JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 24, Number 2, Summer 2012

# A QUADRATURE METHOD FOR SYSTEMS OF CAUCHY SINGULAR INTEGRAL EQUATIONS

M.C. DE BONIS AND C. LAURITA

Communicated by Giovanni Monegato

Dedicated to Professor Giuseppe Mastroianni for his 70th birthday

ABSTRACT. The aim of this paper is to propose a numerical method approximating the solutions of a system of CSIE. The stability and the convergence of the method are proved in weighted $L^2$ spaces. An application to the numerical resolution of CSIE on curves is also given. Finally, some numerical tests confirming the error estimates are shown.

## 1. Introduction.

Systems of singular integral equations with Cauchy type kernels may be found in the formulation of many boundary value problems. In many known physical problems of practical interest, the coefficients of the equations are constant. The general theory of such systems is given in [**11, 14**] (see also the references therein).

In this paper we are interested in the numerical solution of systems of the following type

$$(1.1)\quad a_j F_j(\tau) + \frac{b_j}{\pi}\int_{-1}^{1}\frac{F_j(t)}{t-\tau}\,dt + \sum_{k=1}^{n}\int_{-1}^{1} h_{jk}(\tau,t)F_k(t)dt = G_j(\tau), \quad |\tau|<1, \quad j=1,\dots,n,$$

where $h_{jk}$ and $G_j$, $j,k=1,\dots,n$, are known complex-valued functions defined on $[-1,1]^2$ and $[-1,1]$, respectively, and $F_j$, $j=1,\dots,n$, are the unknowns.

2010 AMS *Mathematics subject classification.* Primary 65R20, 45E05.
*Keywords and phrases.* Cauchy singular integral equation, quadrature method, Lagrange interpolation.
Received by the editors on March 17, 2010, and in revised form on July 1, 2010.
DOI:10.1216/JIE-2012-24-2-241 

Moreover, we assume that the constant coefficients $a_j$ and $b_j$ are real and such that

$$a_j^2 + b_j^2 = 1, \quad b_j \neq 0, \text{ for all } j \in \{1, \dots, n\}.$$

Such kinds of systems arise, for instance, in the resolution of Cauchy singular integral equations of the form

$$au(x) + \frac{1}{\pi} \int_\Gamma \frac{M(x,y)}{y-x} u(y)\, dy = g(x), \quad x \in \Gamma,$$

where $\Gamma = \cup_{j=1}^n \Gamma_j$ is the union of $n$ smooth simple closed arcs $\Gamma_j$ in the complex plane having no common points and being of finite length, $M$ and $g$ are given complex-valued functions on $\Gamma \times \Gamma$ and $\Gamma$, respectively, $a$ and $b = M(x,x)$, for all $x \in \Gamma$, are real numbers such that $a^2+b^2 = 1$, $b \neq 0$ (see Section 6).

In this paper we propose a numerical method in order to approximate the exact solution $(F_1, F_2, \dots, F_n)$ of the system (1.1) by means of polynomial interpolation. The polynomial approximation of the solution is computed by solving a well-conditioned linear system. We prove the convergence of the method in weighted $L^2$ spaces.

The paper is organized as follows. In Section 2 we give some preliminary definitions and notations, and in Section 3 we introduce the spaces in which we are going to study our systems. Section 4 is devoted to the description of the numerical method. In Section 5 we state the results dealing with stability and convergence of the method, and proofs are given in Section 7. In Section 6 we apply the proposed method to the resolution of CSIE on curves. Finally, in Section 8 we show some numerical tests.

**2. Preliminaries.** For any $j \in \{1, \dots, n\}$, let us consider a Jacobi weight

$$v^{\alpha_j, \beta_j}(t) = (1-t)^{\alpha_j}(1+t)^{\beta_j}$$

whose exponents $-1 < \alpha_j, \beta_j < 1$ are related to the real coefficients $a_j$ and $b_j$ by

$$\alpha_j = M_j - \frac{1}{2\pi i} \log\left(\frac{a_j + ib_j}{a_j - ib_j}\right) \tag{2.1}$$

and

$$\beta_j = N_j + \frac{1}{2\pi i} \log\left(\frac{a_j + ib_j}{a_j - ib_j}\right), \tag{2.2}$$

where $i^2 = -1$ and $M_j$, $N_j$ are integers chosen such that the index

$$\chi_j = -(\alpha_j + \beta_j) = -(M_j + N_j) \in \{-1, 0, 1\}.$$

We search the solutions $F_j$, $j = 1, \dots, n$, of (1.1) in the following form

$$F_j(t) = f_j(t) v^{\alpha_j, \beta_j}(t).$$

Then, defining the operators

$$(D_j f)(\tau) = a_j f(\tau) v^{\alpha_j,\beta_j}(\tau) + \frac{b_j}{\pi} \int_{-1}^{1} \frac{f(t)}{t-\tau} v^{\alpha_j,\beta_j}(t)\, dt \tag{2.3}$$

and

$$(K_{jk} f)(\tau) = \int_{-1}^{1} h_{jk}(\tau, t) f(t) v^{\alpha_k,\beta_k}(t)\, dt,$$

we can rewrite (1.1) in operator form as

$$(D_j f_j)(\tau) + \sum_{k=1}^{n} (K_{jk} f_k)(\tau) = G_j(\tau), \quad |\tau| < 1,\ j = 1, \dots, n. \tag{2.4}$$

In order to give a more compact matrix form of the previous system, we introduce the following notations

$$\mathbf{f} = (f_1, f_2, \dots, f_n)^T, \qquad \mathbf{G} = (G_1, G_2, \dots, G_n)^T$$

and define the operator matrices

$$\mathbf{D} = \begin{pmatrix} D_1 & O & \dots & O \\ O & D_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \dots & O & D_n \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{pmatrix},$$

where $O$ denotes the zero operator.

Then system (2.4) can also be represented as follows

$$(\mathbf{D}+\mathbf{K})\mathbf{f}(\tau)=\mathbf{G}(\tau), \quad |\tau|<1. \tag{2.5}$$

In this paper we shall consider the case in which

$$\chi_j=-(\alpha_j+\beta_j)=0, \quad \text{for all } j\in\{1,\dots,n\}.$$

Consequently (see [**11**, page 411], [**14**, page 42]), system (2.5) has index

$$\chi=\sum_{j=1}^{n}\chi_j=0.$$

The case when some of the indices $\chi_j$, $j=1,\dots,n$, are not equal to 0 will be the subject of a forthcoming paper. The idea developed there is the following. If any of the indices $\chi_j$ (say $j=q_1,q_2,\dots,q_N$) is equal to 1, in addition to satisfying system (2.4), the unknown functions $f_j$, $j=q_1,q_2,\dots,q_N$, must satisfy $N$ additional conditions of the following form (see [**7**])

$$\int_{-1}^{1} f_j(t)v^{\alpha_j,\beta_j}(t)\,dt=C_j, \quad j=q_1,q_2,\dots,q_N,$$

where $C_j$ is a constant. If any of the equations has negative index $\chi_j=-1$ (say $j=r_1,r_2,\dots,r_M$), one can proceed for the $j$th equation, $j=r_1,r_2,\dots,r_M$, of (2.4) following the scheme recently proposed in [**5**].

