the characteristic function associated to an interval J. It is clear that the sequence $(f_n)_n$ converges uniformly to f. Let u_n be the function given by (2.3) with f_n instead of f. Then, $u_n \to u$ as $n \to \infty$, uniformly on $[0,a]$. Moreover, by Theorem 5.9, we have that $u_n(\cdot)$ is a classical solution on $[0,a]$ of problem (1.4)–(1.7) with $z = 0$ and f_n instead of f. Consequently, $u_n(t) \in D(A)$ for $0 \le t \le a$.

On the other hand, if we fix $0 < t \le a$ and $n \in \mathbf{N}$, then $t \in (t_{j-1}, t_j]$, for some $j = 1, \dots, n$. From our definitions, we can write

(2.6)
$$\begin{aligned} Au_n(t) &= A\sum_{i=1}^{j-1}\int_{t_{i-1}}^{t_i} \mathrm{R}\,(t-s)f(t_i)\,ds + A\int_{t_{j-1}}^{t} \mathrm{R}\,(t-s)f(t_j)\,ds \\ &= A\sum_{i=1}^{j-1}[V(t-t_{i-1}) - V(t-t_i)]f(t_i) + AV(t-t_{j-1})f(t_j) \\ &= AV(t)f(0) + \sum_{i=1}^{j} AV(t-t_{i-1})[f(t_i) - f(t_{i-1})], \end{aligned}$$

so that
$$\|Au_n(t)\| \le M_1 v(f) + M_1\|f\|_\infty.$$

Hence, it follows that $(Au_n(t))_n$ is a sequence uniformly bounded on $[0,a]$. Consequently, for each t, there is a subsequence which converges to $w(t) \in X$ in the weak topology. This implies that $u(t) \in D(A)$

and $w(t) = Au(t)$. A standard argument shows that the full sequence $(Au_n(t))_n$ converges to $Au(t)$.

Moreover, it follows from (2.6) that

$$Au(t) = \int_0^t AV(t-s)\, d_s f(s) + AV(t) f(0)$$

and Lemma 2.5 implies that $Au(\cdot)$ is a continuous function. If $z \in D(A)$, applying Corollary 5.10, we conclude that $x(t) = \mathrm{R}\,(t)z + u(t)$ is a classical solution of (1.7)–(1.4) on $(0, a]$. Similarly, if $z \in E$, then $x(\cdot)$ is a classical solution on $[0, a]$. □

In a particular case we can omit the reflexivity of space X.

Proposition 2.7. *Let $f(t) = \varphi(t)y$, where $\varphi : [0, a] \to \mathbf{C}$ is a continuous function of bounded variation and $y \in X$. If $z \in D(A)$, then the mild solution $x(\cdot)$ of problem* (1.7)–(1.4) *is a classical solution on* $(0, a]$. *If $z \in E$, then the mild solution $x(\cdot)$ of problem* (1.7)–(1.4) *is a classical solution on* $[0, a]$.

Proof. For the most part we proceed as in the proof of Theorem 2.6. Since $AV(\cdot)y$ is a continuous function and φ has bounded variation, we can use the properties of the integration of Riemann-Stieltjes developed in [**25**] to conclude that the integral $\int_0^t AV(t-s)y\, d\varphi(s)$ exists as a limit in the norm of X and that the function $t \mapsto \int_0^t AV(t-s)y\, d\varphi(s)$ is continuous. It follows from (2.6) that

$$Au_n(t) = \varphi(0)AV(t)y + \sum_{i=1}^{j} [\varphi(t_i) - \varphi(t_{i-1})] AV(t - t_{i-1})y,$$

which implies that $Au_n(t) \to \int_0^t AV(t-s)y\, d\varphi(s) + \varphi(0)AV(t)y$ as $n \to \infty$. We finish the proof by proceeding as in the proof of Theorem 2.6. □

Now we consider the semilinear problem

$$\frac{d}{dt}\left(x(t)+\int_0^t N(t-s)x(s)\,ds\right) = Ax(t)+\int_0^t B(t-s)x(s)\,ds+f(t,x(t)),\quad t\in I, \tag{2.7}$$

with initial condition (1.4), where $f:[0,a]\times X\to X$ is a continuous function.

This type of equation has been used in [**5**, Section 5] to model the problem of heat conduction in materials with fading memory and also represents a generalization of equations used recently in [**15**] to study the heat flux in a conserved system, and in [**14**] for studying systems with memory relaxation.

We recall the definition of solution [**11, 22**].

Definition 2.8. A function $x:[0,a]\to X$ is a mild solution of (2.7)–(1.4) if x is continuous and satisfies the integral equation

$$x(t)=\mathrm{R}\,(t)z+\int_0^t \mathrm{R}\,(t-s)f(s,x(s))\,ds,\quad t\in I.$$

A function $x(\cdot)$ is said to be a classical solution of (2.7)–(1.4) if it fulfills the conditions established in Definition 5.4 for $f(t,x(t))$ instead of $f(t)$.

On the other hand, it is well known that every reflexive space has the Radon-Nikodym property (abbreviated, RNP) and that there are non reflexive spaces that have the RNP. We refer to [**9**] for several characterizations of the RNP. For this reason, in the sequel, we consider spaces that have the RNP. We consider the following Lipschitz condition for f.

(H1) For each $r>0$, there exists a constant $C(r)>0$ such that

$$\|f(t,y)-f(s,x)\|\le C(r)[|t-s|+\|y-x\|]$$

for all $0\le s,\, t\le r$ and $\|x-z\|, \|y-z\|\le r$.

Theorem 2.9. *Assume that X satisfies the* RNP. *Let $z \in E$, and let f be a function that satisfies condition* (H1). *Then there is a unique classical solution $x(\cdot)$ of problems* (1.4)–(2.7) *on* $[0, \beta]$, *for some* $\beta > 0$.

