JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 24, Number 4, Winter 2012

THE FACTORIZATION METHOD FOR A CONDUCTIVE BOUNDARY CONDITION

ANDREAS KIRSCH AND ANDREAS KLEEFELD

Communicated by Ivan Graham

ABSTRACT. In this article, the inverse scattering problem for obstacles with conductive boundary conditions is considered. The problem at hand is solved with the factorization method; that is, the operator and its factorization is derived. To obtain generic far field data for several obstacles, the boundary element method is employed to solve the integral equation numerically. The far field data are highly accurate due to superconvergence. Finally, we show that we are able to reconstruct several obstacles and thus confirm the validity of the new operator factorization.

1. Introduction. In this section, we first describe the problem at hand; that is, scattering from obstacles on which we impose conductive boundary conditions. We describe the boundary integral equation to obtain the far field pattern for a given obstacle and a given incident wave. In the next section, the boundary element collocation method is reviewed to evaluate the far field numerically. Now, the purpose is to reconstruct the obstacles from the knowledge of these generic far field data. This goal is achieved with the factorization method. Therefore, we derive the factorization of our operator in Section 3. Several reconstructions are presented in Section 4 to show that the numerical results are in agreement with the theory. Some interesting facts are observed.

To start with, let $D \subset \mathbf{R}^3$ be a finite union of bounded domains such that the exterior is connected. Furthermore, let $\kappa > 0$ be the wave number, $\lambda \in C(\partial D)$ (or only $\lambda \in L^\infty(\partial D)$) such that $\lambda \geq 0$ on ∂D and $\widehat{\theta} \in S^2$ (the unit sphere in $\mathbf{R}^3$). The time harmonic scalar scattering problem for a conductive boundary value problem has the following form. Given the incident field

$$u^{inc}(x) = \mathrm{e}^{\mathrm{i}\kappa\,\hat{\theta}\cdot x}, \quad x \in \mathbf{R}^3,$$

Received by the editors on January 28, 2011, and in revised form on September 21, 2011.

DOI:10.1216/JIE-2012-24-4-575

find $u \in C^2(\mathbf{R}^3 \setminus \partial D)$ such that the traces of u and its normal derivatives exist from both sides in a suitable sense and the total field $u = u^{inc} + u^s$ of the incident field u^{inc} and the scattered field u^s satisfies the Helmholtz equation

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbf{R}^3 \setminus \partial D \tag{1.1}$$

and the conductive boundary conditions

$$u_+ - u_- = 0 \quad \text{on } \partial D$$

and

$$\frac{\partial u_+}{\partial \nu} - \frac{\partial u_-}{\partial \nu} + \mathrm{i}\,\lambda\, u = 0 \quad \text{on } \partial D, \tag{1.2}$$

and u^s satisfies the Sommerfeld radiation condition

$$\frac{\partial u^s}{\partial r}(x) \; - \mathrm{i}\kappa\, u^s(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty, \tag{1.3}$$

uniformly with respect to $x/|x|$. If λ is continuous, the solution u is required to be continuous in all of $\mathbf{R}^3$ and the normal derivatives have to exist uniformly along the normal.[1] If only $\lambda \in L^\infty(\partial D)$ the solution is sought in the local Sobolev space $H^1_{loc}(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} \left[\nabla u \cdot \nabla \psi - \kappa^2 u\, \psi\right] \mathrm{d}x = \mathrm{i} \int_{\partial D} \lambda\, u\, \psi \,\mathrm{d}s \tag{1.4}$$

for all $\psi \in H^1(\mathbf{R}^3)$ with compact support. We note that, if λ is continuous and the solution u is continuous in all of $\mathbf{R}^3$ and the normal derivatives exist uniformly along the normal, then variational equation (1.4) holds. This is seen in the usual way by multiplying (1.1) with ψ, integrating over $\mathbf{R}^3$ and using Green's first theorem and the conductive transmission conditions (1.2).

Formulating the scattering problem as a transmission problem for the scattered field u^s yields

$$\Delta u^s + \kappa^2 u^s = 0 \quad \text{in } \mathbf{R}^3 \setminus \partial D, \tag{1.5}$$

and the conductive boundary conditions

$$u^s_+ - u^s_- = 0 \quad \text{on } \partial D$$

and

$$\frac{\partial u^s_+}{\partial \nu} - \frac{\partial u^s_-}{\partial \nu} + \mathrm{i}\,\lambda\, u^s = -\mathrm{i}\,\lambda\, f \quad \text{on } \partial D, \tag{1.6}$$

for $f = u^{inc}$ on ∂D and in weak form

$$\int_{\mathbf{R}^3} \big[\nabla u^s \cdot \nabla \psi - \kappa^2 u^s\, \psi\big]\,\mathrm{d}x = \mathrm{i}\int_{\partial D} \lambda\, u^s\, \psi\,\mathrm{d}s + \mathrm{i}\int_{\partial D} \lambda\, f\, \psi\,\mathrm{d}s \tag{1.7}$$

for all $\psi \in H^1(\mathbf{R}^3)$ with compact support. We will consider (1.3), (1.5), (1.6), or in weak form (1.3), (1.7) as a problem for $f \in H^{-1/2}(\partial D)$. The latter space is defined as the dual of the Sobolev space $H^{1/2}(\partial D)$ of fractional order 1/2. In this case, the integral $\int_{\partial D} \lambda\, f\, \psi\,\mathrm{d}s$ has to be understood as the dual form $\langle f, \overline{\psi}\rangle$ in the dual system $\langle H^{-1/2}(\partial D), H^{1/2}(\partial D)\rangle$.

With respect to uniqueness and existence we have the following result.

Theorem 1.1. *For every* (*complex valued*) $f \in H^{-1/2}(\partial D)$ *and real valued* $\lambda \in L^\infty(\partial D)$, *the conductive transmission problem* (1.3), (1.5), (1.6) *or, equivalently,* (1.3), (1.7) *has a unique solution in* $H^1_{loc}(\mathbf{R}^3)$ *provided that* $\lambda \ge 0$. *The solution* u^s *has the form*

$$u^s(x) = \frac{\exp(\mathrm{i}\kappa|x|)}{4\pi|x|}\, u^\infty(\widehat{x}) + \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to \infty \tag{1.8}$$

uniformly with respect to $\widehat{x} = x/|x| \in S^2$.

Here, u^∞ denotes the far field pattern of u^s, which depends upon the direction $\widehat{x} \in S^2$. For scattering problems, i.e., if f is a plane incident wave with direction $\widehat{\theta} \in S^2$ of incidence, the far field pattern u^∞ depends upon both, the direction $\widehat{x} \in S^2$ of observation and the direction $\widehat{\theta} \in S^2$ of the incident wave. We indicate this dependence by writing $u^\infty(\widehat{x};\widehat{\theta})$. The wave number $\kappa > 0$ is kept fixed, and the dependence of u^∞ on κ will not be indicated. For a proof for Hölder continuous λ, we refer the reader to [**9**] or [**6**, Theorem 2.1].

Proof. To prove uniqueness, we assume that u is the difference of two solutions. Then u satisfies (1.7) for $f = 0$ and the radiation condition (1.3). From interior regularity results (see [**7**]) we conclude that u is analytic outside every ball of radius $R > 0$ which contains $\overline{D}$ in its interior. Let $\phi \in C^\infty(\mathbf{R}^3)$ such that $\phi(x) = 1$ for $|x| \le R$ and $\phi(x) = 0$ for $|x| \ge R+1$. Setting $\psi = \phi\,\overline{u}$ in (1.7), we obtain

$$\int_{|x|<R} \left[|\nabla u|^2 - \kappa^2\,|u|^2\right] \mathrm{d}x + \int_{R<|x|<R+1} \left[\nabla u \cdot \nabla \psi - \kappa^2 u\,\psi\right] \mathrm{d}x = \mathrm{i} \int_{\partial D} \lambda\,|u|^2 \,\mathrm{d}s.$$

Application of Green's theorem to the second integral yields (note that ψ vanishes for $|x| = R+1$)

$$\int_{|x|<R} \left[|\nabla u|^2 - \kappa^2 |u|^2\right] \mathrm{d}x - \int_{|x|=R} \overline{u}\,\frac{\partial u}{\partial \nu}\,\mathrm{d}s = \mathrm{i} \int_{\partial D} \lambda\,|u|^2\,\mathrm{d}s,$$

i.e.,

$$\mathrm{i} \int_{\partial D} \lambda\,|u|^2\,\mathrm{d}s = \int_{|x|<R} \left[|\nabla u|^2 - \kappa^2\,|u|^2\right] \mathrm{d}x - \mathrm{i}\kappa \int_{|x|=R} |u|^2\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{R^2}\right) \quad \text{as } R \to \infty$$

where we have used the radiation condition. Taking the imaginary part yields

$$\int_{\partial D} \lambda\,|u|^2\,\mathrm{d}s = -\kappa \int_{|x|=R} |u|^2\,\mathrm{d}s + \mathcal{O}\left(\frac{1}{R^2}\right) \quad \text{as } R \to \infty,$$

and thus

$$\int_{\partial D} \lambda\,|u|^2\,\mathrm{d}s = 0,$$

i.e., $\lambda\,u = 0$ on ∂D. Equation (1.7) reduces to

$$\int_{\mathbf{R}^3} \left[\nabla u \cdot \nabla \psi - \kappa^2 u\,\psi\right] \mathrm{d}x = 0$$

for all $\psi \in H^1(\mathbf{R}^3)$ with compact support. Therefore, u is a radiating solution in all of $\mathbf{R}^3$ and, therefore, has to vanish. This proves uniqueness.