In any event, when the total index $\chi=\chi_1+\chi_2+\cdots+\chi_n$ is not equal to zero, for the solvability of equation (2.5), one also has to take into account the well-known fundamental Noether theorems (see, for instance, [**11**, pages 420, 421], [**14**, pages 63, 64]).

**3. Function spaces.** Let us introduce the function spaces in which we want to study our problem.

Let $\{p_m^{\gamma,\delta}\}_m$ be the sequence of polynomials which are orthonormal with respect to the Jacobi weight $v^{\gamma,\delta}(t)=(1-t)^\gamma(1+t)^\delta$, $\gamma,\delta>-1$. We denote by $L^2_{v^{\gamma,\delta}}$ the Hilbert space of all complex-valued functions

$F$ which are square integrable with respect to $v^{\gamma,\delta}$, equipped with the inner product

$$\langle F, G\rangle_{v^{\gamma,\delta}} = \int_{-1}^{1} F(t)\overline{G(t)}v^{\gamma,\delta}(t)\,dt$$

and the corresponding norm

$$\|F\|_{v^{\gamma,\delta},2} = \left(\int_{-1}^{1} |F(t)|^2 v^{\gamma,\delta}(t)\,dt\right)^{1/2}.$$

For more regular functions, we consider the following weighted Sobolev type spaces

$$\begin{aligned} L^{2,s}_{v^{\gamma,\delta}} = \Bigg\{F \in L^2_{v^{\gamma,\delta}} : \|F\|_{L^{2,s}_{v^{\gamma,\delta}}} \\ = \left(\sum_{i=0}^{\infty}(1+i)^{2s}\left|\langle F, p_i^{\gamma,\delta}\rangle_{v^{\gamma,\delta}}\right|^2\right)^{1/2} < +\infty\Bigg\}, \end{aligned}$$

where $s \in R^+$. Note that $L^{2,0}_{v^{\gamma,\delta}} = L^2_{v^{\gamma,\delta}}$. Moreover, with respect to a set of Jacobi weights $v^{\gamma_j,\delta_j}$, $j = 1, \dots, n$, let us define the vector $\mathbf{v} = (v^{\gamma_1,\delta_1}, v^{\gamma_2,\delta_2}, \dots, v^{\gamma_n,\delta_n})$ and consider the product spaces

$$\mathbf{L}^2_{\mathbf{v}} = \Big\{(F_1, F_2, \dots, F_n) : F_j \in L^2_{v^{\gamma_j,\delta_j}},\ j = 1, 2, \dots, n\Big\}$$

and

$$\mathbf{L}^{2,s}_{\mathbf{v}} = \Big\{(F_1, F_2, \dots, F_n) : F_j \in L^{2,s}_{v^{\gamma_j,\delta_j}},\ j = 1, 2, \dots, n\Big\}$$

equipped with the norms

$$\|\mathbf{F}\|_{\mathbf{L}^2_{\mathbf{v}}} = \left(\sum_{j=1}^{n}\|F_j\|^2_{v^{\gamma_j,\delta_j},2}\right)^{1/2}, \qquad \mathbf{F} = (F_1, F_2, \dots, F_n) \in \mathbf{L}^2_{\mathbf{v}},$$

and

$$\|\mathbf{F}\|_{\mathbf{L}^{2,s}_{\mathbf{v}}} = \left(\sum_{j=1}^{n}\|F_j\|^2_{L^{2,s}_{v^{\gamma_j,\delta_j}}}\right)^{1/2}, \qquad \mathbf{F} = (F_1, F_2, \dots, F_n) \in \mathbf{L}^{2,s}_{\mathbf{v}},$$

respectively.

In the sequel, unless otherwise specified, we shall use for brevity the following notations. Using the exponents $\alpha_j$ and $\beta_j$ defined by (2.1) and (2.2), we set

$$v_j = v^{\alpha_j,\beta_j}, \qquad w_j = v^{-\alpha_j,-\beta_j}, \quad j = 1,\dots,n,$$

and

$$\mathbf{v} = (v_1, v_2, \dots, v_n), \qquad \mathbf{w} = (w_1, w_2, \dots, w_n).$$

Furthermore, we shall denote by $\mathcal{C}$ a positive constant which may have different values in different formulas. We shall write $\mathcal{C} \neq \mathcal{C}(a, b, \dots)$ to indicate that $\mathcal{C}$ is independent of the parameters $a, b, \dots$.

**4. A quadrature method.** In this section we propose a numerical method in order to solve (2.5). We are looking for an array of polynomials which approximates the solution $\mathbf{f}$ (when it exists).

To this end, let us consider the Lagrange projection $L_m^{\gamma,\delta}$ based on the zeros $t_1^{\gamma,\delta} < t_2^{\gamma,\delta} < \cdots < t_m^{\gamma,\delta}$ of $p_m^{\gamma,\delta}$, i.e.,

$$L_m^{\gamma,\delta}(F;t) = \sum_{i=1}^{m} F(t_i^{\gamma,\delta}) l_i^{\gamma,\delta}(t), \qquad l_i^{\gamma,\delta}(t) = \frac{p_m^{\gamma,\delta}(t)}{(p_m^{\gamma,\delta})'(t_i^{\gamma,\delta})(t - t_i^{\gamma,\delta})}.$$

Then, we approximate the original system (2.4) by the following finite-dimensional one

$$L_m^{-\alpha_j,-\beta_j}\left(D_j f_{m,j} + \sum_{k=1}^{n} \widetilde{K}_{m,jk} f_{m,k}; \tau\right) = L_m^{-\alpha_j,-\beta_j}\left(G_j; \tau\right), \tag{4.1}$$

with $|\tau| < 1$ and $j = 1,\dots,n$, whose unknowns are the polynomials $f_{m,j} \in \mathcal{P}_{m-1}$, $j = 1,\dots,n$, ($\mathcal{P}_{m-1}$ denotes the set of all algebraic polynomials of degree at most $m-1$) and where the approximating operators $\widetilde{K}_{m,jk}$, $j,k = 1,\dots,n$, are defined as

$$\left(\widetilde{K}_{m,jk} f\right)(\tau) = \int_{-1}^{1} L_m^{\alpha_k,\beta_k}(h_{jk}(\tau,\cdot);t) f(t) v^{\alpha_k,\beta_k}(t)\, dt.$$

By the subscript $m$ in the notation $f_{m,j}$, as usually done in the literature (see, for instance, [**7, 10, 12**]), we want to emphasize the dimension of the subspace $\mathcal{P}_{m-1}$ where the approximating solution is searched.

