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SINGULAR INTEGRO-DIFFERENTIAL EQUATIONS WITH SMALL KERNELS

T.A. BURTON AND I.K. PURNARAS

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ABSTRACT. In 1975, Grimmer and Seifert studied a linear integro-differential equation with weakly singular kernel, $C(t,s)$, by means of a Razumikhin technique. They obtained bounded solutions from bounded forcing functions. Their conditions centered on small integrals of the kernel with respect to the second coordinate, s. On the last page of their paper they express the desire to obtain L^p solutions from L^p forcing functions. A recent result for singular integral equations makes it possible to answer the question. Here, we study a variety of integro-differential equations with singular kernels including linear, nonlinear, scalar, vector and resolvent equations by means of Liapunov functionals. We do obtain the types of L^p solutions from L^p perturbations. The point here is that there is a loose principle of the following type. Generally, but not always, Razumikhin techniques integrate the second coordinate and obtain bounded solutions, while Liapunov functionals integrate the first coordinate of the kernel and obtain L^p solutions. For decades, investigators have discussed and debated which technique was the "best." In fact, neither is best. They perform different sets of tasks, with a non-empty intersection.

1. Introduction. We study a scalar integro-differential equation of the form

$$x'(t) = f(t) - h(t, x(t)) - \int_0^t C(t,s)q(s, x(s))\,ds, \tag{1}$$

and also a linear vector equation, together with its resolvent. The objective is to determine qualitative properties of solutions when

$$\text{there exists a } p \in [1,\infty) \text{ with } f \in L^p[0,\infty), \tag{2}$$

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$$xh(t,x) \geq 0, \qquad xq(t,x) \geq 0, \tag{3}$$

and C has a weak singularity at $t = s$ with properties to be described later.

In 1975, [**10**] Grimmer and Seifert developed a Razumikhin technique (which utilizes a Liapunov function instead of a Liapunov functional) to deal with a vector equation

$$x'(t) = Ax(t) + \int_0^t B(t,s)x(s)\,ds + f(t), \tag{4}$$

where A is a constant matrix which is negative definite, B is a matrix satisfying

$$\lim_{h\to 0}\int_0^t |B(t,s) - B(t+h,s)|\,ds = 0,$$

and

$$\lim_{h\to 0}\int_t^{t+h} |B(t+h,s)|ds = 0, \quad t \geq 0,$$

as well as a number of other conditions, some of which are listed below. Under the central requirement that for a constant matrix K satisfying

$$A^T K + KA = -I \text{ then } \int_0^t |KB(t,s)|\,ds \leq M,$$

where M is related to the eigenvalues of K and, generally, M is small, they give conditions yielding solutions of (4) that have certain qualitative properties in case f is bounded and continuous while B is allowed to have weak singularities. On the last page of their paper, Grimmer and Seifert express the desire to show that the solution of (4) is in L^p when f is in L^q for some positive integers p and q. To the best of our knowledge, those desired results have never been obtained for equations with singular kernels. On the other hand, soon after the Grimmer-Seifert work was done, Liapunov theory was extended to (4) when B is continuous, and that theory led to a great many L^p results of the desired kind; these may be seen throughout the books [**3–5**], where a positive definite Liapunov functional is found with a derivative satisfying $V'(t) \leq -|x|^p + |f|^q$.

Now, the recent paper [**6**] makes it possible to supply the desired results for weakly singular kernels. We construct Liapunov functionals for (4) which give the desired L^p properties of the solutions of (4). We also consider linear equations and resolvents.

2. Preliminaries. Though in our subsequent work we will only allow discontinuities of C at $t = s$, typified by $C(t-s) = (t-s)^{-1/2}$ which occurs so often in the literature, we mention here a more general result (Theorem 2.2) found in [**8**] which concerns existence of a solution of (1) with a continuous derivative when C has some discontinuities. For the sake of completeness, we give a short proof of Theorem 2.2 and state a lemma (Lemma 2.3) which gives a simple condition in order that the inequality assumed in Theorem 2.2 is satisfied. Our terminology (Definition 2.1) follows that of Becker [**2**] who studied integral equations, not integro-differential equations.

Definition 2.1. Let $\Omega_T := \{(t,s) : 0 \le s \le t \le T\}$. The kernel C of (1) is *weakly singular on the set* Ω_T if it is unbounded in Ω_T; but, for each $t \in [0,T]$, $C(t,s)$ has at most finitely many discrete singularities in the interval $\{0 \le s \le t\}$ and, for every continuous function $\phi : [0,T] \to \Re^n$,

$$\int_0^t C(t,s)\phi(s)\,ds,$$

and

$$\int_0^t |C(t,s)|\,ds,$$

both exist and are continuous on $[0,T]$. If $C(t,s)$ is weakly singular on Ω_T for every $T > 0$, then it is *weakly singular on the set* $\Omega := \{(t,s) : 0 \le s \le t < \infty\}$.