In order to prove existence, we choose the integral equation method. We make an Ansatz of the solution u^s as a single-layer potential

$$u^s(x) = \int_{\partial D} \Phi_\kappa(x,y)\,\phi(y)\,\mathrm{d}s(y), \quad x \in \mathbf{R}^3 \setminus \partial D, \tag{1.9}$$

with density $\phi \in H^{-1/2}(\partial D)$ where Φ_κ denotes the fundamental solution of the Helmholtz equation given by

$$\Phi_\kappa(x,y) = \frac{\exp(\mathrm{i}\kappa|x-y|)}{4\pi|x-y|}, \quad x \neq y \in \mathbf{R}^3.$$

As shown in [**16**] the function u^s belongs to $H^1_{loc}(\mathbf{R}^3)$ and satisfies the Helmholtz equation outside of ∂D and the radiation condition (1.3). The jump in the normal derivative $\partial u^s/\partial\nu$ is given by ϕ, i.e., in weak form

$$\int_{\mathbf{R}^3} \left[\nabla u^s \cdot \nabla\psi - \kappa^2 u^s\,\psi\right] \mathrm{d}x = \int_{\partial D} \phi\,\psi\,\mathrm{d}s,$$

for all $\psi \in H^1(\mathbf{R}^3)$ with compact support. Comparing this with (1.7) yields the boundary integral equation

$$\phi - \mathrm{i}\lambda\, S_\kappa\phi = \mathrm{i}\lambda\, f \quad \text{on } \partial D, \tag{1.10}$$

where the boundary integral operator

$$(S_\kappa\phi)(x) = \int_{\partial D} \Phi_\kappa(x,y)\,\phi(y)\,\mathrm{d}s(y), \quad x \in \partial D,$$

maps $H^{-1/2}(\partial D)$ into $H^{1/2}(\partial D)$. In particular, it is compact considered as an operator from $H^{-1/2}(\partial D)$ into itself. Equation (1.10), considered in $H^{-1/2}(\partial D)$, is an equation of the second kind. In order to prove existence, it is sufficient to prove uniqueness of this equation. Therefore, let ϕ be a solution of (1.10) for $f = 0$ and define u^s by (1.9). Then u^s is the weak solution of (1.3), (1.7) for $f = 0$. By the uniqueness (first part of the proof) we conclude that u^s vanishes in all of $\mathbf{R}^3$. The variational equation (1.7) yields $\lambda u^s = 0$ on ∂D, i.e., $\lambda\, S_\kappa\phi = 0$. Finally, the integral equation (1.10) yields $\phi = 0$ and ends the proof. □

The far field pattern of the scattered wave $u^s(x)$ is obtained by letting $|x|$ tend to infinity in (1.9) and thus yields

$$u^\infty(\widehat{x}) = \int_{\partial D} \mathrm{e}^{-\mathrm{i}\kappa\,\hat{x}\cdot y}\phi(y)\,\mathrm{d}s(y), \quad \widehat{x} \in S^2. \tag{1.11}$$

2. The boundary element collocation method. In this section, we describe the boundary element collocation method to solve the integral equation of the second kind given by (1.10) numerically. Note that this projection method is also discussed in [**2**]. At the end of this section, we explain the approximation of the far field pattern given by (1.11). Throughout this section, we assume that λ is positive and continuous.

First, assume that ∂D is a connected smooth surface of class C^2; that is, ∂D can be written as

$$\partial D = \partial D_1 \cup \cdots \cup \partial D_J, \tag{2.12}$$

where every ∂D_j is divided into a triangular mesh. We denote the collection of those by

$$\mathcal{T}_n = \{\Delta_k \mid 1 \le k \le n\}. \tag{2.13}$$

Let the unit simplex in the st-plane be given by

$$\sigma = \{(s,t) \mid 0 \le s, t,\ s+t \le 1\}.$$

For a given constant α with $0 < \alpha < 1/3$, let

$$(s_i, t_j) = \left(\frac{i + (2-3i)\alpha}{2}, \frac{j + (2-3j)\alpha}{2}\right), \quad 0 \le i, j,\ i+j \le 2 \tag{2.14}$$

be the uniform grid inside of σ with six nodes. We denote the ordering of this grid by the nodes $\{q_1, \dots, q_6\}$. We assume that, for each Δ_k, there exists a map

$$m_k : \sigma \xrightarrow[\text{onto}]{1-1} \Delta_k, \tag{2.15}$$

which is used for interpolation and integration on Δ_k. We refer to [**13**, Appendix A] for the map m_k for a variety of surfaces. Next, we define the node points of Δ_k by

$$v_{k,j} = m_k(q_j), \qquad j = 1, \dots, 6.$$

To obtain a triangulation (2.13) and the mapping (2.15), we use a parametric representation of each region ∂D_j of (2.12) and a triangulation into plane triangles $\widehat{\Delta}_k$ of the planar (polygonal) region of the parametrization. This approach covers fairly general surfaces and yields conforming triangulations in the sense of [**1**, page 188]. The refinement of $\Delta_k \in \mathcal{T}_n$ is achieved by connecting the midpoints of the three sides of $\widehat{\Delta}_k$ yielding four new triangles. This also leads to symmetry in the triangulation and cancellation of errors occur (see [**1**, page 173]).

We consider the following space of piecewise smooth functions:

$$V_n = \big\{\phi \in L^\infty(\partial D) : \phi \circ m_k \in \mathcal{P}_2(\sigma),\ k = 1, \dots, n\big\}, \tag{2.16}$$

where $\mathcal{P}_2(\sigma)$ denotes the space of polynomials of degree at most 2 on the reference triangle σ. We note that, globally, the functions of V_n are not continuous. For interpolation of degree two on σ, the Lagrange basis functions of degree two on σ are obtained by the usual conditions

$$\mathbf{L}_i(q_i) = 1 \quad \text{and} \quad \mathbf{L}_i(q_j) = 0 \text{ if } i \neq j.$$

The interpolation operator P_n of degree two over Δ_k is given by

$$(P_n\phi)\big(m_k(s,t)\big) = \sum_{j=1}^{6} \phi\big(m_k(q_j)\big)\, \mathbf{L}_j(s,t), \quad (s,t) \in \sigma,\ k = 1, \dots, n. \tag{2.17}$$

Recall that we have to solve the Fredholm integral equation of the second kind

$$\phi(x) - \mathrm{i}\lambda(x) \int_{\partial D} \Phi_\kappa(x,y)\, \phi(y)\, \mathrm{d}s(y) = \mathrm{i}\lambda(x) f(x), \quad x \in \partial D. \tag{2.18}$$