Proof. Proceeding as usual, applying the fixed point theorem for contraction maps, we obtain that there is a unique mild solution $x(\cdot)$ of problems (1.4) and (2.7) on $[0, \beta]$, for some $0 < \beta \leq a$. This is also a consequence of results established in [**11, 22**]. We can assume that $\|x(t) - z\| \leq r$ for some $r > 0$ and all $0 \leq t \leq \beta$. Hence, for $0 \leq s \leq t \leq \beta$, and using that $z \in E$, we have

$$\begin{aligned} \|x(t) - x(s)\| &\leq \|\mathrm{R}\,(t)z - \mathrm{R}\,(s)z\| \\ &\quad + \bigg\| \int_0^t \mathrm{R}\,(\xi) f(t-\xi, x(t-\xi))\, d\xi \\ &\qquad - \int_0^s \mathrm{R}\,(\xi) f(s-\xi, x(s-\xi))\, d\xi \bigg\| \\ &\leq \|\mathrm{R}\,(t)z - \mathrm{R}\,(s)z\| \\ &\quad + \bigg\| \int_s^t \mathrm{R}\,(\xi) f(t-\xi, x(t-\xi))\, d\xi \bigg\| \\ &\quad + \bigg\| \int_0^s \mathrm{R}\,(\xi) [f(t-\xi, x(t-\xi)) - f(s-\xi, x(s-\xi))] \bigg\|\, d\xi \\ &\leq C_1(r)(t-s) + M_0 C(r) \int_0^s \|x(t-s+\xi) - x(\xi)\|\, d\xi, \end{aligned}$$

for a certain constant $C_1(r)$. It follows from the above estimate and Gronwall-Bellman's lemma that $x(\cdot)$ is Lipschitz continuous. Consequently, the function $t \mapsto g(t) = f(t, x(t))$ is Lipschitz continuous and, since the space X satisfies the RNP, the function $g \in W^{1,1}([0, \beta], X)$. The assertion is now a consequence of Theorem 5.9. $\square$

3. Existence of solutions for the functional problem. In this section we study the existence of classical solutions for the abstract neutral problem with infinite delays (1.1)–(1.2). In the sequel, we always assume that conditions (P1)–(P4) are verified and that $\mathrm{R}\,(t)$ is the resolvent operator studied in Section 5.

We use an axiomatic definition of the phase space $\mathcal{B}$, which is similar to the one used in [**24**]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\|\cdot\|_{\mathcal{B}}$ and verifying the following axioms.

(A) If $x : (-\infty, \sigma + a) \to X$, $a > 0$, $\sigma \in \mathbf{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$, the following conditions hold:

(i) x_t is in $\mathcal{B}$.

(ii) $\|x(t)\| \leq H\|x_t\|_{\mathcal{B}}$.

(iii) $\|x_t\|_{\mathcal{B}} \leq K(t - \sigma) \sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (A), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

(B) The space $\mathcal{B}$ is complete.

Example 3.1. The phase space $C_r \times L^p(\rho, X)$. Let $r \geq 0$, $1 \leq p < \infty$, and let $\rho : (-\infty, -r] \to \mathbf{R}$ be a nonnegative measurable function which satisfies conditions (g-5) and (g-6) in the terminology of [**24**]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho\|\varphi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The semi-norm in $C_r \times L^p(\rho, X)$ is defined by

$$\|\varphi\|_{\mathcal{B}} = \sup\{\|\varphi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta)\|\varphi(\theta)\|^p d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho, X)$ satisfies axioms (A), (A-1) and (B). Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and

$$K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)\, d\theta \right)^{1/2}, \quad \text{for } t \geq 0.$$

See [**24**, Theorem 1.3.8] for details.

To study problem (1.1)–(1.2) in what follows we assume that f satisfies the following assumption.

(H2) The function $f : I \times \mathcal{B} \to X$ verifies the following conditions:

(i) The function $f(t, \cdot) : \mathcal{B} \to X$ is continuous for every $t \in I$, and for every $\psi \in \mathcal{B}$ the function $f(\cdot, \psi) : I \to X$ is strongly measurable.

(ii) There exist a continuous function $m_f : I \to [0, \infty)$ and a continuous non-decreasing function $\Omega_f : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, \psi)\| \leq m_f(t)\Omega_f(\|\psi\|_{\mathcal{B}}), \ (t, \psi) \in I \times \mathcal{B}.$$

Motivated by the results in Section 5, we introduce the following concepts of mild and classical solutions for the neutral system (1.1)–(1.2).

Definition 3.2. A function $x : (-\infty, a] \to X$ is called a *classical solution of the neutral system* (1.1)–(1.2) on $(0, a]$ if $x_0 = \varphi$, the restriction $x_{|_{[0,a]}} \in C([0, a], [D(A)]) \cap C^1((0, a], X)$ and (1.1) is verified on $[0, a]$. If, further, $x_{|_{[0,a]}} \in C([0, a], [D(A)]) \cap C^1([0, a], X)$, then $x(\cdot)$ is said to be a classical solution of the neutral system (1.1)–(1.2) on $[0, a]$.

Definition 3.3. A function $x : (-\infty, a] \to X$ is called a mild solution of the neutral system (1.1)–(1.2) on $[0, a]$ if $x_0 = \varphi$, the restriction $x_{|_{[0,a]}} \in C([0, a], X)$ and the equation

$$x(t) = \mathrm{R}\,(t)\varphi(0) + \int_0^t \mathrm{R}\,(t - s) f(s, x_s)\, ds, \quad t \in [0, a],$$

is verified.

The proof of the next results is standard [**11, 22**]. For the sake of brevity we omit it.

Theorem 3.4. *Assume that the function $f : I \times \mathcal{B} \to X$ is continuous and there is an $L_f \in L^1([0, a], \mathbf{R}^+)$ such that*

$$\|f(t, \psi_1) - f(t, \psi_2)\| \leq L_f(t)\|\psi_1 - \psi_2\|, \quad t \in I, \psi_1, \psi_2 \in \mathcal{B}.$$

Then there is a unique mild solution of (1.1)–(1.2) *on* $[0, a]$.