Let us observe that, taking into account the well-known property of preserving the polynomials of the dominant operators $D_j$ defined by (2.3) (see [**10**, page 447, equations (33), (34)], [**12**, page 310, (2)]), i.e.,

$$\begin{aligned} D_j p_m^{\alpha_j,\beta_j} &= (-1)^{M_j} p_{m-\chi_j}^{-\alpha_j,-\beta_j} \\ &= -\frac{b_j}{\sin(\pi\alpha_j)} p_{m-\chi_j}^{-\alpha_j,-\beta_j} \quad \left(p_{-1}^{-\alpha_j,-\beta_j}=0\right), \quad m=0,1,\dots, \end{aligned}$$

we have

$$L_m^{-\alpha_j,-\beta_j}\left(D_j f_{m,j}\right) = D_j f_{m,j}, \quad j=1\dots,n.$$

Then, setting

$$\left(K_{m,jk}f\right)(\tau) = L_m^{-\alpha_j,-\beta_j}\left(\widetilde{K}_{m,jk}f;\tau\right), \quad j,k=1,\dots,n,$$

we can introduce the matrix of operators

$$\mathbf{K}_m = \begin{pmatrix} K_{m,11} & K_{m,12} & \dots & K_{m,1n} \\ K_{m,21} & K_{m,22} & \dots & K_{m,2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{m,n1} & K_{m,n2} & \dots & K_{m,nn} \end{pmatrix}$$

and rewrite (4.1) in a more compact form, as follows

$$(\mathbf{D}+\mathbf{K}_m)\mathbf{f}_m(\tau) = \mathbf{G}_m(\tau), \quad |\tau|<1, \tag{4.2}$$

where

$$\mathbf{f}_m = (f_{m,1}, f_{m,2}, \dots, f_{m,n})^T \tag{4.3}$$

and

$$\mathbf{G}_m = \left(L_m^{-\alpha_1,-\beta_1}G_1, L_m^{-\alpha_2,-\beta_2}G_2, \dots, L_m^{-\alpha_n,-\beta_n}G_n\right)^T.$$

Now we will show how to reduce the solution of equation (4.2) to the solution of a system of linear equations. To this end, let us expand the polynomials $f_{m,j}, j=1,\dots,n$, with respect to the basis

$$\left\{(\lambda_i^{\alpha_j,\beta_j})^{-\frac{1}{2}} l_i^{\alpha_j,\beta_j}\right\}_{i=1,\dots,m},$$

where $\lambda_i^{\alpha_j,\beta_j}$ denotes the $i$th Christoffel number related to the weight function $v^{\alpha_j,\beta_j}$. We have

$$f_{m,j}(t) = \sum_{i=1}^{m} c_{ji} (\lambda_i^{\alpha_j,\beta_j})^{-1/2} l_i^{\alpha_j,\beta_j}(t), \quad j = 1, \dots, n, \tag{4.4}$$

with $c_{ji} = (\lambda_i^{\alpha_j,\beta_j})^{1/2} f_{m,j}(t_i^{\alpha_j,\beta_j})$.

It can be proved (see, for instance [**10**, page 448]) that

$$(D_j f_{m,j})\,(t_r^{-\alpha_j,-\beta_j}) = \frac{b_j}{\pi} \sum_{i=1}^{m} (\lambda_i^{\alpha_j,\beta_j})^{1/2} \frac{c_{ji}}{t_i^{\alpha_j,\beta_j} - t_r^{-\alpha_j,-\beta_j}},$$

for $r = 1, \dots, m$ and $j = 1, \dots, n$.

Moreover, let us represent the polynomials appearing on both sides of (4.1) with respect to the basis

$$\left\{ (\lambda_r^{-\alpha_j,-\beta_j})^{-1/2} l_r^{-\alpha_j,-\beta_j} \right\}_{r=1,\dots,m}$$

with $\lambda_r^{-\alpha_j,-\beta_j}$ the $r$th Christoffel number related to the Jacobi weight $v^{-\alpha_j,-\beta_j}$.

Then the vector of polynomials $\mathbf{f}_m = (f_{m,1}, f_{m,2}, \dots, f_{m,n})^T$ with $f_{m,j}$ as in (4.4) is a solution of (4.2) if and only if the array

$$\mathbf{c} = (c_{11}, \dots, c_{1m}, c_{21}, \dots, c_{2m}, \dots, c_{n1}, \dots, c_{nm})^T \in R^{nm}$$

is a solution of the following linear system

$$\begin{aligned} (\lambda_r^{-\alpha_j,-\beta_j})^{1/2} \Bigg[ & \sum_{i=1}^{m} \frac{b_j (\lambda_i^{\alpha_j,\beta_j})^{1/2}}{\pi \left( t_i^{\alpha_j,\beta_j} - t_r^{-\alpha_j,-\beta_j} \right)} c_{ji} \\ & + \sum_{k=1}^{n} \sum_{i=1}^{m} (\lambda_i^{\alpha_k,\beta_k})^{1/2} h_{jk}(t_r^{-\alpha_j,-\beta_j}, t_i^{\alpha_k,\beta_k}) c_{ki} \Bigg] \\ = {} & (\lambda_r^{-\alpha_j,-\beta_j})^{1/2} G_j(t_r^{-\alpha_j,-\beta_j}), \quad r = 1, \dots, m,\ j = 1, \dots, n, \end{aligned} \tag{4.5}$$

of $nm$ equations in $nm$ unknowns. Note that the distance between the zeros $t_i^{\alpha_j,\beta_j}$ and $t_r^{-\alpha_j,-\beta_j}$, for all $i, r \in \{1, \dots, m\}$, is large enough in order to avoid numerical cancelation. In fact, letting $t_i^{\alpha_j,\beta_j} = \cos\tau_{m,i}$ and $t_r^{-\alpha_j,-\beta_j} = \cos\theta_{m,r}$, $i, r = 1, \dots, m$, in [**8**] the authors proved that

$$\min_{i,r=1,\dots,m} |\tau_{m,i} - \theta_{m,r}| \geq \frac{\mathcal{C}}{m},$$

wherefrom it is easy to deduce that there exists a positive constant $\mathcal{C} \neq \mathcal{C}(m, i, r)$ such that

$$\frac{1}{|t_i^{\alpha_j,\beta_j} - t_r^{-\alpha_j,-\beta_j}|} > \mathcal{C}.$$

Summing up, the proposed method consists of solving system (4.5) and in computing the array of polynomials $\mathbf{f}_m = (f_{m,1}, f_{m,2}, \dots, f_{m,n})^T$ using (4.4). In the next section we show that the linear system (4.5) is well conditioned and that the array $\mathbf{f}_m$ converges to the exact solution $\mathbf{f}$ of (2.5), when it exists.

**5. Stability and convergence analysis.** Now we want to state stability and convergence results about the described quadrature method. We first establish compactness of the operator $\mathbf{K}$.