Using the map (2.15), equation (2.18) is equivalent to

$$\begin{aligned} \phi(x) - \mathrm{i}\lambda(x) \sum_{k=1}^{n} \int_\sigma \Phi_\kappa\big(x, m_k(s,t)\big)\, \phi\big(m_k(s,t)\big) \\ \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t) = \mathrm{i}\lambda(x) f(x) \end{aligned}$$

for $x \in \partial D$. Then substitute the approximated solution ϕ_n in (2.18) and force the residual

$$r_n(x) = \phi_n(x) - \mathrm{i}\lambda(x) \int_{\partial D} \Phi_\kappa(x,y)\, \phi_n(y)\, \mathrm{d}s(y) - \mathrm{i}\lambda(x) f(x), \quad x \in \partial D,$$

to be zero at the collocation nodes. Thus, we have to solve the linear system of size $6n \times 6n$ given by

$$\begin{aligned}(2.19)\quad \phi(v_{i,\ell}) - \mathrm{i}\lambda(v_{i,\ell}) \sum_{k=1}^{n} \sum_{j=1}^{6} \phi(v_{k,j}) \int_\sigma \Phi_\kappa\big(v_{i,\ell}, m_k(s,t)\big)\, \mathbf{L}_j(s,t) \\ \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t) = \mathrm{i}\lambda(v_{i,\ell})\, f(v_{i,\ell}),\end{aligned}$$

for $i = 1, \dots, n$ and $\ell = 1, \dots, 6$. This can be written in matrix form as

$$(2.20)\qquad (\mathbf{I} + \mathbf{A})\, \vec{x} = \vec{b},$$

where the (i,ℓ)th element of $\vec{x} \in \mathbf{C}^{6n}$ and $\vec{b} \in \mathbf{C}^{6n}$ are given by

$$x_{i,\ell} = \phi(v_{i,\ell}), \quad b_{i,\ell} = \mathrm{i}\lambda(v_{i,\ell})\, f(v_{i,\ell}),$$

respectively, for $i = 1, \dots, n$ and $\ell = 1, \dots, 6$. The $(i,\ell),(j,k)$th element of $\mathbf{A} \in \mathbf{C}^{6n \times 6n}$ is given by

$$\begin{aligned}(2.21)\quad A_{(i,\ell)(k,j)} = -\mathrm{i}\lambda(v_{i,\ell}) \int_\sigma \Phi_\kappa\big(v_{i,\ell}, m_k(s,t)\big)\, \mathbf{L}_j(s,t) \\ \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t),\end{aligned}$$

for $i, k = 1, \dots, n$ and $\ell, j = 1, \dots, 6$.

Of course, the integrals over σ given in (2.21) have to be approximated numerically. Since the kernel is weakly singular, we must use a different integration routine if $v_i = m_k(s,t)$ for some $(s,t) \in \sigma$. We use the Duffy transformation to remove the singularity. This results in integrating over a unit square. This double integral is approximated with a two-dimensional Gaussian quadrature with N_S quadrature points. The T_2:5–1 rule from Stroud [**17**, page 314] is used for nonsingular integrals (see [**1**, pages 457–459 and 460–462, respectively]). In this case each triangle is refined by connecting the midpoints leading to four new triangles. The number of refinements is denoted by N_{NS}.

We have the following standard result on collocation methods (for subspaces which contain only piecewise continuous functions, see [**3**]).

Theorem 2.2. *Let ∂D be a surface of class C^2 which is parametrized as in* (2.12). *Let $\mathcal{K}$ be the integral operator of* (2.18), *i.e., $\mathcal{K} = \mathrm{i}\lambda S_\kappa$ with $0 \le \lambda \in C(\partial D)$, which is compact on $L^\infty(\partial D)$. Let $\mathcal{T}_n$ be a conforming triangulation of ∂D, as described above, and assume the triangulations $\mathcal{T}_n$ are such that their diameters δ_n tend to zero as $n \to \infty$. Define the quadratic interpolatory projection operator P_n using* (2.17), *and consider the approximate solution of $\phi - \mathcal{K}\phi = g$ by means of the collocation approximation, where $g = \mathrm{i}\lambda f$. Then*:

a) *The inverse operators $(\mathcal{I} - P_n\mathcal{K})^{-1}$ exist and are uniformly bounded for all sufficiently large n, say $n \ge N$.*

b) *The approximation ϕ_n has the error*

$$\phi - \phi_n = (\mathcal{I} - P_n\mathcal{K})^{-1}(\mathcal{I} - P_n)\phi, \tag{2.22}$$

and thus $\phi_n \to \phi$ as $n \to \infty$.

As usual, the order of convergence depends upon the smoothness of the solution. For $\lambda \in C^{3,\alpha}(\partial D)$ for some $\alpha \in (0, 1]$ and sufficiently smooth boundary ∂D, the solution ϕ belongs also to $C^{3,\alpha}(\partial D)$, and the order of convergence is 3.

The far field pattern for an arbitrary point $\widehat{x} \in S^2$ given by (1.8) can be written as

$$\begin{aligned} u^\infty(\widehat{x}) &= \int_{\partial D} \mathrm{e}^{-\mathrm{i}\kappa\hat{x}\cdot y}\phi(y)\,\mathrm{d}s(y) \\ &= \sum_{k=1}^{n} \int_\sigma \mathrm{e}^{-\mathrm{i}\kappa\hat{x}\cdot m_k(s,t)}\phi\big(m_k(s,t)\big) \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t). \end{aligned}$$

To approximate the far field pattern, replace ϕ with

$$\phi_n\big(m_k(s,t)\big) = \sum_{j=1}^{6} \phi\big(m_k(q_j)\big)\,\mathbf{L}_j(s,t),$$

to obtain

$$\begin{aligned} u_n^\infty(\widehat{x}) &= \sum_{k=1}^{n} \int_\sigma \mathrm{e}^{-\mathrm{i}\kappa\hat{x}\cdot m_k(s,t)} \sum_{j=1}^{6} \phi\big(m_k(q_j)\big)\, \mathbf{L}_j(s,t) \\ &\qquad\qquad \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{6} \phi\big(m_k(q_j)\big) \int_\sigma \mathrm{e}^{-\mathrm{i}\kappa\hat{x}\cdot m_k(s,t)}\, \mathbf{L}_j(s,t) \\ &\qquad\qquad \times \left| \left(\frac{\partial m_k}{\partial s} \times \frac{\partial m_k}{\partial t} \right)(s,t) \right| \mathrm{d}(s,t), \end{aligned}$$

where the integrals are approximated numerically with the T_2:5–1 rule from Stroud and the $\phi(m_k(q_j))$ are the solution of linear system (2.19).

3. The far field operator and its factorization.

This section is devoted to the inverse problem to determine the shape of ∂D by the knowledge of the far field pattern $u^\infty(\widehat{x};\widehat{\theta})$ for all $\widehat{x},\widehat{\theta} \in S^2$ or, equivalently, from the knowledge of the far field operator F from $L^2(S^2)$ into itself, defined by

$$(Fp)(\widehat{x}) = \int_{S^2} p(\widehat{\theta})\, u^\infty(\widehat{x};\widehat{\theta})\, \mathrm{d}s(\widehat{\theta}), \quad \widehat{x} \in S^2. \tag{3.23}$$

This operator is compact as an integral operator on a compact domain of integration with smooth kernel. With respect to injectivity, we show:

Theorem 3.3. *Let $\Gamma \subset \partial D$ be relatively open such that $\lambda > 0$ on Γ and $\lambda = 0$ on $\partial D \setminus \Gamma$. The null space of F consists exactly of those $p \in L^2(S^2)$ such that the corresponding* Herglotz function

$$v_p(x) = \int_{S^2} p(\widehat{\theta})\, \mathrm{e}^{\mathrm{i}\kappa\, x\cdot\hat{\theta}} \mathrm{d}s(\widehat{\theta}), \quad x \in \mathbf{R}^3, \tag{3.24}$$

vanishes on Γ. In particular, F is one-to-one if $\lambda > 0$ on all of ∂D and κ is not a Dirichlet eigenvalue of $-\Delta$ in D.

Proof. Let $Fp = 0$, and define v_p by (3.24). By the superposition principle, Fp is the far field pattern of the scattered field w^s which corresponds to the incident field v_p. From $Fp = 0$, we conclude by Rellich's lemma (see [**4**]) and analytic continuation that w^s vanishes in the exterior of D.[2] The total field $w^s + v_p$ satisfies the conductive transmission conditions which reduce to

$$w^s_- = 0 \text{ on } \partial D \quad \text{and} \quad -\frac{\partial w^s_-}{\partial \nu} + \mathrm{i}\,\lambda\, v_p = 0 \text{ on } \partial D.$$

By Green's second theorem, we conclude that

$$\mathrm{i}\int_{\partial D} \lambda\,|v_p|^2\,\mathrm{d}s = \int_{\partial D} \frac{\partial w^s_-}{\partial \nu}\,\overline{v_p}\,\mathrm{d}s = \int_{\partial D} \frac{\partial v_p}{\partial \nu}\,\overline{w^s_-}\,\mathrm{d}s = 0.$$

Therefore, λv_p vanishes on ∂D; that is, v_p vanishes on Γ. $\square$

Now we turn to a factorization of operator F. We make the same assumption as in Theorem 3.3, i.e., we assume that $\lambda \in L^\infty(\partial D)$ with $\lambda > 0$ on Γ and $\lambda = 0$ on $\partial D \setminus \Gamma$ for some relatively open set $\Gamma \subset \partial D$.