For (1) we suppose that $f : [0,\infty) \to \Re^n$ is continuous, $h, q : [0,\infty) \times \Re^n \to \Re^n$ are both continuous and both satisfy a global Lipschitz condition for the same constant $\mathtt{L}$. In the proof below, the mapping follows [**9**], but the details then are precisely those of Becker [**2**] or of [**8**, Theorem 2.2].

Theorem 2.2. *In addition to these continuity conditions, let $C(t,s)$ be weakly singular on Ω. Suppose also that, for each $T > 0$ and each*

$k \in (0,1)$, *there is a constant* $\gamma_0 > 0$ *with*

$$\int_0^t e^{-\gamma_0(t-s)}|C(t,s)|\,ds \le k,$$

for $t \in [0,T]$. *Then, for every* $x_0 \in \Re^n$ (1) *has a unique solution* $x(t)$ *with a continuous derivative and satisfying* $x(0) = x_0$.

Proof. Let $T > 0$ and $x_0 \in \Re^n$ be given, and let $(Y, \|\cdot\|)$ be the Banach space of continuous functions $\phi : [0,T] \to \Re^n$ with the supremum norm. Define $P : Y \to Y$ by $\phi \in Y$ implies that

$$(P\phi)(t) = f(t) - h\Big(t, x_0 + \int_0^t \phi(s)\,ds\Big) - \int_0^t C(t,s)q\Big(s, x_0 + \int_0^s \phi(u)du\Big)ds.$$

By the continuity assumptions and the weak singularity, $P\phi \in Y$. As the existence of γ_0 implies that, for any $\gamma > \gamma_0$, we also have $\int_0^t e^{-\gamma(t-s)}|C(t,s)|\,ds \le k$ (see Lemma 2.3 below), we will define a weighted norm $\|\cdot\|_T$ by $\phi \in Y$ implies that

$$\|\phi\|_T = \sup_{0 \le t \le T} e^{-\gamma t}|\phi(t)|,$$

where $\gamma \ge \gamma_0$ is chosen so large and k is chosen so small so that $(\mathtt{L}/\gamma) + \mathtt{L}Tk \le 1/2$. With this mapping and norm, the details are readily completed as in [**8**]. $\square$

Lemma 2.3 below states that the inequality in Theorem 2.2 is satisfied if condition (5) is satisfied. We omit the routine proof.

Lemma 2.3. *Let* $C(t,s)$ *be a weakly singular kernel on the set* Ω, *and fix* $T > 0$. *Moreover, suppose that, for any* $k \in (0,1)$ *there exists an* $\varepsilon := \varepsilon(k,T) > 0$ *such that*

$$\int_{t-\varepsilon}^t |C(t,s)|\,ds \le k \text{ for all } t \in [0,T], \tag{5}$$

where we have set $C(t,s) = 0, (t,s) \in \Re^2 - \Omega$. *Then there always exists a* $\gamma_{k,T} > 0$ *such that for any* $\gamma \ge \gamma_{k,T}$ *we have*

$$\int_0^t e^{-\gamma(t-s)}|C(t,s)|\,ds \le k \text{ for all } t \in [0,T].$$

Though there are many other existence results (Grimmer and Seifert [**10**] and Grossman and Miller [**11**] deal with some far more complicated ones) we believe that Theorem 2.2 is simple, general and very instructive concerning existence ideas. In the following material, we will assume that the Liapunov results are being applied to problems in which existence has been established.

We will also be looking at a resolvent equation

$$\frac{d}{dt}z(t,s) = A(t)z(t,s) - \int_s^t C(t,u)z(u,s)\,du,$$

where A is a continuous $n \times n$ matrix and existence theory for it will be the same. Indeed, in this case, q is linear, and we automatically have a global Lipschitz condition. When C is continuous, Becker [**1**] has shown that, if $Z(t,s)$ is the $n \times n$ matrix solution of that equation satisfying $Z(s,s) = I$, then the solution of

$$x' = Ax - \int_0^t C(t,s)x(s)\,ds + f(t), \quad x(0) = x_0,$$

is given by

$$x(t) = Z(t,0)x_0 + \int_0^t Z(t,s)f(s)\,ds.$$

It is not difficult to verify that, when C satisfies Definition 2.1, then Z and Z_t are continuous and so all the steps in Becker's proof are valid and the same variation of parameters formula holds. This is used in Section 4.

3. A simple result. To see what is happening in order to get the desired L^p property, note that all of our integral conditions on $C(t,s)$ are with respect to t, while all of the Grimmer-Seifert integral conditions yielding boundedness are with respect to s. Our conclusion will be that $q(\cdot, x(\cdot)) \in L^1[0,\infty)$, as a result of $f \in L^1[0,\infty)$, a direct solution to the Grimmer-Seifert question. But we also get $x(t)$ bounded.