Let us assume that the kernels $h_{jk}$ satisfy the conditions

$$\sup_{|t|\leq 1} \|h_{jk}(\cdot, t)\|_{L^{2,s}_{w_j}} < +\infty, \quad j, k = 1, \dots, n, \tag{5.1}$$

for some $s \geq 0$. Then one can prove the following proposition.

**Proposition 5.1.** *Under the assumptions* (5.1) *the operator* $\mathbf{K} : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^{2,t}_{\mathbf{w}}$ *is bounded for all* $t \leq s$. *Moreover, it is compact for all* $t < s$.

For the stability and convergence of the method, we need to make the following additional assumptions on the known functions appearing in (2.4). For some $s > 1/2$, we suppose that

$$\sup_{|\tau|\leq 1} \|h_{jk}(\tau, \cdot)\|_{L^{2,s}_{v_k}} < +\infty, \quad j, k = 1, \dots, n, \tag{5.2}$$

$$G_j \in L^{2,s}_{w_j}, \quad j = 1, \dots, n. \tag{5.3}$$

**Theorem 5.1.** *Let the conditions* (5.1), (5.2) *and* (5.3) *be fulfilled for* $s > 1/2$, *and let* $\mathrm{Ker}(\mathbf{D}+\mathbf{K}) = \{\mathbf{0}\}$ *in* $\mathbf{L}^2_{\mathbf{v}}$. *Then, for any sufficiently large* $m$ (*say* $m > m_0$), *system* (4.5) *has a unique solution* $\mathbf{c}$ *and the corresponding array* $\mathbf{f}_m$, *defined by* (4.3) *and* (4.4), *is the unique solution of* (4.2).

*Moreover, if* $\mathbf{A}_m$ *denotes the matrix of the coefficients of* (4.5) *and* $\mathrm{cond}\,(\mathbf{A}_m)$ *its condition number in the spectral norm, then we have*

$$\lim_m \mathrm{cond}\,(\mathbf{A}_m) = \mathrm{cond}\,(\mathbf{D}+\mathbf{K}). \tag{5.4}$$

*Finally,* $\mathbf{f}_m$ *converges to the unique solution* $\mathbf{f}$ *of* (2.5) *in* $\mathbf{L}^2_{\mathbf{v}}$ *with the error*

$$\|\mathbf{f}-\mathbf{f}_m\|_{\mathbf{L}^2_{\mathbf{v}}} \le \frac{\mathcal{C}}{m^s}\,\|\mathbf{G}\|_{L^{2,s}_{\mathbf{w}}}\,, \tag{5.5}$$

*where* $\mathcal{C} \neq \mathcal{C}(m,\mathbf{f})$.

We emphasize that, according to estimate (5.5), the smoother the kernels and the known terms, the higher is the convergence order of the error of the proposed method.

As announced in the introduction, in the next section we will show an application of the method to the solution of some Cauchy singular integral equations on curves.

**6. An application.** Let us consider the following Cauchy singular integral equations with real constant coefficients

$$au(x) + \frac{1}{\pi}\int_\Gamma \frac{M(x,y)}{y-x}u(y)\,dy = g(x), \quad x \in \Gamma, \tag{6.1}$$

where the integral is understood in the Cauchy principal value sense and is taken over the curve $\Gamma = \cup_{j=1}^n \Gamma_j$ that is the union of a finite number $n$ of smooth simple closed arcs $\Gamma_j$ in the complex plane which have no common points and are of finite length. Assume that $M$ and $g$ are given complex-valued functions on $\Gamma\times\Gamma$ and $\Gamma$, respectively, and $M(x,x) = b$, for all $x \in \Gamma$, with the coefficients $a$ and $b$ real and satisfying $a^2+b^2 = 1$, $b \neq 0$.

If we denote by $u_j := u|_{\Gamma_j}$ and $g_j := g|_{\Gamma_j}$ the restrictions of the functions $u$ and $g$ to $\Gamma_j$, respectively, and by $M_{jk} := M|_{\Gamma_j \times \Gamma_k}$, for $j, k = 1, \dots, n$, the restriction of the function $M$ to $\Gamma_j \times \Gamma_k$, we can rewrite the equation (6.1) as the equivalent system

$$
\begin{gathered}
a u_j(x) + \frac{1}{\pi} \sum_{k=1}^{n} \int_{\Gamma_k} \frac{M_{jk}(x,y)}{y-x} u_k(y)\, dy = g_j(x), \\
x \in \Gamma_j, \quad j = 1, \dots, n.
\end{gathered} \tag{6.2}
$$

In order to transform the curvilinear 2D integrals appearing in (6.2) into 1D integrals on the same reference interval, for each arc $\Gamma_j, j = 1, \dots, n$, we introduce a parametrization $\sigma_j$ defined on $[-1,1]$:

$$
\sigma_j : t \in [-1,1] \longrightarrow \sigma_j(t) \in \Gamma_j, \quad j = 1, \dots, n,
$$

and, from (6.1), we get

$$
\begin{gathered}
a \overline{u}_j(\tau) + \frac{1}{\pi} \sum_{k=1}^{n} \int_{-1}^{1} \frac{\widehat{M}_{jk}(\tau,t)\sigma_k'(t)}{\sigma_k(t) - \sigma_j(\tau)} \overline{u}_k(t)\, dt = \overline{g}_j(\tau), \\
|\tau| < 1, \quad j = 1, \dots, n,
\end{gathered} \tag{6.3}
$$

where, for $j, k \in \{1, \dots, n\}$, we set

$$
\begin{cases}
\overline{u}_j(t) = u_j(\sigma_j(t)) & t \in (-1,1), \\
\overline{g}_j(t) = g_j(\sigma_j(t)) & t \in (-1,1), \\
\widehat{M}_{jk}(\tau,t) = M_{jk}(\sigma_j(\tau), \sigma_k(t)) & \tau, t \in (-1,1).
\end{cases}
$$

Moreover if we define, for $j, k \in \{1, \dots, n\}$,

$$
\overline{M}_{jk}(\tau,t) = \begin{cases}
\frac{\widehat{M}_{jk}(\tau,t)\sigma_k'(t)(t-\tau)}{\sigma_k(t)-\sigma_j(\tau)} & j = k, \\
\frac{\widehat{M}_{jk}(\tau,t)\sigma_k'(t)}{\sigma_k(t)-\sigma_j(\tau)} & j \neq k,
\end{cases}
$$

we can rewrite (6.3) as follows

$$
\begin{aligned}
a \overline{u}_j(\tau) + \frac{1}{\pi} \int_{-1}^{1} \frac{\overline{M}_{jj}(\tau,t)}{t-\tau} \overline{u}_j(t)\, dt& \\
+ \frac{1}{\pi} \sum_{\substack{k=1 \\ k \neq j}}^{n} \int_{-1}^{1} \overline{M}_{jk}(\tau,t) \overline{u}_k(t)\, dt &= \overline{g}_j(\tau),
\end{aligned} \tag{6.4}
$$

where $j = 1, \dots, n$.