We note that the scattered field u^s satisfies the conductive transmission conditions

$$u^s_- = u^s_+ \text{ on } \partial D$$

and

$$\frac{\partial u^s_+}{\partial \nu} - \frac{\partial u^s_-}{\partial \nu} + \mathrm{i}\,\lambda\, u^s = -\mathrm{i}\,\lambda\, u^{inc} \quad \text{on } \partial D.$$

Therefore, we define operators $H : L^2(S^2) \to L^2(\Gamma)$ and $G : L^2(\Gamma) \to L^2(S^2)$ by

$$\begin{aligned}(Hp)(x) &= \sqrt{\lambda(x)} \int_{S^2} p(\widehat{\theta})\,\mathrm{e}^{\mathrm{i}\kappa\, x\cdot\hat{\theta}}\,\mathrm{d}s(\widehat{\theta}), \quad x \in \partial D,\\ Gf &= v^\infty\end{aligned}$$

where v^∞ is the far field pattern of the radiating solution $v \in H^1_{loc}(\mathbf{R}^3)$ of the Helmholtz equation in $\mathbf{R}^3 \setminus \partial D$ satisfying

$$v_- = v_+ \quad \text{on } \partial D \tag{3.25}$$

and

$$\frac{\partial v_+}{\partial \nu} - \frac{\partial v_-}{\partial \nu} + \mathrm{i}\,\lambda\, v = -\mathrm{i}\sqrt{\lambda}\, f \quad \text{on } \partial D.$$

From the definitions of F, H and G, we observe that $F = GH$.

Next, we compute the adjoint $H^* : L^2(\Gamma) \to L^2(S^2)$ as

$$(H^*\varphi)(\widehat{x}) = \int_\Gamma \varphi(y)\, \sqrt{\lambda(y)}\, \mathrm{e}^{-\mathrm{i}\kappa\, \hat{x}\cdot y}\, \mathrm{d}s(y), \quad \widehat{x} \in S^2, \tag{3.26}$$

which is the far field pattern of the single layer potential

$$w(x) = \int_\Gamma \varphi(y)\, \sqrt{\lambda(y)}\, \Phi(x,y)\, \mathrm{d}s(y), \quad x \in \mathbf{R}^3, \tag{3.27}$$

with density $\varphi\sqrt{\lambda}$. From the jump conditions of the normal derivative of the single layer potential (see [**16**]) we observe that $w \in H^1_{loc}(\mathbf{R}^3)$ is a radiating solution of the Helmholtz equation in $\mathbf{R}^3 \setminus \partial D$ and satisfies

$$w_- = w_+ \quad \text{on } \partial D$$

and

$$\frac{\partial w_-}{\partial \nu} - \frac{\partial w_+}{\partial \nu} = \varphi\, \sqrt{\lambda} \quad \text{on } \partial D.$$

We rewrite this as

$$\frac{\partial w_+}{\partial \nu} - \frac{\partial w_-}{\partial \nu} + \mathrm{i}\,\lambda\, w = \mathrm{i}\sqrt{\lambda}\, (\mathrm{i}\varphi + \sqrt{\lambda}\, w) \quad \text{on } \partial D,$$

and observe from the definition of G that $-G(\mathrm{i}\varphi + \sqrt{\lambda}\, w) = H^*\varphi$. Therefore, defining the operator T from $L^2(\Gamma)$ into itself by $T\varphi = \mathrm{i}\,\varphi + \sqrt{\lambda}\, w|_\Gamma$ where $w \in H^1_{loc}(\mathbf{R}^3)$ is given by (3.27), we arrive at $H^* = -GT$. Substituting $H = -T^*G^*$ into $F = GH$, we derive the factorization of F in the form $F = -G\,T^*G^*$. We formulate the result as a theorem.

Theorem 3.4. *Let $F : L^2(S^2) \to L^2(S^2)$ be the far field operator, $G : L^2(\Gamma) \to L^2(S^2)$ and $T : L^2(\Gamma) \to L^2(\Gamma)$ defined by $Gf = v^\infty$ and*

$T\varphi = \mathrm{i}\,\varphi + \sqrt{\lambda}\,w$, respectively, where v^∞ is the far field pattern of the radiating solution $v \in H^1_{loc}(\mathbf{R}^3)$ of the Helmholtz equation in $\mathbf{R}^3 \setminus \partial D$ satisfying (3.25) *and $w \in H^1_{loc}(\mathbf{R}^3)$ is given by* (3.27)*. Then*

$$F = -G\,T^* G^*. \tag{3.28}$$

The following property of T is needed for the factorization method.

Lemma 3.5. *The imaginary part* $\operatorname{Im} T = (1/2\mathrm{i})(T - T^*)$ *of operator T is self-adjoint and coercive; that is,*

$$\operatorname{Im}\langle T\varphi, \varphi\rangle_{L^2(\Gamma)} \geq \|\varphi\|^2_{L^2(\Gamma)} \quad \text{for all } \varphi \in L^2(\Gamma).$$

Proof. Let $\varphi \in L^2(\Gamma)$. Then

$$\begin{aligned}
\langle T\varphi, \varphi\rangle_{L^2(\Gamma)} &= \int_\Gamma \big[\mathrm{i}\,\varphi + \sqrt{\lambda}\,w\big]\,\overline{\varphi}\,\mathrm{d}s \\
&= \mathrm{i}\,\|\varphi\|^2_{L^2(\Gamma)} + \int_{\partial D} w\,\sqrt{\lambda}\,\overline{\varphi}\,\mathrm{d}s \\
&= \mathrm{i}\|\varphi\|^2_{L^2(\Gamma)} + \int_{\partial D} w\left(\frac{\partial \overline{w}_-}{\partial\nu} - \frac{\partial \overline{w}_+}{\partial\nu}\right)\mathrm{d}s \\
&= \mathrm{i}\,\|\varphi\|^2_{L^2(\Gamma)} + \int_{|x|<R} \big[|\nabla w|^2 - \kappa^2|w|^2\big]\,\mathrm{d}x \\
&\qquad - \int_{|x|=R} w\,\frac{\partial\overline{w}}{\partial\nu}\,\mathrm{d}s \\
&= \mathrm{i}\,\|\varphi\|^2_{L^2(\Gamma)} + \int_{|x|<R} \big[|\nabla w|^2 - \kappa^2|w|^2\big]\,\mathrm{d}x \\
&\qquad + \mathrm{i}\,\kappa \int_{|x|=R} |w|^2\,\mathrm{d}s + \mathcal{O}(1/R^2)
\end{aligned}$$

as $R \to \infty$ where we have used Green's first theorem in D and in $\{x \notin D : |x| < R\}$ and the radiation condition in the form $\partial\overline{w}/\partial\nu = -\mathrm{i}\,\kappa\overline{w} + \mathcal{O}(1/R^2)$ as $R \to \infty$. This yields the assertion. $\square$

Therefore, the imaginary part $\operatorname{Im} F = (1/2\mathrm{i})(F - F^*)$ of the operator F has factorization in the form

$$\operatorname{Im} F = G\,(\operatorname{Im} T)\,G^* \tag{3.29}$$

where $\operatorname{Im} T$ is coercive.