Theorem 3.1. *Let* (2) *hold with* $p = 1$. *Suppose there is a* $\gamma > 0$ *with* $|h(t,x)| \geq \gamma |q(t,x)|$ *on* $[0,\infty) \times \Re$. *Suppose also that there is a* $\beta > 0$ *so that, for each* $\varepsilon > 0$, *we have* $\int_\varepsilon^\infty |C(u+t,t)|\,du \leq \beta$ *for all*

$t \geq 0$, where $\gamma - \beta =: \mu > 0$. Finally, if there is an $\eta < \mu$ and a fixed $\varepsilon > 0$ with

$$\int_s^t |C(u+\varepsilon, s) - C(u,s)|\, du \leq \eta, \quad \text{for } 0 \leq s \leq t < \infty,$$

then any solution $x(t)$ of (1) on $[0,\infty)$ satisfies $q(\cdot, x(\cdot)) \in L^1[0,\infty)$.

Proof. For the fixed $\varepsilon > 0$, define a Liapunov functional

$$V(t,\varepsilon) = |x(t)| + \int_0^t \left[\int_{t-s+\varepsilon}^{\infty} |C(u+s,s)|\, du \right] |q(s,x(s))|\, ds, \quad t \geq 0,$$

so that, since

$$-|C(t+\varepsilon, s)| \leq -|C(t,s)| + |C(t+\varepsilon,s) - C(t,s)|,$$

we have

$$\begin{aligned}
V'(t,\varepsilon) &\leq |f(t)| - |h(t,x(t))| + \int_0^t |C(t,s)q(s,x(s))|\, ds \\
&\quad + \int_\varepsilon^\infty |C(u+t,t)|\, du |q(t,x(t))| \\
&\quad - \int_0^t |C(t+\varepsilon,s)||q(s,x(s))|\, ds \\
&\leq |f(t)| - \gamma |q(t,x(t))| + \int_0^t |C(t,s)q(s,x(s))|\, ds \\
&\quad + \beta |q(t,x(t))| - \int_0^t |C(t,s)q(s,x(s))|\, ds \\
&\quad + \int_0^t |C(t+\varepsilon,s) - C(t,s)||q(s,x(s))|\, ds \\
&= |f(t)| - \mu |q(t,x(t))| \\
&\quad + \int_0^t |C(t+\varepsilon,s) - C(t,s)||q(s,x(s))|\, ds.
\end{aligned}$$

In preparation for integration of this expression, we calculate

$$\begin{aligned}\int_\varepsilon^t \int_0^u |C(u+\varepsilon,s)-C(u,s)||q(s,x(s))|\,ds\,du \\ &\le \int_0^t \int_s^t |C(u+\varepsilon,s)-C(u,s)|\,du|q(s,x(s))|\,ds \\ &\le \int_0^t \eta|q(s,x(s))|\,ds.\end{aligned}$$

With this conclusion in hand, we now integrate V', obtaining

$$\begin{aligned}V(t,\varepsilon) &\le V(\varepsilon,\varepsilon)+\int_\varepsilon^t |f(u)|\,du-\mu\int_\varepsilon^t |q(s,x(s))|\,ds \\ &\quad+\int_\varepsilon^t\int_0^u |C(u+\varepsilon,s)-C(u,s)||q(s,x(s))|\,ds\,du \\ &\le V(\varepsilon,\varepsilon)+\int_0^t |f(u)|\,du \\ &\quad-(\mu-\eta)\int_\varepsilon^t |q(s,x(s))|\,ds+\eta\int_0^\varepsilon |q(s,x(s))|\,ds.\end{aligned}$$

This completes the proof. □

This case with $p = 1$ is very simple, and the proof is very short. Yet, it contains most of the properties and techniques involved in the case of an arbitrary, even, positive number p which is the topic of our last theorem. That proof makes repeated use of Young's and Schwarz's inequalities and, consequently, goes on for several pages. All of this involves small kernels in which the sign of the kernel is never employed.

4. The resolvent. Let C be an $n\times n$ matrix of functions with weak singularities, and consider

$$x'(t) = Ax(t) - \int_0^t C(t,s)x(s)\,ds + f(t), \quad x(0) = x_0, \tag{6}$$

where A is an $n \times n$ constant matrix, all of whose characteristic roots have negative real parts. There is then an $n \times n$ symmetric matrix B with

$$A^T B + BA = -I. \tag{7}$$

Associated with (6) is the resolvent equation

$$\frac{d}{dt}Z(t,s) = AZ(t,s) - \int_s^t C(t,u)Z(u,s)\,du, \quad Z(s,s) = I,$$

whose columns are the vector equations

$$z'(t,s) = Az(t,s) - \int_s^t C(t,u)z(u,s)\,du. \tag{8}$$

There is then the variation of parameters formula

$$x(t) = Z(t,0)x_0 + \int_0^t Z(t,s)f(s)\,ds.$$

We focus on three fundamental results. (i) If we can show that there is an $M > 0$ with $\int_0^t |Z(t,s)|\,ds \le M$, then for $f \in L^\infty[0,\infty)$, we see that for $x(0) = 0$, there is the bounded solution of (6), $x(t) = \int_0^t Z(t,s)f(s)\,ds$.