Now, let us consider the following transformation:

$$\frac{\overline{M}_{jj}(\tau,t)}{t-\tau} = \frac{\overline{M}_{jj}(\tau,t)-\overline{M}_{jj}(\tau,\tau)}{t-\tau} + \frac{\overline{M}_{jj}(\tau,\tau)}{t-\tau}. \quad j=1,2,$$

and, moreover, note that

$$\overline{M}_{jj}(\tau,\tau) = \widehat{M}_{jj}(\tau,\tau) = M_{jj}(\sigma_j(\tau),\sigma_j(\tau)) = b, \quad \text{for all } \tau \in (-1,1).$$

Then, setting

$$\overline{h}_{jk}(\tau,t) = \begin{cases} \frac{1}{\pi}\frac{\overline{M}_{jk}(\tau,t)-\overline{M}_{jk}(\tau,\tau)}{t-\tau} & j=k, \\ \frac{1}{\pi}\overline{M}_{jk}(\tau,t) & j\neq k, \end{cases}$$

(6.4) becomes

$$a\overline{u}_j(\tau) + \frac{b}{\pi}\int_{-1}^{1}\frac{\overline{u}_j(t)}{t-\tau}\,dt + \sum_{k=1}^{n}\int_{-1}^{1}\overline{h}_{jk}(\tau,t)\overline{u}_k(t)\,dt = \overline{g}_j(\tau), \quad j=1,\dots,n. \tag{6.5}$$

System (6.5) represents a particular case of the more general system of Cauchy singular integral equations of type (1.1). If the functions $M$ and $g$ appearing in (6.1) and the parametrizations $\sigma_j, j=1,\dots,n$, are sufficiently smooth to assure that the functions $\overline{h}_{jk}$ and $\overline{g}_j, j,k = 1,\dots,n$, satisfy the hypotheses of Theorem 5.1, then system (6.5) can be solved using the method proposed in Section 4.

## 7. Proofs.

We first give some preliminary definitions and results.

In the sequel we will denote by $\mathbf{P}_m$ the following set

$$\mathbf{P}_m = \{(p_1,\dots,p_n) : p_j \in \mathcal{P}_m,\ j=1,\dots,n\}. \tag{7.1}$$

For a Jacobi weight $v^{\gamma,\delta}, \gamma,\delta > -1$ and a real number $s \geq 0$, let us denote by

$$E_m(F)_{L^{2,s}_{v^{\gamma,\delta}}} = \inf_{P\in\mathcal{P}_m}\|F-P\|_{L^{2,s}_{v^{\gamma,\delta}}}$$

the error of best approximation of a function $F \in L^{2,s}_{v^{\gamma,\delta}}$ by means of polynomials of degree at most $m$.

Moreover, let us denote by $S_m^{\gamma,\delta} : L^2_{v^{\gamma,\delta}} \to L^2_{v^{\gamma,\delta}}$ the Fourier projection

$$S_m^{\gamma,\delta} F = \sum_{k=0}^{m-1} \langle F, p_k^{\gamma,\delta} \rangle_{v^{\gamma,\delta}} p_k^{\gamma,\delta}.$$

The following two lemmas provide Fourier and Lagrange interpolation error estimates for functions belonging to $L^{2,s}_{v^{\gamma,\delta}}$ spaces (see [**2, 3, 9**]).

**Lemma 7.1.** *Let $v^{\gamma,\delta}$, $\gamma, \delta > -1$, be a Jacobi weight. For $s \geq 0$ and $F \in L^{2,s}_{v^{\gamma,\delta}}$, we have*

$$\|F - S_m^{\gamma,\delta} F\|_{L^{2,t}_{v^{\gamma,\delta}}} \leq \frac{\mathcal{C}}{m^{s-t}} \|F\|_{L^{2,s}_{v^{\gamma,\delta}}}, \quad 0 \leq t \leq s, \tag{7.2}$$

*where $\mathcal{C} \neq \mathcal{C}(m, F)$.*

*Proof.* The proof is given in [**2**]. □

**Lemma 7.2.** *Let $v^{\gamma,\delta}$, $\gamma, \delta > -1$, be a Jacobi weight. For $s > 1/2$ and $F \in L^{2,s}_{v^{\gamma,\delta}}$, the estimate*

$$\|F - L_m^{\gamma,\delta} F\|_{L^{2,t}_{v^{\gamma,\delta}}} \leq \frac{\mathcal{C}}{m^{s-t}} \|F\|_{L^{2,s}_{v^{\gamma,\delta}}}, \quad 0 \leq t \leq s, \tag{7.3}$$

*holds true, where $\mathcal{C} \neq \mathcal{C}(m, F)$.*

*Proof.* We first observe that, for $F = \Re F + i \Im F$, one has

$$\|F\|^2_{L^{2,s}_{v^{\gamma,\delta}}} = \|\Re F\|^2_{L^{2,s}_{v^{\gamma,\delta}}} + \|\Im F\|^2_{L^{2,s}_{v^{\gamma,\delta}}}.$$

Consequently, since $L_m^{\gamma,\delta} F = L_m^{\gamma,\delta} \Re F + i L_m^{\gamma,\delta} \Im F$,

$$\|F - L_m^{\gamma,\delta}\|^2_{L^{2,s}_{v^{\gamma,\delta}}} = \|\Re F - L_m^{\gamma,\delta} \Re F\|^2_{L^{2,s}_{v^{\gamma,\delta}}} + \|\Im F - L_m^{\gamma,\delta} \Im F\|^2_{L^{2,s}_{v^{\gamma,\delta}}}$$

holds true. Then, by applying Theorem 3.4 in [**9**] to estimate both norms on the right-hand side, the thesis follows. □

*Proof of Proposition* 5.1. It is sufficient to prove that $K_{jk} : L^2_{v_k} \to L^{2,t}_{w_j}$ is bounded for $t \le s$ and compact for $t < s$, for all $j, k = 1, \dots, n$. To this end, we can apply Lemma 4.2 in [**2**]. Nevertheless, we want to give here an alternative proof of the compactness of $K_{jk}$.

Let $t < s$; since $K_{jk} f \in L^{2,t}_{w_j}$ for any $f \in L^2_{v_k}$, we can use estimate (7.2) obtaining

$$\begin{aligned} E_{m-1}(K_{jk} f)_{L^{2,t}_{w_j}} &\le \|K_{jk} f - S_m^{-\alpha_j,-\beta_j}(K_{jk} f)\|_{L^{2,t}_{w_j}} \\ &\le \mathcal{C} m^{t-s} \|K_{jk} f\|_{L^{2,s}_{w_j}} \\ &\le \mathcal{C} m^{t-s} \|f\|_{v_k,2} \end{aligned}$$

with $\mathcal{C} \ne \mathcal{C}(m, f)$. Then, setting $S = \{f \in L^2_{v_k} : \|f\|_{v_k,2} \le 1\}$, we have

$$\lim_m \sup_{f \in S} E_m(K_{jk} f)_{L^{2,t}_{w_j}} = 0$$

wherefrom we deduce that $K_{jk} : L^2_{v_k} \to L^{2,t}_{w_j}$ is a compact operator (see [**13**, page 44]) for all $t < s$. $\square$

In order to prove Theorem 5.1, we need the following results.