Applying a simple functional analytic result (see [**11**, Theorem 1.21]) we conclude that the ranges of G and $\sqrt{\operatorname{Im} F}$ coincide. Note that the compact and self-adjoint operator $\operatorname{Im} F$ has a spectral decomposition in the form

$$(\operatorname{Im} F)g = \sum_j \lambda_j\,\langle g, \psi_j\rangle_{L^2(S^2)}\,\psi_j, \quad g \in L^2(S^2),$$

where λ_j and $\psi_j \in L^2(S^2)$ are the eigenvalues and eigenfunctions, respectively, of $\operatorname{Im} F$ such that $\{\psi_j : j\}$ form an orthonormal system in $L^2(S^2)$. Note that $\lambda_j \geq 0$ since $\operatorname{Im} T$ is coercive. Therefore, we can define the square root of $\operatorname{Im} F$ as

$$\sqrt{\operatorname{Im} F}\,g = \sum_j \sqrt{\lambda_j}\,\langle g, \psi_j\rangle_{L^2(S^2)}\,\psi_j, \quad g \in L^2(S^2). \tag{3.30}$$

The next part of the factorization method is to construct test functions $\phi_z \in L^2(S^2)$ dependent upon some geometrical test object z (such as a point or arc) such that ϕ_z belongs to the range of G, and thus to the range of $\sqrt{\operatorname{Im} F}$, if and only if test object z belongs to the unknown geometric shape. We distinguish between two cases:

3.1. Case A: $\Gamma = \partial D$. We make the assumption that $\lambda \in L^\infty(\partial D)$ such that $\lambda \geq c$ on all of ∂D for some constant $c > 0$. In particular, $\Gamma = \partial D$. In this case we can take points $z \in \mathbf{R}^3$ as our test objects and define $\phi_z \in L^2(S^2)$ by

$$\phi_z(\widehat{x}) = \mathrm{e}^{-\mathrm{i}\,\kappa\,z\cdot\hat{x}}, \quad \widehat{x} \in S^2. \tag{3.31}$$

Lemma 3.6. *Let κ^2 not be a Dirichlet eigenvalue of $-\Delta$ in D. For any $z \in \mathbf{R}^3$, define $\phi_z \in L^2(S^2)$ by* (3.31). *Then ϕ_z belongs to the range of G if and only if $z \in D$.*

Proof. First, let $z \in D$. Define $v \in H^1(D)$ by the solution of the interior boundary value problem for the Helmholtz equation in D with Dirichlet boundary data $\Phi(z,\cdot)$ on ∂D. Extend v by $\Phi(z,\cdot)$ in the exterior of D. Then $v \in H^1_{loc}(\mathbf{R}^3)$, and the far field pattern v^∞ of v is given by the far field pattern of $\Phi(z,\cdot)$, that is, $v^\infty = \phi_z$. Defining $f \in L^2(\partial D)$ by

$$f = \frac{\mathrm{i}}{\sqrt{\lambda}}\left(\frac{\partial v_+}{\partial \nu} - \frac{\partial v_-}{\partial \nu} + \mathrm{i}\,\lambda\, v\right) \quad \text{on } \partial D$$

yields $Gf = v^\infty = \phi_z$, that is, ϕ_z is in the range of G.

Second, let $z \notin D$ and assume, on the contrary, that $\phi_z = Gf$ for some $f \in L^2(\partial D)$. Let $v \in H^1_{loc}(\mathbf{R}^3)$ be the corresponding radiating solution of the Helmholtz equation in $\mathbf{R}^3 \setminus \partial D$ satisfying (3.25). Since the far field patterns of v and $\Phi(z,\cdot)$ coincide, we conclude by Rellich's lemma and unique continuation that v and $\Phi(z,\cdot)$ coincide on $\mathbf{R}^3 \setminus (D \cup \{z\})$. This is a contradiction since $v \in H^1(K \setminus \overline{D})$ but $\Phi(z,\cdot) \notin H^1(K \setminus \overline{D})$ where K is any ball centered at z. ◻

Combining the previous results leads to the first main theorem of this section.

Theorem 3.7. *Let κ^2 not be a Dirichlet eigenvalue of $-\Delta$ in D. For any $z \in \mathbf{R}^3$, define $\phi_z \in L^2(S^2)$ by* (3.31)*. Then ϕ_z belongs to the range of $\sqrt{\operatorname{Im} F}$ if and only if $z \in D$.*

Furthermore, let λ_j and $\psi_j \in L^2(S^2)$, $j \in \mathbf{N}$, be the eigenvalues with corresponding orthonormalized eigenfunctions, respectively, of $\operatorname{Im} F$*. Then $\lambda_j > 0$ for all j and $\{\psi_j : j \in \mathbf{N}\}$ is a complete orthonormal system in $L^2(S^2)$. A point z belongs to D if and only if the series*

$$\sum_{j\in\mathbf{N}} \frac{|\langle \phi_z, \psi_j\rangle_{L^2(S^2)}|^2}{\lambda_j} \tag{3.32}$$

converges. Therefore, if we agree on the setting $1/\infty = 0$ *and* $\operatorname{sign}(t) = 1$ *for* $t > 0$ *and* $\operatorname{sign}(t) = 0$ *for* $t = 0$*, then*

$$\chi_D(z) = \operatorname{sign}\left[\sum_{j\in\mathbf{N}} \frac{|\langle \phi_z, \psi_j\rangle_{L^2(S^2)}|^2}{\lambda_j}\right]^{-1} \tag{3.33}$$

is the characteristic function of D.

Proof. The equivalence $z \in D \Leftrightarrow \phi_z \in \mathcal{R}(\sqrt{\operatorname{Im} F})$ follows from the previous lemma and the observation that the ranges of G and $\sqrt{\operatorname{Im} F}$ coincide. From the factorization $F = -GT^*G^*$ and Theorem 3.3, we conclude that G^* is one-to-one. Using the factorization $\operatorname{Im} F = G(\operatorname{Im} T)G^*$ and Lemma 3.5, we conclude, first, that $\operatorname{Im} F$ is non-negative and, second, that $\operatorname{Im} F$ is one-to-one. Indeed, $(\operatorname{Im} F)g = 0$ implies

$$\begin{aligned} 0 &= \langle (\operatorname{Im} F)g, g\rangle_{L^2(S^2)} \\ &= \langle G(\operatorname{Im} T)G^*g, g\rangle_{L^2(S^2)} \\ &= \langle (\operatorname{Im} T)G^*g, G^*g\rangle_{L^2(\Gamma)} \\ &\geq \|G^*g\|^2_{L^2(\Gamma)}, \end{aligned}$$

and thus $g = 0$. Therefore, all eigenvalues λ_j are positive, and $\{\psi_j : j \in \mathbf{N}\}$ is a complete orthonormal system in $L^2(S^2)$. From the form (3.30) of $\sqrt{\operatorname{Im} F}$, we observe that the equation $\sqrt{\operatorname{Im} F}\, g = \phi_z$ is solvable if and only if $\sum_{j\in\mathbf{N}} |\langle \phi_z, \psi_j\rangle_{L^2(S^2)}|^2/\lambda_j$ converges. This latter property is sometimes known as Picard's criterion (see [**10**]). $\square$

3.2. Case B: $\Gamma \neq \partial D$**.** Now we assume that Γ is a relatively open strict subset of ∂D, $\lambda \geq c > 0$ on Γ and $\lambda = 0$ on $\partial D \setminus \Gamma$. We note that u and $\partial u/\partial \nu$ are both continuous $\partial D \setminus \Gamma$. Therefore, u satisfies the Helmholtz equation in all of $\mathbf{R}^3 \setminus \Gamma$. Thus, in this case, the scatterer has the form of an open surface Γ, and there is no distinction between the exterior and interior of D anymore. The best we can hope is to determine Γ. As is known for the factorization method for the scattering by an open arc in $\mathbf{R}^2$ (see [**12**]) the far field patterns ϕ_z of point sources are too strongly singular to distinguish between Γ from $\mathbf{R}^3 \setminus \Gamma$. Therefore, instead of points for the test objects, we take small test surfaces S from a set $\mathcal{S}$ of open C^2 surfaces with C^2-boundaries. For such $S \in \mathcal{S}$, we define $\phi_S \in L^2(S^2)$ by

$$\phi_S(\widehat{x}) = \int_S \mathrm{e}^{-\mathrm{i}\,\kappa\, z\cdot\hat{x}}\,\mathrm{d}s(z), \quad \widehat{x} \in S^2. \tag{3.34}$$

We note that ϕ_S is the far field pattern of the single layer potential on S with density 1. Analogously to Lemma 3.6, we can show the following characterization of Γ.