(ii) If we can show that there is an $M > 0$ with $\int_s^t |Z(u,s)|\,du \le M$ and if $f \in L^1[0,\infty)$, then for $x(0) = 0$, we have $|x(t)| \le \int_0^t |Z(t,s)f(s)|\,ds$. Thus, we would have

$$\begin{aligned}\int_0^t |x(s)|\,ds &\le \int_0^t \int_0^u |Z(u,s)||f(s)|\,ds\,du \\ &= \int_0^t \int_s^t |Z(u,s)|\,du|f(s)|\,ds \le M\int_0^t |f(s)|\,ds\end{aligned}$$

so that $x \in L^1[0,\infty)$.

(iii) If C is scalar, if there is an $M > 0$ with $\int_s^t Z^2(u,s)\,du \le M$ and if $f \in L^1[0,\infty)$, then for $x(0) = 0$ we have

$$|x(t)|^2 \le \left(\int_0^t |Z(t,s)f(s)|\,ds\right)^2 \le \int_0^t |f(s)|\,ds \int_0^t Z^2(t,s)|f(s)|\,ds,$$

and $x \in L^2[0,\infty)$ by the argument in (ii).

There are endless other uses for the resolvent and asking $x(0) = 0$ is not necessary. But these properties now direct our work. We have two choices for a Liapunov functional for (8). For the B of (7), for a positive constant ε to be determined, and for $0 \le s \le t$, define

$$\begin{aligned} V_1(t,s;\varepsilon) &= z^T(t,s)Bz(t,s) \\ &\quad + \int_s^t \int_{t-u+\varepsilon}^{\infty} |C^T(v+u,u)B|\,dv|z(u,s)|^2 du. \end{aligned} \tag{9}$$

We will also have occasion to ask for an $r > 0$ with

$$r|z| \le [z^T Bz]^{1/2},$$

and then, for $0 \le s \le t$, define

$$\begin{aligned} V_2(t,s;\varepsilon) &= [z^T(t,s)Bz(t,s)]^{1/2} \\ &\quad + \frac{1}{r}\int_s^t \int_{t-u+\varepsilon}^{\infty} |C^T(v+u,u)B|\,dv|z(u,s)|\,du. \end{aligned} \tag{10}$$

It should be obvious to the reader that the subscripts on V do not refer to partial derivatives. It is assumed that there exists an $\varepsilon > 0$ such that $C(t,s)$ is continuous for $0 \le s \le t-\varepsilon$. Here, we have $v \ge t-u+\varepsilon \ge \varepsilon$ so these integrands are continuous. With the V_1 we will obtain $\int_s^t z^2(u,s)\,du$ bounded. The second functional yields $\int_s^t |z(u,s)|\,du$ bounded; it also satisfies a global Lipschitz condition.

Lemma 4.1. *The derivative of $z^T(t,s)Bz(t,s)$ with respect to t along a solution of (8) satisfies*

$$\begin{aligned} [z^T(t,s)Bz(t,s)]' &\le -|z(t,s)|^2 \\ &\quad + \int_s^t |C^T(t,u)B|(|z(u,s)|^2 + |z(t,s)|^2)\,du. \end{aligned} \tag{11}$$

Proof. Differentiating by the product rule yields

$$
\begin{aligned}
&(z^T(t,s))'Bz(t,s) + z^T(t,s)Bz'(t,s)\\
&\quad = (z'(t,s))^T Bz(t,s) + z^T(t,s)Bz'(t,s)\\
&\quad = \Big[Az(t,s) - \int_s^t C(t,u)z(u,s)\,du\Big]^T Bz(t,s)\\
&\qquad + z^T(t,s)B\Big[Az(t,s) - \int_s^t C(t,u)z(u,s)\,du\Big]\\
&\quad = z^T(t,s)[A^TB + BA]z(t,s) - 2\int_s^t z^T(u,s)C^T(t,u)Bz(t,s)\,du\\
&\quad \le -z^T(t,s)z(t,s) + 2\int_s^t |C^T(t,u)B||z(u,s)||z(t,s)|\,du\\
&\quad \le -z^T(t,s)z(t,s) + \int_s^t |C^T(t,u)B|(|z(u,s)|^2 + |z(t,s)|^2)\,du,
\end{aligned}
$$

as required. □

We will now have two parallel results.