**Proposition 7.1.** *Under assumptions* (5.1) *and* (5.2)*, we have*

$$\|\mathbf{K} - \mathbf{K}_m\|_{\mathbf{L}^2_{\mathbf{v}} \longrightarrow \mathbf{L}^2_{\mathbf{w}}} \le \frac{\mathcal{C}}{m^s}, \tag{7.4}$$

*where $\mathcal{C}$ is a positive constant that does not depend upon $m$ but depends linearly upon order $n$ of matrix $\mathbf{K}$.*

*Proof.* In this proof, for brevity, we use the following notations

$$\|K_{jk} - K_{m,jk}\| = \|K_{jk} - K_{m,jk}\|_{L^2_{v_k} \to L^2_{w_j}}.$$

For $\mathbf{f} = (f_1, f_2, \dots, f_n) \in \mathbf{L}^2_{\mathbf{v}}$, by applying at first Minkowski's inequality and then Holder's inequality, we have

$$\|(\mathbf{K} - \mathbf{K}_m)\mathbf{f}\|_{\mathbf{L}^2_{\mathbf{w}}} = \left( \sum_{j=1}^n \left\| \sum_{k=1}^n (K_{jk} - K_{m,jk}) f_k \right\|^2_{w_j,2} \right)^{1/2}$$

$$\leq \left( \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \| (K_{jk} - K_{m,jk}) \, f_k \|_{w_j,2} \right)^2 \right)^{1/2}$$
$$\leq \left( \sum_{j=1}^{n} \left( \sum_{k=1}^{n} \| K_{jk} - K_{m,jk} \| \, \| f_k \|_{v_k,2} \right)^2 \right)^{1/2}$$
$$\leq \sum_{k=1}^{n} \| f_k \|_{v_k,2} \left( \sum_{j=1}^{n} \| K_{jk} - K_{m,jk} \|^2 \right)^{1/2}$$
$$\leq \left( \sum_{k=1}^{n} \| f_k \|_{v_k,2}^2 \right)^{1/2} \left( \sum_{k=1}^{n} \sum_{j=1}^{n} \| K_{jk} - K_{m,jk} \|^2 \right)^{1/2}$$
$$= \| \mathbf{f} \|_{\mathbf{L}^2_{\mathbf{v}}} \left( \sum_{k=1}^{n} \sum_{j=1}^{n} \| K_{jk} - K_{m,jk} \|^2 \right)^{1/2}.$$

Then, in order to prove the thesis, it is sufficient to show that

$$\| K_{jk} - K_{m,jk} \| \leq \frac{\mathcal{C}}{m^s}, \quad \text{for all } j, k \in 1, \dots, n.$$

To this end, using Lemma 7.2, one can proceed as in [**4**, Proof of Lemma 3.1], and the proof is complete. □

**Lemma 7.3.** *Let* $v^{\gamma_j,\delta_j}$ *and* $v^{\rho_j,\theta_j}$, $\gamma_j, \delta_j, \rho_j, \theta_j > -1$, $j = 1, 2, \dots, n$, *be Jacobi weights, and set*

$$\mathbf{v} = (v^{\gamma_1,\delta_1}, v^{\gamma_2,\delta_2}, \dots, v^{\gamma_n,\delta_n}) \quad \text{and} \quad \mathbf{w} = (v^{\rho_1,\theta_1}, v^{\rho_2,\theta_2}, \dots, v^{\rho_n,\theta_n}).$$

*Assume that* $\mathbf{B} : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}$ *and* $\mathbf{B}_m : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{P}_{m-1} \subset \mathbf{L}^2_{\mathbf{w}}$, $m \in \mathbf{N}$, *are bounded linear operators such that*

$$\lim_{m\to\infty} \| \mathbf{B} - \mathbf{B}_m \|_{\mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}} = 0. \tag{7.5}$$

*Then*

$$\lim_{m\to\infty} \left\| \mathbf{B}_m |_{\mathbf{P}_{m-1}} \right\|_{\mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}} = \| \mathbf{B} \|_{\mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}}, \tag{7.6}$$

*where* $\mathbf{B}_m|_{\mathbf{P}_{m-1}}$ *denotes the restriction of* $\mathbf{B}_m$ *to the subspace* $\mathbf{P}_{m-1}$.

*Proof*. We give the main idea of the proof following step-by-step the proof of Theorem 2.3 in [**6**]. In order to simplify the notations, we set $\|\mathbf{B}\| := \|\mathbf{B}\|_{\mathbf{L}^2_\mathbf{v}\to\mathbf{L}^2_\mathbf{w}}$.

Let $\varepsilon > 0$ be arbitrarily chosen, but fixed. By the definition of the operator norm, there exists a function $\mathbf{F}_\varepsilon \in \mathbf{L}^2_\mathbf{v}$ such that

$$\|\mathbf{B}\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_\mathbf{w}} > \|\mathbf{B}\| - \frac{\varepsilon}{2}, \qquad \|\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_\mathbf{v}} = 1. \tag{7.7}$$

For $j = 1, 2, \ldots, n$, let $T_{m,j}$ be a projection of $L^2_{v^{\gamma_j,\delta_j}}$ onto $\mathcal{P}_{m-1}$ such that

$$\sup_m \|T_{m,j}\|_{L^2_{v^{\gamma_j,\delta_j}}\to L^2_{v^{\gamma_j,\delta_j}}} < \infty. \tag{7.8}$$

If we denote by $\mathbf{T}_m$ the matrix

$$\mathbf{T}_m = \begin{pmatrix} T_{m,1} & O & \ldots & O \\ O & T_{m,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \ldots & O & T_{m,n} \end{pmatrix},$$

then, for any $\mathbf{F} = (F_1, F_2, \ldots, F_n) \in \mathbf{L}^2_\mathbf{v}$, we have

$$\|\mathbf{T}_m\mathbf{F} - \mathbf{F}\|_{\mathbf{L}^2_\mathbf{v}} = \left( \sum_{j=1}^n \|T_{m,j}F_j - F_j\|^2_{v^{\gamma_j,\delta_j},2} \right)^{1/2}.$$