Similarly, for functions f that satisfy a local Lipschitz condition we can establish a result of existence of local solutions. We consider the following Lipschitz condition for f.

(H3) For each $r > 0$, there exists a constant $C(r) > 0$ such that

$$\|f(t,\psi_2) - f(s,\psi_2)\| \leq C(r)[|t-s| + \|\psi_2 - \psi_1\|_{\mathcal{B}}],$$

for $0 \leq s$, $t \leq r$ and $\psi_1, \psi_2 \in \mathcal{B}$ with $\|\psi_i - \varphi\|_{\mathcal{B}} \leq r$ for $i = 1, 2$.

Proposition 3.5. *Assume that the function $f : I \times \mathcal{B} \to X$ satisfies condition* (H3)*. Then there is a unique mild solution of* (1.1)–(1.2) *on* $[0, \beta]$*, for some* $0 < \beta \leq a$*.*

Arguing as usual, using the compactness of the resolvent operator and the Schauder-Tikhonov theorem, we can avoid the Lipschitz condition.

Theorem 3.6. *Assume that condition* (H2) *is fulfilled, and that* $\mathrm{R}\,(t)$ *is compact for every* $t > 0$*. Then there exists a mild solution of problem* (1.1)–(1.2) *on* $[0, \beta]$ *for some* $0 < \beta \leq a$*.*

We refer to Lemma 5.3 for a criterion to guarantee the compactness of $\mathrm{R}\,(t)$. On the other hand, we define the operator $S(t) : \mathcal{B} \to \mathcal{B}$ for $t \geq 0$ by

$$[S(t)\psi](\theta) = \begin{cases} \psi(0) & -t \leq \theta \leq 0, \\ \psi(t+\theta) & \theta \leq -t. \end{cases}$$

It follows from the axioms of the phase space that $S(\cdot)$ is a strongly continuous semigroup. We denote by $\mathcal{B}_{\mathrm{Lip}}$ the subspace of $\mathcal{B}$ consisting of functions ψ such that

$$\|S(h)\psi - \psi\|_{\mathcal{B}} \leq L_\psi h, \quad h \geq 0,$$

where $L_\psi \geq 0$ is a constant.

Example 3.7. Let $\mathcal{B} = C_0 \times L^2(\rho, X)$ be the space defined in Example 3.1. Assume, in addition, that the function ρ is integrable on $(-\infty, 0]$. It is clear that, if $\psi \in \mathcal{B}$ is a uniformly Lipschitz continuous function, then $\psi \in \mathcal{B}_{\mathrm{Lip}}$.

The next property can be proved using a standard argument based upon the phase space axioms and the Gronwall-Bellman lemma. We omit the proof.

Lemma 3.8. *Assume that condition* (H3) *is fulfilled,* $\varphi \in \mathcal{B}_{\mathrm{Lip}}$ *and* $\varphi(0) \in E$. *Let* $x(\cdot)$ *be the mild solution of* (1.1)–(1.2) *on* $[0, \beta]$. *Then the functions* $x(\cdot)$ *and* $s \to x_s$ *are Lipschitz continuous on* $[0, a]$.

Corollary 3.9. *Assume that* X *satisfies the* RNP *and that the function* $f : I \times \mathcal{B} \to X$ *verifies condition* (H3). *If* $\varphi \in \mathcal{B}_{\mathrm{Lip}}$ *and* $\varphi(0) \in E$, *then the mild solution* $x(\cdot)$ *of* (1.1)–(1.2) *on* $[0, \beta]$ *is a classical solution on* $[0, \beta]$.

Proof. It follows from Lemma 3.8 that the function $t \mapsto x_t$ is Lipschitz continuous. Therefore, the function $t \mapsto f(t, x_t)$ is also Lipschitz continuous. We complete the proof arguing as in the proof of Theorem 2.9. □

We complete this result by studying the differentiability of function $t \mapsto x_t$ when f is only Lipschitz continuous. To establish this result, we will need additional properties of the phase space $\mathcal{B}$. We next denote by $C_{00}(X)$ the space of continuous functions from $(-\infty, 0]$ to X with compact support. We consider the following axiom for the phase space $\mathcal{B}$ ([**24**]).

(C2) If a uniformly bounded sequence $(\varphi^n)_n$ in $C_{00}(X)$ converges to a function φ in the compact-open topology, then φ belongs to $\mathcal{B}$ and $\|\varphi^n - \varphi\|_{\mathcal{B}} \to 0$, as $n \to \infty$.

It is easy to see ([**24**]) that if axiom (C2) holds, then the space of continuous and bounded functions $C_b = C_b((-\infty, 0], X)$ is continuously included in $\mathcal{B}$. Thus, there exists a constant $Q > 0$ such that

$$\|\varphi\|_{\mathcal{B}} \leq Q\|\varphi\|_\infty, \quad \varphi \in C_b((-\infty, 0], X).$$

Furthermore, in this case ([**24**, Proposition 7.1.5]) the function $K(\cdot)$ involved in axiom (A) can be chosen as the constant Q. As an example, we mention that, if the function ρ is integrable on $(-\infty, -r]$, then the space $C_r \times L^p(\rho, X)$ defined in Example 3.1 satisfies axiom (C2). We also use the notation $UC_b = UC_b((-\infty, 0], X)$ to represent the subspace of C_b consisting of uniformly continuous functions.