Theorem 4.2. *Let V_1 be defined in* (9)*, and let $z(t,s)$ be a solution of* (8)*. Suppose there is a $\widehat{\beta} > 0$ with $\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv \le \widehat{\beta}$. Then the derivative of V_1 along $z(t,s)$ with respect to t satisfies*

$$
\begin{aligned}
V_1'(t,s;\varepsilon) \le{}& -|z(t,s)|^2\Big[1 - \widehat{\beta} - \int_s^t |C^T(t,u)B|\,du\Big]\\
&+ \int_s^t |[C^T(t+\varepsilon,u) - C^T(t,u)]B||z(u,s)|^2 du.
\end{aligned}
\tag{12}
$$

Proof. In view of (11) we have, for $t \ge 0$,

$$
\begin{aligned}
V_1'(t,s;\varepsilon) \le{}& -|z(t,s)|^2\\
&+ \int_s^t |C^T(t,u)B|(|z(u,s)|^2 + |z(t,s)|^2)\,du\\
&+ \int_\varepsilon^\infty |C^T(v+t,t)B|\,dv|z(t,s)|^2\\
&- \int_s^t |C^T(t+\varepsilon,u)B||z(u,s)|^2 du
\end{aligned}
$$

$$\leq -|z(t,s)|^2 + |z(t,s)|^2 \int_s^t |C^T(t,u)B|\,du + \widehat{\beta}|z(t,s)|^2$$
$$+ \int_s^t |[C^T(t+\varepsilon,u) - C^T(t,u)]B||z(u,s)|^2 du,$$

as required. □

We can now see exactly what is needed to conclude that $\int_s^t z^2(u,s)\,du$ is bounded. If we integrate the last term from s to t and interchange the order of integration, we have

$$\int_s^t \int_s^w [C^T(w+\varepsilon,u) - C^T(w,u)]B||z(u,s)|^2 du\,dw$$
$$= \int_s^t \int_u^t |[C^T(w+\varepsilon,u) - C^T(w,u)]|B|\,dw|z(u,s)|^2 du.$$

The required condition is that there exist $\varepsilon > 0$, $\alpha > 0$, $\beta > 0$, $\widehat{\beta} > 0$ and $\alpha + \beta + \widehat{\beta} < 1$, with

$$\int_s^t |C^T(t,u)B|\,du \leq \alpha, \quad 0 \leq s \leq t < \infty, \tag{13}$$

$$\int_u^t |[C^T(w+\varepsilon,u) - C^T(w,u)]B|\,dw \leq \beta, \tag{14}$$
$$0 \leq u \leq t < \infty,$$

and

$$\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv \leq \widehat{\beta}. \tag{15}$$

Theorem 4.3. *If* (13)*,* (14) *and* (15) *hold, then for V_1 defined in* (9) *and $z(t,s)$ a solution of* (8)*, we have*

$$V_1(t,s;\varepsilon) - V_1(s,s;\varepsilon) \leq -(1-\alpha-\beta-\widehat{\beta}) \int_s^t |z(u,s)|^2 du,$$

and there is an $M > 0$ with $\int_s^t |z(u,s)|^2 du \leq M$ for $0 \leq s \leq t$.

Proof. By (12) and (13) upon integration of (12), we have from the above interchange of order of integration

$$
\begin{aligned}
V_1(t,s;\varepsilon)-V_1(s,s;\varepsilon)& \\
&\le -(1-\alpha-\widehat{\beta})\int_s^t |z(u,s)|^2du+\int_s^t\int_u^t |[C^T(w+\varepsilon,u) \\
&\qquad - C^T(w,u)]B|\,dw|z(u,s)|^2du \\
&\le -(1-\alpha-\beta-\widehat{\beta})\int_s^t |z(u,s)|^2du. \qquad \square
\end{aligned}
$$

Remark. Quantities α and $\widehat{\beta}$ are of an essentially different character than β, which is a measure of the singularity and in many significant problems can be made arbitrarily small by taking ε small enough. Thus, the essential part of the inequality is that $\alpha+\widehat{\beta}<1$. In fractional differential equations the kernel $(t-s)^{q-1}$ appears for $0<q<1$ and the equation is transformed into two integral equations, one of which has a kernel $R(t-s)$ for which it is easily shown that β tends to zero as $\varepsilon\to 0$. See, for example, [**7**, Lemma 8.1].

We come now to (10) and prepare V_2. When the characteristic roots of A all have negative real parts, then we find the symmetric matrix B with (7) holding. There are then positive constants r,k,K (not unique) with

$$
|z|\ge 2k[z^TBz]^{1/2},\quad |Bz|\le K[z^TBz]^{1/2},\quad r|z|\le [z^TBz]^{1/2}. \tag{16}
$$

Lemma 4.4. *If $z(t,s)$ is a solution of* (8)*, then for $z(t,s)\ne 0$ and for $W(t,s)=[z^T(t,s)Bz(t,s)]^{1/2}$, we have*

$$
\frac{d}{dt}W(t,s)\le -k|z(t,s)|+\frac{1}{r}\int_s^t |C^T(t,u)B||z(u,s)|\,du.
$$