Lemma 3.8. *Let $S \in \mathcal{S}$ be such that the complement of $\Gamma \cup S$ is connected, and define $\phi_S \in L^2(S^2)$ by (3.34). Then ϕ_S belongs to the range of G if and only if $S \subset \Gamma$.*

Proof. First let $S \subset \Gamma$. As just mentioned, ϕ_S is the far field pattern $\widetilde{v}^\infty$ of

$$\widetilde{v}(x) = \int_S \Phi(z, x)\,\mathrm{d}s(z), \quad x \notin \Gamma. \tag{3.35}$$

Since $\widetilde{v} \in H^1_{loc}(\mathbf{R}^3)$ (see again [**16**]) we define $f \in L^2(\Gamma)$ by

$$f = \frac{\mathrm{i}}{\sqrt{\lambda}}\left(\frac{\partial \widetilde{v}_+}{\partial \nu} - \frac{\partial \widetilde{v}_-}{\partial \nu} + \mathrm{i}\,\lambda\,\widetilde{v}\right) \quad \text{on } \Gamma,$$

and observe that $Gf = \widetilde{v}^\infty = \phi_S$.

Now let $S \setminus \Gamma \neq \varnothing$. Then there exists a closed ball B such that $B \cap \overline{\Gamma} = \varnothing$ and $B \cap S \neq \varnothing$. We can choose B so small such that S separates B into two connected parts B_1 and B_2, that is, $B \setminus S = B_1 \cup B_2$ and $B_1 \cap B_2 = \varnothing$. We assume, on the contrary, that $\phi_S = Gf$ for some $f \in L^2(\Gamma)$. Let $v \in H^1_{loc}(\mathbf{R}^3)$ be the corresponding radiating solution of the Helmholtz equation in $\mathbf{R}^3 \setminus \partial D$ satisfying (3.25). Again let $\widetilde{v}$ be defined by (3.35). Since the far field patterns of v and $\widetilde{v}$ coincide, we conclude by Rellich's lemma and unique continuation that v and $\widetilde{v}$ coincide on the complement[3] of $\Gamma \cup S$ and, in particular, on $B \setminus S = B_1 \cup B_2$. Since v is analytic in all of $\mathbf{R}^3 \setminus \overline{\Gamma}$, it is analytic in the ball, B. Therefore, the jump of the normal derivative of v vanishes on $B \cap S$. This contradicts the fact that it coincides with $\widetilde{v}$ in $B \setminus S$ since the normal derivative of the single layer potential $\widetilde{v}$ has a jump ± 1 (the sign depends on the orientation of the normal vector). $\square$

Combining the previous results leads to the second main theorem of this section.

Theorem 3.9. *Let $S \in \mathcal{S}$ be such that the complement of $\Gamma \cup S$ is connected, and define $\phi_S \in L^2(S^2)$ by (3.34). Then ϕ_S belongs to the range of $\sqrt{\operatorname{Im} F}$ if and only if $S \subset \Gamma$.*

4. Numerical results. In this section, we will show that we are able to reconstruct different surfaces from the knowledge of the far field pattern with the factorization method.

First, we demonstrate that our boundary element collocation solver yields highly accurate far field data due to superconvergence. Note that this solver is a modification of the solver that has been implemented and used in [**13**] to solve the exterior Neumann and Robin problem for Helmholtz's equation. In addition, superconvergence has been proved at the collocation points (see also [**14, 15**]).

We choose the wave number κ as well as λ to be 1, the latter on all of ∂D. We denote the number of faces of the triangulation with n and the number of node points of the triangulation with n_v. We use 66 incident waves $\widehat{\theta} \in \mathcal{A} := \{\widehat{\theta}_j : j = 1, \ldots, 66\}$ and measure 66 far field data in the same directions $\widehat{x} \in \mathcal{A}$ for each incident wave. We refer to the appendix for our choice of the $\widehat{\theta}_j$. This yields our set of data $f_{jl} \in \mathbf{C}$ for $j, l = 1, \ldots, 66$. Figure 1 illustrates the 66 elements of the set $\mathcal{A}$. They correspond to the dots on the unit sphere. As shown in the Appendix, the far field of a conductive sphere centered at the origin with radius R and with constant impedance λ can be calculated via

$$u^\infty(\widehat{x};\widehat{\theta}) = 4\pi\mathrm{i}\,\lambda \sum_{n=0}^{\infty}(2n+1)\,\frac{j_n(\kappa R)^2}{1/R^2 + \lambda\kappa\, j_n(\kappa R)\, h_n^{(1)}(\kappa R)}\, P_n(\widehat{x}\cdot\widehat{\theta}), \tag{4.36}$$

where $\widehat{x}, \widehat{\theta} \in S^2$ again denote the directions of observation and incidence, respectively (see [**5**, page 53] for a series expansion of a sound-soft sphere). In (4.36) j_n denotes the spherical Bessel function of the first kind of order n and $h_n^{(1)}$ the spherical Hankel function of the first kind of order n. P_n is the Legendre polynomial of order n. Thus, we are able to compare the approximated far field data with the series expansion (4.36).

We denote the error between the calculated solution f_{ij} and the true solution $u^\infty(\widehat{x}_i;\widehat{\theta}_j)$ at the point $\widehat{x}_i \in S^2$ and direction $\widehat{\theta}_j$ by $\mathcal{E}_n(\widehat{x}_i, \widehat{\theta}_j)$, that is,

$$\mathcal{E}_n(\widehat{x}_i, \widehat{\theta}_j) = \left|u(\widehat{x}_i, \widehat{\theta}_j) - f_{ij}\right|.$$

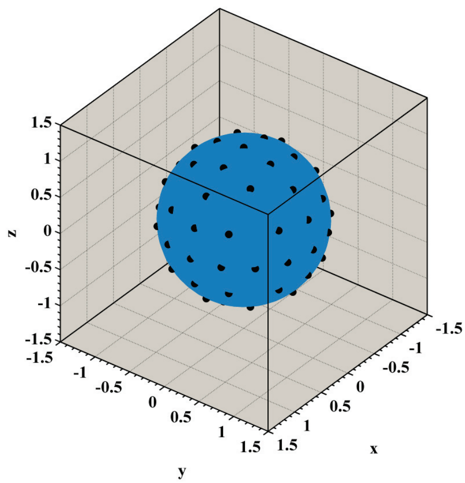


FIGURE 1. The 66 elements of $\mathcal{A}$.

TABLE 1. Far field errors for a conductive sphere with radius one.

	Quadratic interpolation				
n (n_v)	$\max_{i,j} \mathcal{E}_n(\widehat{x}_i, \widehat{\theta}_j)$	EOC	n (n_v)	$\max_{i,j} \mathcal{E}_n(\widehat{x}_i, \widehat{\theta}_j)$	EOC
4 (24)	3.0374D-01	3.48	8 (48)	3.1813D-02	5.15
16 (96)	2.7309D-02	5.83	32 (192)	8.9745D-04	5.44
64 (384)	4.8142D-04	4.80	128 (768)	2.0610D-05	4.18
256 (1536)	1.7326D-05		512 (3072)	1.1407D-06	

Define the estimated order of convergence (EOC) by

$$\mathrm{EOC} = \log_2 \Big(\max_{i,j} \big\{ \mathcal{E}_n(\widehat{x}_i, \widehat{\theta}_j) \big\} / \max_{i,j} \big\{ \mathcal{E}_{4n}(\widehat{x}_i, \widehat{\theta}_j) \big\} \Big).$$

Verification of superconvergence for quadratic interpolation is shown in Table 1 for two different triangulations. Note that, in general, we should have a rate of at least three and, for the superconvergence, a rate of almost four.

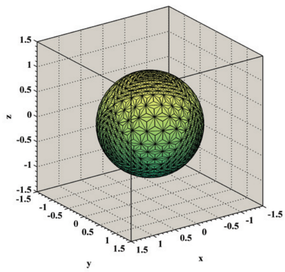
Next, we reconstruct some surfaces with the factorization method, where we use the far field data generated with the boundary element collocation solver. We consider a total of five different surfaces. In all cases we have chosen $\lambda = 1$ on ∂D, that is, we consider only Case A. The first surface is the unit sphere, the second surface a peanut. Its surface in spherical coordinates is given through $x = \varrho \sin(\phi)\cos(\theta)$, $y = \varrho \sin(\phi)\sin(\theta)$ and $z = \varrho\cos(\phi)$, where $\varrho^2 = 9\{\cos^2(\phi) + \sin^2(\phi)/4\}/4$. The third surface is an acorn and in spherical coordinates given by $\varrho^2 = 9\{17/4 + 2\cos(3\phi)\}/25$. The fourth surface is a cushion. Its surface can be described in spherical coordinates with $\varrho = 1-\cos(2\phi)/2$. The fifth surface is a cube with edge length two centered at the origin. A triangulation of each surface is given in Figure 2 a)–e). Note that we used 3072 collocation nodes for the first four surfaces and 4608 collocation nodes for the cube. To generate the far field data we used the parameters $N_S = 128$ and $N_{NS} = 4$.