Thus, since by (7.8), for any $j \in \{1, 2, \ldots, n\}$, it follows that

$$\|T_{m,j}F_j - F_j\|_{v^{\gamma_j,\delta_j},2} \longrightarrow 0, \quad \text{as } m \to \infty,$$

we get

$$\|\mathbf{T}_m\mathbf{F} - \mathbf{F}\|_{\mathbf{L}^2_\mathbf{v}} \longrightarrow 0, \quad \text{as } m \to \infty, \tag{7.9}$$

for any $\mathbf{F} \in \mathbf{L}^2_\mathbf{v}$. By (7.9) applied to $\mathbf{F}_\varepsilon$, we deduce that there exists an index $m_0 \in \mathbf{N}$ such that

$$\|\mathbf{T}_m\mathbf{F}_\varepsilon - \mathbf{F}_\varepsilon\|_{\mathbf{L}^2_\mathbf{v}} < \frac{\varepsilon}{2\|\mathbf{B}\|}, \quad \text{for all } m \geq m_0, \tag{7.10}$$

and then, by (7.7) and (7.10), we obtain

$$\begin{aligned}\|\mathbf{B}\mathbf{T}_m\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_{\mathbf{w}}} &\geq \|\mathbf{B}\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_{\mathbf{w}}} - \|\mathbf{B}(\mathbf{T}_m\mathbf{F}_\varepsilon - \mathbf{F}_\varepsilon)\|_{\mathbf{L}^2_{\mathbf{w}}}\\ &> \|\mathbf{B}\| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = \|\mathbf{B}\| - \varepsilon.\end{aligned}$$

Consequently, we get

$$\|\mathbf{B}\| - \varepsilon < \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\mathbf{T}_m\mathbf{F}_\varepsilon\right\|_{\mathbf{L}^2_{\mathbf{w}}} \leq \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| \cdot \|\mathbf{T}_m\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_{\mathbf{v}}}$$

and, taking into account (7.7) and (7.9), we obtain

$$\|\mathbf{B}\| - \varepsilon \leq \liminf_m \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| \|\mathbf{F}_\varepsilon\|_{\mathbf{L}^2_{\mathbf{v}}} = \liminf_m \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\|$$

and then

$$\|\mathbf{B}\| - \varepsilon \leq \liminf_m \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| \leq \limsup_m \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| \leq \|\mathbf{B}\|.$$

Now, since $\varepsilon$ is arbitrarily chosen, we can deduce that there exists $\lim_m \|\mathbf{B}|_{\mathbf{P}_{m-1}}\|$ and

$$\lim_m \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| = \|\mathbf{B}\|. \tag{7.11}$$

Finally, since

$$\begin{aligned}\left|\left\|\mathbf{B}_m|_{\mathbf{P}_{m-1}}\right\| - \|\mathbf{B}\|\right| &\leq \left|\left\|\mathbf{B}_m|_{\mathbf{P}_{m-1}}\right\| - \left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\|\right|\\ &\quad + \left|\left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| - \|\mathbf{B}\|\right|\\ &\leq \left\|\mathbf{B}_m|_{\mathbf{P}_{m-1}} - \mathbf{B}|_{\mathbf{P}_{m-1}}\right\|\\ &\quad + \left|\left\|\mathbf{B}|_{\mathbf{P}_{m-1}}\right\| - \|\mathbf{B}\|\right|,\end{aligned}$$

by (7.11) and taking into account that, by hypothesis (7.5),

$$\left\|\mathbf{B}_m|_{\mathbf{P}_{m-1}} - \mathbf{B}|_{\mathbf{P}_{m-1}}\right\| \longrightarrow 0, \quad \text{as } m \to \infty,$$

holds true, then (7.6) follows. □

Now we can prove Theorem 5.1.

*Proof of Theorem* 5.1. Since the operator $\mathbf{D} : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}$ is invertible (see for example [**12**, page 311]) with

$$\mathbf{D}^{-1} = \begin{pmatrix} D_1^{-1} & O & \dots & O \\ O & D_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & O \\ O & \dots & O & D_n^{-1} \end{pmatrix},$$

and, by Proposition 5.1, the operator $\mathbf{K} : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}$, is compact, the Fredholm alternative is true for equation (2.5) and, then, the operator $\mathbf{D}+\mathbf{K} : \mathbf{L}^2_{\mathbf{v}} \to \mathbf{L}^2_{\mathbf{w}}$ is invertible. By (7.4) and some well-known results (see for example [**1**, page 55]), we deduce that, for a sufficiently large $m$ (say $m > m_0$), the inverse operators $(\mathbf{D}+\mathbf{K}_m)^{-1}$ exist and are uniformly bounded with respect to $m$. More precisely, one has

$$\begin{aligned}(7.12)\quad & \left\|(\mathbf{D}+\mathbf{K}_m)^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}} \\ & \qquad \le \frac{\left\|(\mathbf{D}+\mathbf{K})^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}}}{1-\|(\mathbf{D}+\mathbf{K})^{-1}\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}}\cdot\|\mathbf{K}-\mathbf{K}_m\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}}},\end{aligned}$$

which assures the stability of the method. Then (4.2) has a unique solution $\mathbf{f}_m$ which is a vector of polynomials since we have

$$\mathbf{f}_m = \mathbf{D}^{-1}\left(\mathbf{G}_m - \mathbf{K}_m\mathbf{f}_m\right),$$

and, for all $j$, operator $D_j^{-1}$ satisfies the following property (see [**10**, page 447, (33), (34)], [**12**, page 310, (2)]),

$$D_j^{-1}p_m^{-\alpha_j,-\beta_j} = -\frac{b_j}{\sin(\pi\alpha_j)}p_{m+\chi_j}^{\alpha_j,\beta_j}, \quad m = 0,1,2,\dots .$$

Hence, in virtue of the equivalence between (4.2) and (4.5), we can deduce also that the linear system (4.5) has a unique solution $\mathbf{c}$ related to $\mathbf{f}_m$ according to (4.3) and (4.4).

In order to prove (5.5), we use the identity

$$\mathbf{f}-\mathbf{f}_m = (\mathbf{D}+\mathbf{K}_m)^{-1}\left[(\mathbf{G}-\mathbf{G}_m) - (\mathbf{K}-\mathbf{K}_m)(\mathbf{D}+\mathbf{K})^{-1}\mathbf{G}\right]$$

and, by (7.12), for $m > m_0$ and $\mathcal{C} \neq \mathcal{C}(m)$, we get

$$\|\mathbf{f} - \mathbf{f}_m\|_{\mathbf{L}^2_\mathbf{v}} \leq \mathcal{C}\left[\|\mathbf{G} - \mathbf{G}_m\|_{\mathbf{L}^2_\mathbf{w}} + \|\mathbf{K} - \mathbf{K}_m\|_{\mathbf{L}_{\mathbf{v}^2} \to \mathbf{L}^2_\mathbf{w}}\|\mathbf{G}\|_{\mathbf{L}^2_\mathbf{w}}\right]. \tag{7.13}$$

Thus, taking into account (7.4), we have only to estimate $\|\mathbf{G}-\mathbf{G}_m\|_{\mathbf{L}^2_\mathbf{w}}$. But, applying hypothesis (5.3) and (7.3), we have

$$\begin{aligned}
\|\mathbf{G} - \mathbf{G}_m\|_{\mathbf{L}^2_\mathbf{w}} &= \left(\sum_{j=1}^{n} \|G_j - L_m^{-\alpha_j,-\beta_j} G_j\|^2_{w_j,2}\right)^{1/2} \\
&\leq \frac{\mathcal{C}}{m^s}\left(\sum_{j=1}^{n} \|G_j\|^2_{L^{2,s}_{w_j}}\right)^{1/2} \\
&= \frac{\mathcal{C}}{m^s}\,\|\mathbf{G}\|_{\mathbf{L}_{\mathbf{w}}{}^{2,s}}.
\end{aligned}$$

Finally, combining the last inequality with (7.4) and (7.13), since

$$\|\mathbf{G}\|_{\mathbf{L}^2_\mathbf{w}} \leq \|\mathbf{G}\|_{\mathbf{L}^{2,s}_\mathbf{w}},$$

then (5.5) follows.