Proof. By the proof of Lemma 4.1 and (7), we have

$$
\frac{d}{dt}W(t,s)=\frac{-z^T(t,s)z(t,s)-2\int_s^t z^T(u,s)C^T(t,u)Bz(t,s)\,du}{2[z^T(t,s)Bz(t,s)]^{1/2}}.
$$

By (16),

$$\frac{z^T z}{2[z^T Bz]^{1/2}} \geq \frac{z^T z}{(|z|/k)} = k|z|, \quad \text{and} \quad \frac{|z|}{[z^T Bz]^{1/2}} \leq \frac{1}{r},$$

so the conclusion is verified. □

Theorem 4.5. *Let B satisfy* (7)*, $z(t,s)$ satisfy* (8) *and let V_2 be defined by* (10)*. If* (15) *holds for some $\widehat{\beta} > 0$, and k and r satisfy* (16)*, then the derivative of V_2 along $z(t,s)$ with respect to t satisfies*

$$\begin{aligned} V_2'(t,s;\varepsilon) &\leq -k|z(t,s)| + \frac{1}{r}\int_s^t |C^T(t,u)B||z(u,s)|\,du \\ &\quad + \frac{1}{r}\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv|z(t,s)| \\ &\quad - \frac{1}{r}\int_s^t |C^T(t+\varepsilon,u)B||z(u,s)|\,du \\ &\leq -k|z(t,s)| + \frac{1}{r}\widehat{\beta}|z(t,s)| \\ &\quad + \frac{1}{r}\int_s^t |[C^T(t+\varepsilon,u) - C^T(t,u)]B||z(u,s)|\,du. \end{aligned}$$

Proof. From (10) and Lemma 4.4, we have

$$\begin{aligned} V_2'(t,s;\varepsilon) &= \frac{d}{dt}W(t,s) + \frac{1}{r}\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv|z(t,s)| \\ &\quad - \frac{1}{r}\int_s^t |C^T(t+\varepsilon,u)B||z(u,s)|\,du \\ &\leq -k|z(t,s)| + \frac{1}{r}\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv|z(t,s)| \\ &\quad + \frac{1}{r}\int_s^t |C^T(t,u)B||z(u,s)|\,du \\ &\quad - \frac{1}{r}\int_s^t |C^T(t+\varepsilon,u)B||z(u,s)|\,du \\ &\quad - \frac{1}{r}\int_s^t |C^T(t,u)B||z(u,s)|\,du \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{r}\int_s^t |C^T(t,u)B||z(u,s)|\,du\\
&\le -k|z(t,s)| + \frac{1}{r}\int_\varepsilon^\infty |C^T(v+t,t)B|\,dv|z(t,s)|\\
&+\frac{1}{r}\int_s^t |[C^T(t+\varepsilon,u)-C^T(t,u)]B||z(u,s)|\,du,
\end{aligned}
$$

as required. □

Theorem 4.6. *Let B satisfy* (7)*, $z(t,s)$ satisfy* (8) *and let V_2 be defined in* (10)*. Suppose also that* (14) *and* (15) *hold with*

$$-\mu := -k + \frac{\beta}{r} + \frac{\widehat{\beta}}{r} < 0.$$

Then

$$V_2(t,s;\varepsilon) - V_2(s,s;\varepsilon) \le -\mu\int_s^t |z(u,s)|\,du.$$

Proof. Integration of V_2' in Theorem 4.5 and interchange of the order of integration will yield

$$V_2(t,s;\varepsilon) - V_2(s,s;\varepsilon) \le -\mu\int_s^t |z(u,s)|\,du,$$

upon application of (14) and (15), as required. □

5. Scalar equations and arbitrary p**.** It is possible to take f, h and g to be vectors and C to be an $n\times n$ matrix. Care must be taken in multiplication, but most of the absolute values translate easily into norms. For $p=1$, there is no real distinction between the vector and scalar notation.

While the proof of our main theorem here is long, we view this as our main result. Here, we have great flexibility and are able to treat a much wider variety of forcing functions.

Theorem 5.1. *In* (1) *and* (3)*, let $q(t,x)$ be independent of t, and write $q(t,x)=g(x)$. Assume that*

(17) *there exists a $\delta>0$ with $|h(t,x)|\ge\delta|g(x)|$, $(t,x)\in[0,\infty)\times\Re$.*

Suppose that (2) *holds for some even integer* p *and there are positive numbers* α, β *with*

$$\beta + (p-1)\alpha < p\delta, \tag{18}$$

so that, for each $\varepsilon > 0$ *and for any* $t \geq 0$, *we have*

$$\int_{\varepsilon}^{\infty} |C(u+t,t)|\, du \leq \beta, \tag{19}$$

and for $t \geq 0$, *then*

$$\int_{0}^{t} |C(t,s)|\, ds \leq \alpha. \tag{20}$$

Moreover, assume that there exists a $\mu > 0$ *with*

$$\mu \in (0, p\delta - \beta - (p-1)\alpha), \tag{21}$$

such that, for all sufficiently small $\varepsilon > 0$, *we have*

$$\sup_{s\in[0,\infty)} \int_{s}^{\infty} |C(u+\varepsilon,s) - C(u,s)|\, du < \mu. \tag{22}$$

If $f \in L^p[0,\infty)$, *and if* x *solves* (1) *on* $[0,\infty)$, *then* $g(x(\cdot)) \in L^p[0,\infty)$.