Next, we describe the implementation of the factorization method for solving the inverse problem. Let the grid $\mathcal{G}$ be defined as the cube $\mathfrak{C} = [-1.5, 1.5]^3$ with 55 equidistant points in each direction. As mentioned above, we know approximate values f_{jl} of the far field pattern $u^\infty(\widehat{\theta}_j;\widehat{\theta}_l)$, $j,l = 1,\dots,66$, as our data. Next, we compute a singular value decomposition $\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$ of $\mathbf{A} = \operatorname{Im}\mathbf{F} = (\mathbf{F}-\mathbf{F}^*)/(2\mathrm{i})$ where the matrix $\mathbf{F} \in \mathbf{C}^{66\times 66}$ is given by the data $\mathbf{F} := (f_{jl})$. Note that we do not need to know the value of λ. However, to be consistent with the theory, we test the objects with points z. For each z from the grid $\mathcal{G}$ ($55\times 55\times 55$ values) we compute the expansion coefficients of $r_z = (\exp(-\mathrm{i}\,\kappa z\cdot\widehat{\theta}_j))_{j=1,\dots,66} \in \mathbf{C}^{66}$ with respect to the columns of V by

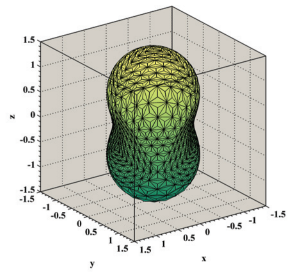
$$\rho_l^{(z)} = \sum_{j=1}^{66} V_{j,l}\,\mathrm{e}^{-\mathrm{i}\,\kappa z\cdot\widehat{\theta}_j}, \quad l = 1,\dots,66,$$

which is a matrix-vector multiplication $\rho^{(z)} = \mathbf{V}^{\mathrm{T}} r_z$ of $\mathbf{V}^{\mathrm{T}}$ and r_z. For each z, we compute

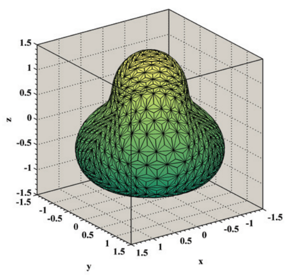
$$W(z) := \left[\sum_{l=1}^{66} \frac{|\rho_l^{(z)}|^2}{|\lambda_l|}\right]^{-1}$$



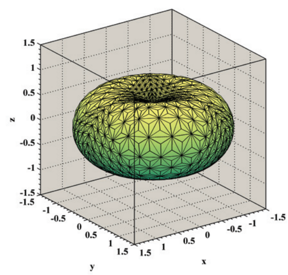
(a) Triangulation of a unit sphere with 512 faces.



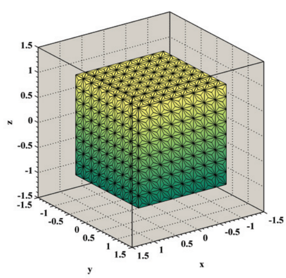
(b) Triangulation of a peanut with 512 faces.



(c) Triangulation of an acorn with 512 faces.



(d) Triangulation of a cushion with 512 faces.



(e) Triangulation of a cube with 768 faces.

FIGURE 2. Different surfaces.

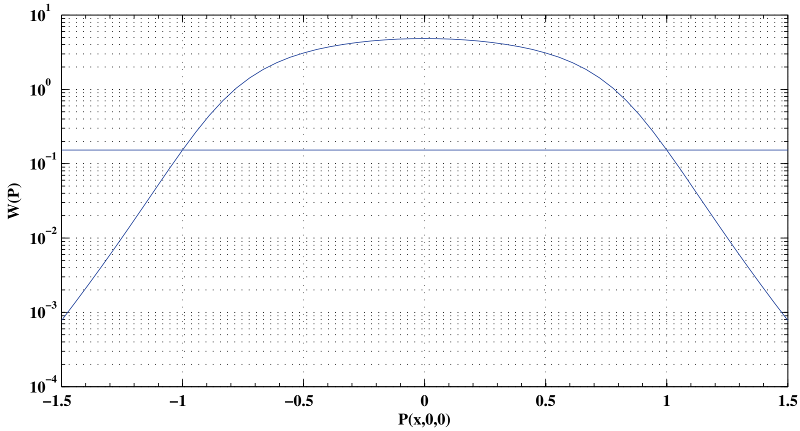


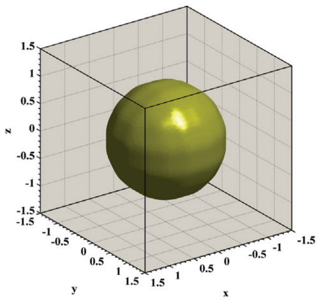
FIGURE 3. The slice $W(x_i, 0, 0)$ for $i = 1, \ldots, 55$ of W for the far field data of a unit sphere.

and plot one isosurface of $z \mapsto W(z)$ for a given threshold ε. The values of $W(z)$ should be much smaller for $z \notin D$ than for those lying within D. To get an idea of how to pick the threshold, we consider the following procedure. Given the cube $\mathfrak{C}$, we generate the corresponding W for the far field data of a unit sphere. The slice $W(x_i, 0, 0)$ for $i = 1, \ldots, 55$ of W is shown in Figure 3.

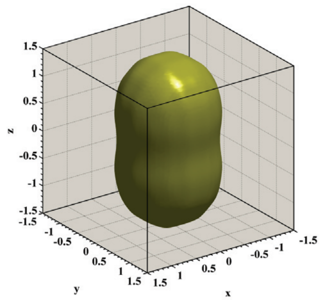
As we see, the threshold ε lies between 0.1 and 0.2. Thus, for the reconstruction, we will pick $\varepsilon = 0.18$ as the threshold value.

The reconstruction of the five surfaces is shown in Figure 4 a)–e). As we observe, the reconstructions of the surfaces are very accurate. Apparently, the global smoothness of the surface is not important as the example of the cube shows. We also changed the parameters of the problems. For example, we observed that an increase in the number of incident waves and directions of observation increased the quality of the reconstruction. In particular, 18 incident waves are not sufficient for a good reconstruction. We also observed that the threshold parameter ε does not seem to depend upon the wave number κ–in contrast to the impedance λ. We finally note that the $(F^*F)^{1/4}$-factorization method also works for our experiments despite the fact that it is not justified in theory.[4]

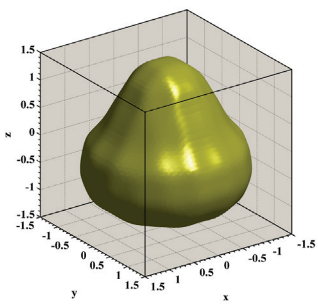
To present numerical results also for case B, we consider the unit sphere, where we choose $\lambda = 1$ on the upper and $\lambda = 0$ on the lower



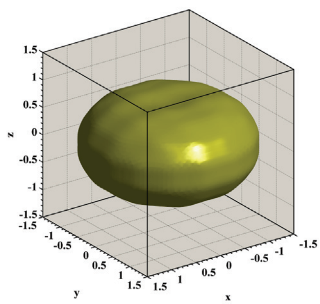
(a) Reconstructed surface of a unit sphere.



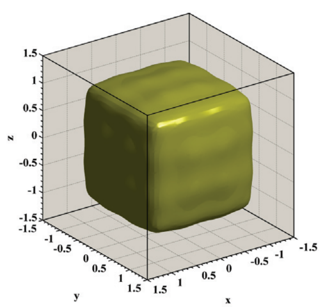
(b) Reconstructed surface of a peanut.



(c) Reconstructed surface of an acorn.

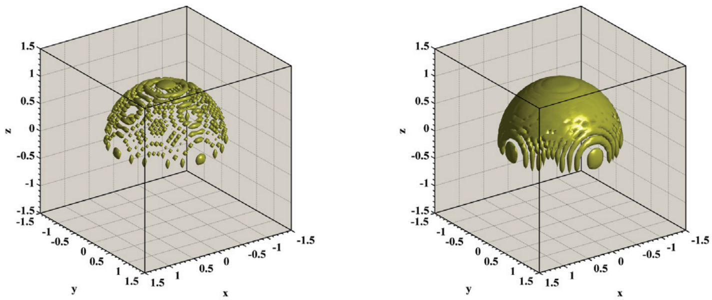


(d) Reconstructed surface of a cushion.