Corollary 3.10. *Assume that X satisfies the* RNP*, the space $\mathcal{B}$ satisfies axiom* (C2) *and the function $f : I \times \mathcal{B} \to X$ verifies condition* (H3)*. Assume, further, that $\varphi \in C_b$ is a continuously differentiable function such that $\varphi' \in UC_b$, $\varphi(0) \in E$ and*

$$\varphi'(0) = A\varphi(0) + f(0, \varphi) - N(0)\varphi(0).$$

Let $x(\cdot)$ be the mild solution of (1.1)–(1.2) *on* $[0, \beta]$*. Then the function $t \mapsto x_t$ is continuously differentiable on* $[0, \beta]$*.*

Proof. Assumptions on φ and axiom (C2) imply that $\varphi \in \mathcal{B}_{\mathrm{Lip}}$. It follows from Corollary 3.9 that $x(\cdot)$ is continuously differentiable on $[0, \beta]$ and the right derivative $x'(0^+)$ verifies

$$x'(0^+) = A\varphi(0) + f(0, \varphi) - N(0)\varphi(0).$$

Turning to use the properties of φ, we conclude that $x(\cdot)$ is continuously differentiable on $(-\infty, a]$. The assertion is now a consequence of [**21**, Lemma 1.1]. $\square$

Finally, we will study the existence of classical solutions when X is a general Banach space and f is a smooth function. We introduce some additional notations. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. For a differentiable function $g : I \times W \to Z$, we denote by $Dg(t, w) : \mathbf{R} \times W \to Z$ the derivative of g at (t, w). We decompose

$$Dg(t, w)(h, w_1) = hD_1 g(t, w) + D_2 g(t, w)(w_1).$$

Theorem 3.11. *Assume that $\mathcal{B}$ satisfies the axiom* (C-2)*. Let $f \in C^1([0, a] \times \mathcal{B}, X)$ be such that Df satisfies the Lipschitz condition*

$$\|Df(s, \psi_2) - Df(s, \psi_1)\| \le L(s)\|\psi_2 - \psi_1\|_{\mathcal{B}}, \tag{3.1}$$

where $L(\cdot)$ is a locally bounded function. Let $\varphi \in \mathcal{B}$ be a continuously differentiable function such that $\varphi' \in \mathcal{B}$, $\varphi(0) \in E$, and

$$\varphi'(0) = A\varphi(0) + f(0, \varphi) - N(0)\varphi(0). \tag{3.2}$$

Then there is a classical solution of (1.1) *and* (1.2) *on* $[0, \beta]$ *for some* $0 < \beta \le a$.

Proof. By Theorem 2.1 we can affirm that there is a mild solution $x(\cdot)$ of (1.1) and (1.2) on $[0, \beta_1]$ for some $0 < \beta_1 \le a$.

We consider the initial value problem

$$
\begin{aligned}
w(t) = \mathrm{R}\,(t)\varphi'(0) + h(t) + \int_0^t \mathrm{R}\,(t-s)B(s)\varphi(0)\,ds \\
+ \int_0^t \mathrm{R}\,(t-s)D_1 f(s, x_s)\,ds \\
+ \int_0^t \mathrm{R}\,(t-s)D_2 f(s, x_s)(w_s)\,ds,
\end{aligned}
\tag{3.3}
$$

$$
w_0 = \varphi', \tag{3.4}
$$

where $h(t) = d/dt \int_0^t \mathrm{R}\,(t-s)[N(0) - N(s)]\varphi(0)\,ds$. Using the condition $\varphi(0) \in E$, it follows from Definition 1.1 and Remark 5.6 that h is a continuous function such that $h(0) = 0$. Hence, equation (3.3) has the form

$$
w(t) = g(t) + \int_0^t \mathrm{R}\,(t-s)P(s)(w_s)\,ds,
$$

where g is a continuous function and the bounded linear map $P(s) = D_2 f(s, x_s)$ is continuous at s. Applying the contraction mapping principle, we can show that there exist $0 < \beta < \beta_1$, a unique function $w : (-\infty, b] \to X$ which is continuous on $[0, \beta]$ and a solution of problem (3.3)–(3.4).

We define $v(t) = \varphi(0) + \int_0^t w(s)\,ds$ for $t \ge 0$ and $v(\theta) = \varphi(\theta)$ for $\theta \le 0$. It follows from (3.3) and condition (3.2) that

(3.5)

$$
\begin{aligned}
v(t) = \varphi(0) + \int_0^t \mathrm{R}\,(s)[A\varphi(0) + f(0, \varphi) - N(0)\varphi(0)]\,ds + \int_0^t h(s)\,ds \\
+ \int_0^t \int_0^s \mathrm{R}\,(s-\xi)B(\xi)\varphi(0)\,d\xi\,ds
\end{aligned}
$$

$$+\int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_1 f(\xi, x_\xi)\, d\xi\, ds$$
$$+\int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_2 f(\xi, x_\xi) w_\xi\, d\xi\, ds.$$

On the other hand, since the function $s \mapsto w_s$ is continuous, from the theory of integration of vector functions with values in Banach spaces ([**28**]) and axiom (C-2), we obtain that $v_t = \varphi + \int_0^t w_s\, ds$ for $t \geq 0$. Consequently, functions $t \mapsto v_t$ and $t \mapsto \int_0^t \mathrm{R}\,(t-s) f(s, v_s)\, ds$ are continuously differentiable, and

$$\begin{aligned}
&\frac{d}{dt}\int_0^t \mathrm{R}\,(t-s) f(s, v_s)\, ds \\
&\qquad = \frac{d}{dt}\int_0^t \mathrm{R}\,(s) f(t-s, v_{t-s})\, ds \\
&\qquad = \mathrm{R}\,(t) f(0,\varphi) + \int_0^t \mathrm{R}\,(s)[D_1 f(t-s, v_{t-s}) \\
&\qquad\qquad + D_2 f(t-s, v_{t-s}) w_{t-s}]\, ds \\
&\qquad = \mathrm{R}\,(t) f(0,\varphi) + \int_0^t \mathrm{R}\,(t-s)[D_1 f(s, v_s) + D_2 f(s, v_s) w_s]\, ds.
\end{aligned}$$