Proof. For $\varepsilon > 0$ satisfying (22), and for $t \geq 0$, define

$$V(t,\varepsilon) = p\int_{0}^{x(t)} g^{p-1}(s)\, ds + \int_{0}^{t} \left[\int_{t-s+\varepsilon}^{\infty} |C(u+s,s)|\, du \right] g^{p}(x(s))\, ds,$$

so that $u \geq t - s + \varepsilon \geq \varepsilon$ since $0 \leq s \leq t$; that is, the integrand is continuous.

Notice that, by the assumption $xg(x) \geq 0$ and that p is an even integer, it follows that $\int_0^{x(t)} g^{p-1}(s)\, ds \geq 0$ for any $t \geq 0$, and so

$$0 \leq V(t,\varepsilon), \quad t \geq 0 \text{ for any } \varepsilon > 0.$$

Using

$$-|C(t+\varepsilon,s)| \leq -|C(t,s)| + |C(t+\varepsilon,s) - C(t,s)|,$$

we find

$$\begin{aligned}V'(t,\varepsilon) &= pg^{p-1}(x(t))x'(t) + \int_\varepsilon^\infty |C(u+t,t)|\,du g^p(x(t))\\ &\quad - \int_0^t |C(t+\varepsilon,s)|g^p(x(s))\,ds\\ &\le pg^{p-1}(x(t))x'(t) + g^p(x(t))\int_\varepsilon^\infty |C(u+t,t)|\,du\\ &\quad - \int_0^t |C(t,s)|g^p(x(s))\,ds\\ &\quad + \int_0^t |C(t+\varepsilon,s)-C(t,s)|g^p(x(s))\,ds,\end{aligned}$$

from which, in view of (19), we find

$$\begin{aligned}V'(t,\varepsilon) &\le pg^{p-1}(x(t))x'(t) + \beta g^p(x(t))\\ &\quad - \int_0^t |C(t,s)|g^p(x(s))\,ds\\ &\quad + \int_0^t |C(t+\varepsilon,s)-C(t,s)|g^p(x(s))\,ds.\end{aligned} \tag{23}$$

Since x is a solution of (1), it is true that

$$H := pg^{p-1}(x(t))\left[f(t) - x'(t) - h(t,x(t)) - \int_0^t C(t,s)g(x(s))\,ds\right] = 0,$$

and we have

$$\begin{aligned}H = pg^{p-1}(x(t))f(t) - pg^{p-1}(x(t))x'(t) - pg^{p-1}(x(t))h(t,x(t))&\\ - pg^{p-1}(x(t))\int_0^t C(t,s)g(x(s))\,ds.&\end{aligned}$$

First, we may note that, by (3), it follows that

$$pg^{p-1}(x(t))h(t,x(t)) \ge 0,$$

and so, by (17), we have

$$-pg^{p-1}(x(t))h(t,x(t)) = -p|g^{p-1}(x(t))h(t,x(t))| \le -p\delta g^p(x(t)).$$

Next, note that, for $p \geq 2$, we have

$$\frac{1}{p/(p-1)} + \frac{1}{p} = 1,$$

for use in Young's inequality

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

where $a \geq 0$, $b \geq 0$ and $q = p/(p-1)$. In view of (21) for

$$\gamma \in \left(0, \frac{p\delta - (p-1)\alpha - \beta - \mu}{p-1}\right),$$

and for M satisfying

$$M^{1/p}\gamma^{(p-1)/p} \geq 1,$$

we apply the inequality to

$$M^{1/p}|f(t)| \cdot \gamma^{(p-1)/p}|g(x(t))|^{p-1},$$

obtaining

$$\begin{aligned} |g(x(t))|^{p-1}|f(t)| &\leq M^{1/p}|f(t)| \cdot \gamma^{(p-1)/p}|g(x(t))|^{p-1} \\ &\leq M\frac{f^p(t)}{p} + \gamma\frac{g^p(x(t))}{p/(p-1)}. \end{aligned}$$