(e) Reconstructed surface of a cube.

FIGURE 4. Different reconstructed surfaces with the factorization method with $\varepsilon = 0.18$.



(a) Reconstructed unit half sphere with $\varepsilon = 0.001$.

(b) Reconstructed unit half sphere with $\varepsilon = 0.0001$.

FIGURE 5. Reconstructed unit half sphere with the factorization method with various choices of ε.

half. As mentioned above, the best we can hope for is to reconstruct the upper half of the sphere with the factorization method. By violating the theory of Theorem 3.9, we tested cube $\mathfrak{C}$ with points z rather than test surfaces S, that is, we took ϕ_z rather that ϕ_S. As we observe in Figure 5, we are able to reconstruct the upper half of the sphere, where we picked threshold values $\varepsilon = 0.001$ and $\varepsilon = 0.0001$. We note, however, that the reconstruction crucially depends on the choice of ε.

APPENDIX

A.1. The construction of $\mathcal{A}$. The set $\mathcal{A}$ illustrated in Figure 1 and used at the beginning of Section 4 is constructed as follows. We inscribe a regular octahedron (eight triangles) inside the unit sphere such that its vertices given by

$$\begin{array}{ll} V_1(0,0,1), & V_2(1,0,0), \\ V_3(0,1,0), & V_4(-1,0,0), \\ V_5(0,-1,0), & V_6(0,0,-1) \end{array}$$

are located on the unit sphere. One can imagine that we now have eight curved triangles on the unit sphere with a total of six vertices. Next, we calculate the 12 midpoints between those 6 vertices leading to 18 points on the unit sphere. Finally, we refine the eight curved triangles by connecting the midpoints with a line which leads to 32 curved triangles with 18 vertices and 48 midpoints. Those 66 points generate our set $\mathcal{A}$.

A.2. Series expansion of the far field pattern for a conductive sphere. We will derive a series expansion of the far field pattern for a conductive sphere with constant λ centered at the origin with radius R that corresponds to equation (4.36). We start with the Jacobi-Anger expansion of the incident field (see [**5**, subsection 2.4]) in polar coordinates $x = r\widehat{x}$, $r \geq 0$, $\widehat{x} \in S^2$:

$$u^{inc}(r\widehat{x}) = \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1)\, j_n(\kappa r)\, P_n(\widehat{x} \cdot \widehat{\theta}),$$

where $\widehat{x} \cdot \widehat{\theta}$ is the angle between direction $\widehat{x}$ and the direction $\widehat{\theta}$ of the incident field. We make an Ansatz of the total field u in the interior (that is, $r < R$) and the exterior (that is, $r > R$) of the ball by

$$u(r\widehat{x}) = \begin{cases} \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1)\, b_n\, j_n(\kappa r)\, P_n(\widehat{x} \cdot \widehat{\theta}), & r < R, \\ \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1) \big[j_n(\kappa r) + a_n\, h_n^{(1)}(\kappa r) \big]\, P_n(\widehat{x} \cdot \widehat{\theta}), & r > R. \end{cases}$$

Transmission conditions (1.2) on the sphere $r = R$ yield the following system of two equations for a_n and b_n:

$$\begin{aligned} h_n^{(1)}(\kappa R)\, a_n - j_n(\kappa R)\, b_n &= -j_n(\kappa R), \\ \kappa \left(h_n^{(1)}\right)'(\kappa R)\, a_n - \left[\kappa\, j_n'(\kappa R) - \mathrm{i}\lambda\, j_n(\kappa R)\right] b_n &= -\kappa\, j_n'(\kappa R). \end{aligned}$$

Solving this system for a_n by using the Wronskian $(h_n^{(1)})'(t)\, j_n(t) - j_n'(t)\, h_n^{(1)}(t) = \mathrm{i}/t^2$ yields

$$a_n = -\frac{\kappa\lambda\, j_n(\kappa R)^2}{1/R^2 + \kappa\lambda\, j_n(\kappa R)\, h_n^{(1)}(\kappa R)}.$$

Inserting this into the Ansatz yields the representation of the scattered field in the form

$$\begin{aligned} &u^s(r\widehat{x}; \widehat{\theta}) \\ &= -\kappa\lambda \sum_{n=0}^{\infty} \mathrm{i}^n (2n+1)\, \frac{j_n(\kappa R)^2}{1/R^2 + \kappa\lambda\, j_n(\kappa R)\, h_n^{(1)}(\kappa R)}\, h_n^{(1)}(\kappa r)\, P_n(\widehat{x} \cdot \widehat{\theta}), \\ &\qquad r > R. \end{aligned}$$

Finally, we note the asymptotic behavior of the spherical Bessel function in the form

$$h_n^{(1)}(t) = \frac{1}{t}\, \mathrm{e}^{\mathrm{i}\, t}(-\mathrm{i})^{n+1} + \mathcal{O}(1/t^2)$$

as t tends to infinity, which finally yields

$$u^\infty(\widehat{x};\widehat{\theta}) = 4\pi \mathrm{i}\,\lambda \sum_{n=0}^{\infty} (2n+1)\, \frac{j_n(\kappa R)^2}{1/R^2 + \lambda\kappa\, j_n(\kappa R)\, h_n^{(1)}(\kappa R)}\, P_n(\widehat{x}\cdot\widehat{\theta}).$$

ENDNOTES

1. For a precise statement, we refer to [**5**].
2. Here we need the assumption that the exterior is connected.
3. Here we need the assumption that this complement is connected.
4. In this form of the factorization method one replaces $\operatorname{Im} F$ by $(F^*F)^{1/2}$.

REFERENCES

1. K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge University Press, Cambridge, 1997.

2. K.E. Atkinson and G. Chandler, *The collocation method for solving the radiosity equation for unoccluded surfaces*, J. Integral Equations Appl. **10** (1998), 253–290.

3. K.E. Atkinson, I. Graham and I. Sloan, *Piecewise continuous collocation for integral equations*, SIAM J. Numer. Anal. **20** (1983), 172–186.

4. D. Colton and R. Kress, *Integral equation methods in scattering theory*, John Wiley & Sons, New York, 1983.

5. ———, *Inverse acoustic and electromagnetic scattering theory*, Second ed., Springer-Verlag, New York, 1998.

6. T. Gerlach and R. Kress, *Uniqueness in inverse obstacle scattering with conductive boundary condition*, Inverse Problems **12** (1996), 619–625.

7. D. Gilbarg and N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer, New York, 1983.

8. W. Hackbusch, *Integral equations–Theory and numerical treatment*, Birkhäuser, Berlin, 1995.

9. F. Hettlich, *On the uniqueness of the inverse conductive scattering problem for the Helmholtz equation*, Inverse Problems **10** (1994), 129–144.

10. A. Kirsch, *An introduction to the mathematical theory of inverse problems*, Springer, New York, 1996.

11. A. Kirsch and N. Grinberg, *The factorization method for inverse problems*, Oxford University Press, Oxford, 2008.

12. A. Kirsch and S. Ritter, *A linear sampling method for inverse scattering from an open arc*, Inverse Problems **16** (2000), 89–105.

13. A. Kleefeld, *Direct and inverse acoustic scattering for three-dimensional surfaces*, Ph.D. thesis, University of Wisconsin, Milwaukee, 2009.

14. A. Kleefeld and T.-C. Lin, *Boundary element collocation method for solving the exterior Neumann problem for Helmholtz's equation in three dimensions*, Electron. Trans. Numer. Anal. **39** (2012), 113–143.

15. ———, *Boundary element collocation method for solving the exterior Robin problem for Helmholtz's equation in three dimensions*, submitted.

16. W. McLean, *Strongly elliptic systems and boundary integral equations*, Cambridge University Press, Cambridge, 2000.

17. A. Stroud, *Approximate calculation of multiple integrals*, Prentice-Hall, Englewood Cliffs, New York, 1971.

Karlsruhe Institute of Technology (KIT), Department of Mathematics, 76131 Karlsruhe, Germany
Email address: andreas.kirsch@kit.edu

Brandenburgische Technische Universität, Department I (Mathematics, Natural Sciences, Computer Science), P.O. Box 101344, 03013 Cottbus, Germany
Email address: andreas.kleefeld@tu-cottbus.de