Integrating on the interval $[0, t]$ in the above expression, we obtain

$$\begin{aligned}
\int_0^t \mathrm{R}\,(t-s) f(s, v_s)\, ds &= \int_0^t \mathrm{R}\,(s) f(0,\varphi)\, ds \\
&\quad + \int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_1 f(\xi, v_\xi)\, d\xi\, ds \\
&\quad + \int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_2 f(\xi, v_\xi) w_\xi\, d\xi\, ds.
\end{aligned} \tag{3.6}$$

On the other hand, applying resolvent equation (1.6), we can write

$$\begin{aligned}
&\mathrm{R}\,(t)\varphi(0) + \int_0^t \mathrm{R}\,(t-s) N(s)\varphi(0)\, ds \\
&\qquad = \varphi(0) + \int_0^t \mathrm{R}\,(s) A\varphi(0)\, ds \\
&\qquad\quad + \int_0^t \int_0^s \mathrm{R}\,(s-\xi) B(\xi)\varphi(0)\, d\xi\, ds,
\end{aligned} \tag{3.7}$$

and, combining (3.6) with (3.7), we have

$$
\begin{aligned}
\varphi(0) + \int_0^t \mathrm{R}\,(s) f(0,\varphi)\, ds
&= \mathrm{R}\,(t)\varphi(0) + \int_0^t \mathrm{R}\,(t-s) N(s)\varphi(0)\, ds \\
&\quad - \int_0^t \mathrm{R}\,(s) A\varphi(0)\, ds - \int_0^t \int_0^s \mathrm{R}\,(s-\xi) B(\xi)\varphi(0)\, d\xi\, ds \\
&\quad + \int_0^t \mathrm{R}\,(t-s) f(s, v_s)\, ds \\
&\quad - \int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_1 f(\xi, v_\xi)\, d\xi\, ds \\
&\quad - \int_0^t \int_0^s \mathrm{R}\,(s-\xi) D_2 f(\xi, v_\xi) w_\xi\, d\xi\, ds.
\end{aligned}
$$

Substituting this expression in (3.5) yields

$$
\begin{aligned}
v(t) = \mathrm{R}\,(t)\varphi(0) &+ \int_0^t \mathrm{R}\,(t-s) f(s, v_s)\, ds \\
&+ \int_0^t \int_0^s \mathrm{R}\,(s-\xi)[D_1 f(\xi, x_\xi) - D_1 f(\xi, v_\xi)]\, d\xi\, ds \\
&+ \int_0^t \int_0^s \mathrm{R}\,(s-\xi)[D_2 f(\xi, x_\xi) - D_2 f(\xi, v_\xi)] w_\xi\, d\xi\, ds.
\end{aligned}
$$

Now, using Definition 3.3. we get

$$
\begin{aligned}
v(t) - x(t) = &\int_0^t \mathrm{R}\,(t-s)[f(s, v_s) - f(s, x_s)]\, ds \\
&+ \int_0^t \int_0^s \mathrm{R}\,(s-\xi)[D_1 f(\xi, x_\xi) - D_1 f(\xi, v_\xi)]\, d\xi\, ds \\
&+ \int_0^t \int_0^s \mathrm{R}\,(s-\xi)[D_2 f(\xi, x_\xi) - D_2 f(\xi, v_\xi)] w_\xi\, d\xi\, ds.
\end{aligned}
$$

Combining [**1**, Proposition 2.4.7] with condition (3.1) and the fact that x and v are continuous functions, we can state that

$$
\|f(s, v_s) - f(s, x_s)\| \le C \|v_s - x_s\|_{\mathcal{B}},
$$

where C is a constant independent of x, v and s for $s \in [0, \beta]$. Therefore, again using condition (3.1), we can affirm that

$$\|v(t) - x(t)\| \leq M_0 C Q \int_0^t \max_{0 \leq s \leq t} \|v(s) - x(s)\| \, ds + M_0 C_1 Q \int_0^t \max_{0 \leq s \leq t} \|v(s) - x(s)\| \, ds,$$

where C_1 is also a constant independent of x, v and t for $t \in [0, \beta]$. Applying the Gronwall-Bellman lemma we conclude that $x(t) = v(t)$ and the function $t \mapsto x_t = v_t$ is continuously differentiable. Consequently, the function $t \mapsto f(t, x_t)$ is also continuously differentiable. From Theorem 5.9, it follows that $x(\cdot)$ is a classical solution of (1.1)–(1.2) on $[0, \beta]$. □

4. Applications. In this section we apply our abstract results to study a neutral integro-differential equation that arises in the theory of heat flow in materials with fading memory ([**5**]). To simplify the exposition we assume the domain is $[0, \pi]$ and that the external source is modeled by a general expression which is used by several authors to describe systems with past dependence. Specifically, we consider the system

(4.1)
$$\frac{\partial}{\partial t}\left[u(t,\xi) + \int_0^t a(t-s) u(s,\xi)\, ds\right] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + \int_0^t b(t-s) \frac{\partial^2 u(s,\xi)}{\partial \xi^2}\, ds + H\left(\int_0^\infty q(s) u(t-s,\xi)\, ds\right) + p(u(t,\xi)) + \widetilde{h}(t,\xi),$$

(4.2)
$$u(t,\pi) = u(t,0) = 0,$$

(4.3)
$$u(\theta,\xi) = \varphi(\theta,\xi),$$

for $(t, \xi) \in [0, \infty) \times [0, \pi]$, $\theta \leq 0$. In this system, $a, b, q : [0, \infty) \to \mathbf{R}$, $H, p : \mathbf{R} \to \mathbf{R}$ and $\widetilde{h} : [0, \infty) \times [0, \pi] \to \mathbf{R}$ are continuous functions that satisfy appropriate conditions to be specified later. Moreover, we have identified $\varphi(\theta)(\xi) = \varphi(\theta, \xi)$.