Then this, along with Young's inequality also applied to the integrand below, yields

$$\begin{aligned} H &\leq p|g(x(t))|^{p-1}|f(t)| - pg^{p-1}(x(t))x'(t) - pg^{p-1}(x(t))h(t,x(t)) \\ &\quad + p\int_0^t |C(t,s)||g(x(s))||g(x(t))|^{p-1}\,ds \\ &\leq pM\frac{f^p(t)}{p} + p\gamma\frac{g^p(x(t))}{p/(p-1)} - pg^{p-1}(x(t))x'(t) - p\delta g^p(x(t)) \\ &\quad + p\int_0^t |C(t,s)|\left(\frac{g^p(x(t))}{p/(p-1)} + \frac{g^p(x(s))}{p}\right)ds \\ &= Mf^p(t) + \gamma(p-1)g^p(x(t)) - pg^{p-1}(x(t))x'(t) - p\delta g^p(x(t)) \end{aligned}$$

$$+ (p-1)\int_0^t |C(t,s)|\,ds g^p(x(t)) + \int_0^t |C(t,s)| g^p(x(s))\,ds,$$

from which by the use of (20), we take

$$\begin{aligned} H \le\, & Mf^p(t) + \gamma(p-1)g^p(x(t)) - pg^{p-1}(x(t))x'(t) - p\delta g^p(x(t)) \\ & + (p-1)\alpha g^p(x(t)) + \int_0^t |C(t,s)| g^p(x(s))\,ds. \end{aligned} \tag{24}$$

In view of (23) and (24), we have

$$\begin{aligned} V'(t,\varepsilon) &\le pg^{p-1}(x(t))x'(t) + \beta g^p(x(t)) - \int_0^t |C(t,s)| g^p(x(s))\,ds \\ &\quad + \int_0^t |C(t+\varepsilon,s) - C(t,s)| g^p(x(s))\,ds \\ &\le Mf^p(t) + \gamma(p-1)g^p(x(t)) - p\delta g^p(x(t)) \\ &\quad + (p-1)\alpha g^p(x(t)) + \int_0^t |C(t,s)| g^p(x(s))\,ds \\ &\quad + \beta g^p(x(t)) - \int_0^t |C(t,s)| g^p(x(s))\,ds \\ &\quad + \int_0^t |C(t+\varepsilon,s) - C(t,s)| g^p(x(s))\,ds, \end{aligned}$$

that is,

$$\begin{aligned} V'(t,\varepsilon) \le\, & Mf^p(t) + [\beta + \gamma(p-1) + (p-1)\alpha - p\delta] g^p(x(t)) \\ & + \int_0^t |C(t+\varepsilon,s) - C(t,s)| g^p(x(s))\,ds. \end{aligned} \tag{25}$$

If we integrate the last term from 0 to t and interchange the order of integration, taking into consideration (21) and (22), we obtain

$$\begin{aligned} & \int_0^t \int_0^u |C(u+\varepsilon,s) - C(u,s)| g^p(x(s))\,ds\,du \\ & \qquad = \int_0^t \int_s^t |C(u+\varepsilon,s) - C(u,s)|\,du g^p(x(s))\,ds \\ & \qquad \le \mu \int_0^t g^p(x(s))\,ds. \end{aligned} \tag{26}$$

Set $\mu^* := \beta+(p-1)\alpha-p\delta+\gamma(p-1)+\mu$ and note that, by the definition of γ, we have $\mu^* < 0$. Using (25) and (26), we obtain

$$\begin{aligned}
V(t,\varepsilon) - V(0,\varepsilon) &\le M \int_0^t f^p(s)\,ds \\
&\quad + [\beta + \gamma(p-1) + (p-1)\alpha - p\delta] \int_0^t g^p(x(s))\,ds \\
&\quad + \int_0^t \int_0^u |C(u+\varepsilon, s) - C(u,s)| g^p(x(s))\,ds\,du \\
&\le M \int_0^t f^p(s)\,ds \\
&\quad + [\beta + \gamma(p-1) + (p-1)\alpha - p\delta + \mu] \int_0^t g^p(x(s))\,ds \\
&= M \int_0^t f^p(s)\,ds + \mu^* \int_0^t g^p(x(s))\,ds,
\end{aligned}$$

and so,

$$0 \le V(t,\varepsilon) \le V(0,\varepsilon) + \mu^* \int_0^t g^p(x(s))\,ds + M \int_0^t f^p(s)\,ds.$$

Since $V(0,\varepsilon) = p\int_0^{x(0)} g^{p-1}(s)\,ds < \infty$, it follows that

$$0 \le \int_0^t g^p(x(s))\,ds \le \frac{1}{-\mu^*}\left[p \int_0^{x(0)} g^{p-1}(s)\,ds + M \int_0^t f^p(s)\,ds\right],$$

as required. $\square$

Notes. Assume that $\delta > \alpha$. Clearly, (18) holds true for any positive even integer p with $p > (\beta-\alpha)/(\delta-\alpha)$. In addition to $\delta > \alpha$, if $\alpha + \beta < 2\delta$, then (18) holds true for all positive even integers p. It is not difficult to see that, for any $\beta > 0$, there always exists a positive even integer p_0 such that (18) holds true for all integers $p \ge p_0$.

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Northwest Research Institute, 732 Caroline St., Port Angeles, WA
Email address: taburton@olypen.com

Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
Email address: ipurnara@uoi.gr