To represent this system in the abstract form (1.1)–(1.2), we choose the spaces $X = L^2([0,\pi])$ and $\mathcal{B} = C_0 \times L^2(\rho, X)$, where $\rho : (-\infty, 0] \to [0, \infty)$ is integrable. In the sequel, $A : D(A) \subseteq X \to X$ is the operator given by $Ax = x''$ with domain $D(A) = \{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues of A are $-n^2$, $n \in \mathbf{N}$, with corresponding eigenvectors $z_n(\xi) = (2/\pi)^{1/2}\sin(n\xi)$. Moreover, the set of functions $\{z_n : n \in \mathbf{N}\}$ is an orthonormal basis of X and

$$T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n \quad \text{for } x \in X.$$

We assume that the following conditions hold:

(i) The functions $a, b \in L^1([0,\infty))$, and there is a $\vartheta \in (\pi/2, \pi)$ such that $|\widehat{a}(\lambda)| \leq C/|\lambda|$ for $\lambda \in \Lambda_\vartheta = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta\}$ and $|\widehat{b}(\lambda)| \to 0$ as $\lambda \in \Lambda_\vartheta$ and $|\lambda| \to \infty$.

(ii) $\int_{-\infty}^{0} (q^2(-\theta))/\rho(\theta)\, d\theta < \infty$.

(iii) The functions H, p are uniformly Lipschitz continuous on $\mathbf{R}$.

(iv) The function $\widetilde{h}$ satisfies Caratheódory conditions to ensure that $h(t) = \widetilde{h}(t,\cdot) \in L^2([0,\pi]$ for $t \geq 0$, and there is a positive function $g \in L^2([0,\pi])$ such that

$$|\widetilde{h}(t_2,\xi) - \widetilde{h}(t_1,\xi)| \leq g(\xi)|t_2 - t_1|$$

for all $t_2, t_1 \geq 0$ and $\xi \in [0,\pi]$.

We define the operators $N(t) : X \to X$, $B(t) : D(A) \subseteq X \to X$ and $f : [0,\infty) \times \mathcal{B} \to X$ by the expressions

$$\begin{aligned}
(B(t)x)(\xi) &= b(t)Ax(\xi),\\
(N(t)x)(\xi) &= a(t)x(\xi),\\
f(t,\psi)(\xi) &= H\left(\int_{-\infty}^{0} q(-s)\psi(s,\xi)\, ds\right) + p(\psi(0,\xi)) + \widetilde{h}(t,\xi).
\end{aligned}$$

It follows from (i) that conditions (P1)–(P4) are satisfied with $\widehat{N}(\lambda) = \widehat{a}(\lambda)I$, $\widehat{B}(\lambda) = \widehat{b}(\lambda)A$, and where we take $D = C_0^\infty([0,\pi])$ the space consisting of infinite differentiable functions that vanish at $\xi = 0$ and $\xi = \pi$. Moreover, it is not difficult to see from (ii)–(iv) that $f(\cdot,\cdot)$ is a uniformly Lipschitz continuous function.

With this notation the system (4.1)–(4.3) can be considered as an abstract neutral system (1.1)–(1.2). From the expression for $\widehat{N}(\lambda)$, it follows that the space $E = D(A)$.

Since X is a Hilbert space, the next results are a direct consequence of Corollaries 3.9 and 3.10.

Proposition 4.1. *Assume that $\varphi \in \mathcal{B}$ is uniformly Lipschitz continuous and $\varphi(0,\cdot) \in D(A)$. Then there exists a classical solution $u(\cdot)$ of* (4.1)–(4.3) *on* $[0,\infty)$.

Proposition 4.2. *Assume that φ satisfies the following conditions*:

(i) $\varphi(0,\cdot) \in D(A)$.

(ii) $\varphi(\cdot,\xi)$ *is continuous almost everywhere* $\xi \in [0,\pi]$ *and* $\sup_{\theta\le 0} \int_0^\pi |\varphi(\theta,\xi)|^2 \, d\xi < \infty$.

(iii) $(\partial/\partial\theta)\varphi(\cdot,\xi)$ *is uniformly continuous almost everywhere* $\xi \in [0,\pi]$, *and* $\sup_{\theta\le 0} \int_0^\pi |(\partial/\partial\theta)\varphi(\theta,\xi)|^2 \, d\xi < \infty$.

(iv)

$$\begin{aligned}\frac{\partial}{\partial\theta}\varphi(0,\xi) = {} & \frac{\partial^2}{\partial\xi^2}\varphi(0,\xi) + H\bigg(\int_{-\infty}^0 q(-\theta)\varphi(\theta,\xi)\, d\theta\bigg) \\ & + p(\varphi(0,\xi)) + \widetilde{h}(0,\xi) \\ & - a(0)\varphi(0,\xi).\end{aligned}$$

Let $u(\cdot)$ be the classical solution of (4.1)–(4.3) *on* $[0,\infty)$. *Then the function $t \mapsto u_t$ is differentiable on* $[0,\infty)$.

APPENDIX

5. In this section we collect some properties about the resolvent operator for the problem (1.3)–(1.4). This section includes only a brief review, mostly without proof, of properties of the resolvent operator to make the text self-contained. For details, we refer to [**10, 11, 22**].

We consider the following conditions:

(P1) The operator $A : D(A) \subseteq X \to X$ is the infinitesimal generator of an analytic semigroup $T(t)$ on X, and there are constants $M_0 > 0$ and $\vartheta \in (\pi/2, \pi)$ such that $\rho(A) \supseteq \Lambda_\vartheta = \{\lambda \in \mathbf{C} \setminus \{0\} : |\arg(\lambda)| < \vartheta\}$ and $\|R(\lambda, A)\| \leq M_0/|\lambda|$ for every $\lambda \in \Lambda_\vartheta$.

(P2) The function $N : [0, \infty) \to \mathcal{L}(X)$ is strongly continuous and $\widehat{N}(\lambda)x$ is absolutely convergent for $x \in X$ and $\mathrm{Re}(\lambda) > 0$. There are an $\alpha > 0$ and an analytical extension of $\widehat{N}(\lambda)$ (still denoted by $\widehat{N}(\lambda)$) to Λ_ϑ such that $\|\widehat{N}(\lambda)\| \leq N_0/|\lambda|^\alpha$ for every $\lambda \in \Lambda_\vartheta$, and $\|\widehat{N}(\lambda)x\| \leq N_1/|\lambda|\|x\|_1$ for every $\lambda \in \Lambda_\vartheta$ and $x \in D(A)$.

(P3) For all $t \geq 0$, $B(t) : D(B(t)) \subseteq X \to X$ is a closed linear operator, $D(A) \subseteq D(B(t))$ and $B(\cdot)x$ is strongly measurable on $(0, \infty)$ for each $x \in D(A)$. There exists a $b(\cdot) \in L^1_{\mathrm{loc}}(\mathbf{R}^+)$ such that $\widehat{b}(\lambda)$ exists for $\mathrm{Re}(\lambda) > 0$ and $\|B(t)x\| \leq b(t)\|x\|_1$ for all $t > 0$ and $x \in D(A)$. Moreover, the operator valued function $\widehat{B} : \Lambda_{\pi/2} \to \mathcal{L}([D(A)], X)$ has an analytical extension (still denoted by $\widehat{B}$) to Λ_ϑ such that $\|\widehat{B}(\lambda)x\| \leq \|\widehat{B}(\lambda)\| \, \|x\|_1$ for all $x \in D(A)$, and $\|\widehat{B}(\lambda)\| \to 0$ as $|\lambda| \to \infty$.

(P4) There exists a subspace $D \subseteq D(A)$ dense in $[D(A)]$ such that $A(D) \subseteq D(A)$, $\widehat{B}(\lambda)(D) \subseteq D(A)$, $\widehat{N}(\lambda)(D) \subseteq D(A)$, and the following estimates are verified:

$$\|A\widehat{B}(\lambda)x\| \leq C_1(x),$$
$$\|\widehat{N}(\lambda)x\|_1 \leq \frac{C_2}{|\lambda|^\alpha}\|x\|_1$$

for every $x \in D$ and $\lambda \in \Lambda_\vartheta$.

In what follows, we always assume that conditions (P1)–(P4) are verified.

In the rest of this section, $r > 0$, $\theta \in ((\pi/2), \vartheta)$ are fixed numbers, and we represent for $\Lambda_{r,\theta}$ the set $\{\lambda \in \mathbf{C}\setminus\{0\} : |\lambda| > r, \, |\arg(\lambda)| < \theta\}$. Additionally, we denote by $\Gamma^i_{r,\theta}$ for $i = 1, 2, 3$ the curves

$$\Gamma^1_{r,\theta} = \{te^{i\theta} : t \geq r\},$$
$$\Gamma^2_{r,\theta} = \{\mathrm{r}e^{i\xi} : -\theta \leq \xi \leq \theta\}$$

and

$$\Gamma^3_{r,\theta} = \{te^{-i\theta} : t \geq r\},$$

and $\Gamma_{r,\theta} = \cup_{i=1}^{3} \Gamma^i_{r,\theta}$. We always assume that these curves are oriented so that $\mathrm{Im}\,(\lambda)$ is increasing.

Lemma 5.1. *There exists a constant* $r > 0$ *such that the operator*

$$G(\lambda) = (\lambda I + \lambda \widehat{N}(\lambda) - A - \widehat{B}(\lambda))^{-1} \in \mathcal{L}(X)$$

for $\lambda \in \Lambda_{r,\vartheta}$. *Moreover, the following properties hold*:

(a) *The function* $G : \Lambda_{r,\vartheta} \to \mathcal{L}(X)$ *is analytic and there exists a constant* M_1 *such that*

$$\|G(\lambda)\| \leq \frac{M_1}{|\lambda|}.$$

(b) *The space* $\mathcal{R}(G(\lambda)) \subseteq D(A)$ *and the function* $AG : \Lambda_{r,\vartheta} \to \mathcal{L}(X)$ *is analytic, and there exist constants* M_2, M_3 *such that*

$$\begin{aligned} \|AG(\lambda)x\| &\leq \frac{M_2}{|\lambda|}\|x\|_1, \quad x \in D(A), \\ \|AG(\lambda)\| &\leq M_3, \end{aligned}$$

for every $\lambda \in \Lambda_{r,\vartheta}$.

Theorem 5.2. *The function*

$$\mathrm{R}\,(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} G(\lambda)\, d\lambda & t > 0, \\ I & t = 0 \end{cases} \tag{5.1}$$

is a resolvent operator for the integro-differential system (1.3)–(1.4).

The following property is a direct consequence of the construction of the resolvent operator.

Lemma 5.3. *If* $R(\lambda_0, A)$ *is a compact operator for some* $\lambda_0 \in \rho(A)$, *then* $\mathrm{R}\,(t)$ *is compact for all* $t > 0$.

Proof. It follows from Lemma 5.1 that $G(\lambda)$ is compact for all $\lambda \in \Lambda_{r,\vartheta}$. The assertion is now a consequence of expression (5.1). □

For the convenience of the reader, we restate the non-homogeneous problem

$$\frac{d}{dt}\left(x(t) + \int_0^t N(t-s)x(s)\,ds\right) = Ax(t) + \int_0^t B(t-s)x(s)\,ds + f(t), \quad t \in I, \tag{5.2}$$

$$x(0) = z, \tag{5.3}$$

where $f : [0,a] \to X$ is a continuous function.

Definition 5.4. A function $x : [0,a] \to X$ is called a classical solution of problem (5.2)–(5.3) on $(0,a]$ if $x \in C([0,a],[D(A)]) \cap C^1((0,a],X)$, the condition (5.3) holds and equation (5.2) is verified on $[0,a]$. If, further, $x \in C([0,a],[D(A)]) \cap C^1([0,a],X)$, the function x is said to be a classical solution of problem (5.2)–(5.3) on $[0,a]$.

It is clear from the preceding definition that $\mathrm{R}(\cdot)z$ is a classical solution of problem (1.3)–(1.4) on $(0,\infty)$ for $z \in D(A)$.

Initially, we establish that the solutions of problem (5.2)–(5.3) are given by the variation of constants formula.

Theorem 5.5. *Let $z \in D(A)$. Assume that $f \in C([0,a],X)$ and $x(\cdot)$ is a classical solution of problem* (5.2)–(5.3) *on* $(0,a]$*. Then*

$$x(t) = \mathrm{R}(t)z + \int_0^t \mathrm{R}(t-s)f(s)\,ds, \quad t \in [0,a].$$

Proof. For $\varepsilon > 0$, we consider $t \geq \varepsilon$, and we define

$$w(t) = \mathrm{R}(\varepsilon)x(t-\varepsilon) - \mathrm{R}(t)z - \int_0^{t-\varepsilon} \mathrm{R}(t-s)f(s)\,ds.$$

Taking the limit at the above expression as $\varepsilon \to 0$, and using the properties of $\mathrm{R}(\cdot)$, the assertion follows. □

Remark 5.6. We define the space E consisting of vectors $x \in X$ such that the function $\mathrm{R}(\cdot)x \in C([0,\infty),[D(A)]) \cap C^1([0,\infty),X)$. It is clear that $E \subseteq D(A)$ and $d/dt\mathrm{R}(t)x|_{t=0} = Ax - N(0)x$ for $x \in E$.

Motivated by the variation of constants formula, we introduce the following concept of the mild solution.

Definition 5.7. The function $x(\cdot)$ given by (5.4) is said to be the mild solution of problem (5.2)–(5.3).

Next we will study several conditions under which the mild solution of problem (5.2)–(5.3) is a classical solution. We begin with the following lemma.

Lemma 5.8. *Let $V : [0,\infty) \to \mathcal{L}(X)$ be the operator-valued function defined by $V(t)x = \int_0^t \mathrm{R}(s)x\,ds$. Then $\mathcal{R}(V(t)) \subseteq D(A)$ for all $t \geq 0$, the map $AV(\cdot) : [0,\infty) \to \mathcal{L}(X)$ is strongly continuous, and*

$$\begin{aligned} AV(t)x &= \mathrm{R}(t)x - x + \int_0^t N(t-s)\mathrm{R}(s)x\,ds \\ &\quad - \int_0^t B(t-s)V(s)x\,ds, \quad t \geq 0,\ x \in X. \end{aligned} \tag{5.5}$$

Proof. For $x \in D(A)$, the assertion is an easy consequence of the properties of the resolvent operator. For $x \in X$, we select a sequence $(x_n)_n$ in $D(A)$ such that $x_n \to x$ as $n \to \infty$. Consequently, $V(t)x_n \to V(t)x$ as $n \to \infty$. It follows from our initial assertion that $(AV(t)x_n)_n$ is a Cauchy sequence. Since A is closed, we obtain that $V(t)x \in D(A)$. Moreover, $B(t-s)V(s)$ is a bounded linear operator and $B(t-s)V(s)x_n \to B(t-s)V(s)x$ as $n \to \infty$. In view of $\|B(t-s)V(s)x_n\| \leq b(t-s)\|V(s)x_n\|_1$, from the Lebesgue dominated convergence theorem, we can affirm that $\int_0^t B(t-s)V(s)x_n\,ds \to \int_0^t B(t-s)V(s)x\,ds$ as $n \to \infty$.

Using the resolvent equation (1.5) with x_n instead of x, we get

$$AV(t)x_n \longrightarrow \mathrm{R}\,(t)x - x + \int_0^t N(t-s)\mathrm{R}\,(s)x\,ds \\ - \int_0^t B(t-s)V(s)x\,ds, \quad n \to \infty,$$

which implies that (5.5) holds. Since the function $t \mapsto \int_0^t B(t-s)V(s)x\,ds$ is continuous, from the above expression we conclude that $AV(\cdot)x \in C([0,\infty),X)$. This completes the proof. □

Theorem 5.9. *Let $z \in D(A)$, and let $f \in W^{1,1}([0,a],X)$. Then the mild solution $x(\cdot)$ of problem* (5.2)–(5.3) *is a classical solution on* $(0,a]$*. Further, if $z \in E$, then $x(\cdot)$ is a classical solution on* $[0,a]$.

Proof. We may assume that $z = 0$. Let u be the function given by (2.3). Applying [**2**, Proposition 1.3.6], we can assert that functions $u(\cdot)$ and $N * u(\cdot)$ are of class C^1 on $[0,a]$ and that

$$u'(t) = \int_0^t \mathrm{R}\,(t-s)f'(s)\,ds + \mathrm{R}\,(t)f(0),$$
$$\frac{d}{dt}\left(\int_0^t N(t-s)u(s)\,ds\right) = \int_0^t N(t-s)u'(s)\,ds + N(t)u(0),$$

for each $t \in [0,a]$. Using these expressions, we can establish that $u(\cdot)$ is a classical solution of problem (5.2)–(5.3) on $[0,a]$ with initial condition $u(0) = 0$. □

Let $f \in C([0,a],X)$. Approximating f by continuously differentiable functions, and applying Theorem 5.9, we can establish the following consequence.

Corollary 5.10. *Let $z \in D(A)$ and $f \in C([0,a],X)$. Let $x(\cdot)$ be the mild solution of problem* (5.2)–(5.3)*. If $x \in C([0,a],[D(A)])$, then $x(\cdot)$ is a classical solution on* $(0,a]$.

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