It remains to prove (5.4). To this end, let us introduce some notation. Let $\mathbf{P}_{m-1}$ be defined as in (7.1), and let $\|\mathbf{d}\|_2 = (\sum_{i=1}^N |d_i|^2)^{1/2}$ denote the Euclidean norm of an array $\mathbf{d} \in R^N$.

Taking $\mathbf{c} = (c_{11}, \dots, c_{1m}, \dots, c_{n1}, \dots, c_{nm})^T \in R^{nm}$ as an arbitrary array, then the vector $\mathbf{g} = (g_{11}, \dots, g_{1m}, \dots, g_{n1}, \dots, g_{nm})^T$ satisfies $\mathbf{A}_m\mathbf{c} = \mathbf{g}$ if and only if $(\mathbf{D} + \mathbf{K}_m)\mathbf{f}_m = \mathbf{G}_m$ with

$$\begin{aligned}
\mathbf{f}_m &= (f_{m,1}, f_{m,2}, \dots, f_{m,n})^T, \\
f_{m,j}(t) &= \sum_{i=1}^{m} c_{ji}(\lambda_i^{\alpha_j,\beta_j})^{-1/2} l_i^{\alpha_j,\beta_j}(t),
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{G}_m &= (G_{m,1}, G_{m,2}, \dots, G_{m,n})^T, \\
G_{m,j}(t) &= \sum_{i=1}^{m} g_{ji}(\lambda_i^{-\alpha_j,-\beta_j})^{-1/2} l_i^{-\alpha_j,-\beta_j}(t).
\end{aligned}$$

Since

$$\begin{aligned}\|\mathbf{f}_m\|^2_{\mathbf{L}^2_{\mathbf{v}}} &= \sum_{j=1}^{n} \|f_{m,j}\|^2_{v_j,2}\\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} \lambda_i^{\alpha_j,\beta_j} |f_{m,j}(t_i^{\alpha_j,\beta_j})|^2\\ &= \sum_{j=1}^{n}\sum_{i=1}^{m} |c_{ji}|^2\\ &= \|\mathbf{c}\|_2^2\end{aligned}$$

and, analogously, $\|\mathbf{G}_m\|_{\mathbf{L}^2_{\mathbf{w}}} = \|\mathbf{g}\|_2$, we can deduce

$$\begin{aligned}\|\mathbf{A}_m\| &= \sup_{\substack{\mathbf{c}\in R^{nm}\\ \mathbf{c}\neq 0}} \frac{\|\mathbf{A}_m\mathbf{c}\|_2}{\|\mathbf{c}\|_2}\\ &= \sup_{\substack{\mathbf{f}_m\in \mathbf{P}_{m-1}\\ \mathbf{f}_m\neq 0}} \frac{\|(\mathbf{D}+\mathbf{K}_m)\mathbf{f}_m\|_{\mathbf{L}^2_{\mathbf{w}}}}{\|\mathbf{f}_m\|_{\mathbf{L}^2_{\mathbf{v}}}}\\ &= \left\|(\mathbf{D}+\mathbf{K}_m)|_{\mathbf{P}_{m-1}}\right\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}},\end{aligned}$$

where the matrix norm is the spectral one. Now, since by (7.4), one has

$$\|(\mathbf{D}+\mathbf{K}) - (\mathbf{D}+\mathbf{K}_m)\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}} \longrightarrow 0, \quad \text{as } m\to\infty,$$

by applying (7.6), we get

$$\|\mathbf{A}_m\| \longrightarrow \|\mathbf{D}+\mathbf{K}\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}}, \quad \text{as } m\to\infty. \tag{7.14}$$

In the same way, for the inverse matrix, we can write

$$\begin{aligned}\|\mathbf{A}_m^{-1}\| &= \sup_{\substack{\mathbf{g}\in R^{nm}\\ \mathbf{g}\neq 0}} \frac{\|\mathbf{A}_m^{-1}\mathbf{g}\|_2}{\|\mathbf{g}\|_2}\\ &= \sup_{\substack{\mathbf{G}_m\in \mathbf{P}_{m-1}\\ \mathbf{G}_m\neq 0}} \frac{\|(\mathbf{D}+\mathbf{K}_m)^{-1}\mathbf{G}_m\|_{\mathbf{L}^2_{\mathbf{v}}}}{\|\mathbf{G}_m\|_{\mathbf{L}^2_{\mathbf{w}}}}\\ &= \left\|[(\mathbf{D}+\mathbf{K}_m)^{-1}]|_{\mathbf{P}_{m-1}}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}}.\end{aligned}$$

Now, taking into account that

$$\begin{aligned}&\left\|(\mathbf{D}+\mathbf{K}_m)^{-1}-(\mathbf{D}+\mathbf{K})^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}}\\ &\le \left\|(\mathbf{D}+\mathbf{K})^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}}\left\|\mathbf{K}-\mathbf{K}_m\right\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}}\left\|(\mathbf{D}+\mathbf{K}_m)^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}},\end{aligned}$$

and also using (7.4) and (7.12), we obtain

$$\left\|(\mathbf{D}+\mathbf{K}_m)^{-1}-(\mathbf{D}+\mathbf{K})^{-1}\right\|_{\mathbf{L}^2_{\mathbf{v}}\to\mathbf{L}^2_{\mathbf{w}}}\longrightarrow 0,\quad \text{as } m\to\infty.$$

Then, by applying (7.6) again, we get

$$\|\mathbf{A}_m^{-1}\|\longrightarrow\left\|(\mathbf{D}+\mathbf{K})^{-1}\right\|_{\mathbf{L}^2_{\mathbf{w}}\to\mathbf{L}^2_{\mathbf{v}}},\quad \text{as } m\to\infty. \tag{7.15}$$

Hence (5.4) follows from (7.14) and (7.15). □

**8. Numerical examples.** In the following examples we show that our theoretical results are confirmed by numerical tests.

Where the exact solutions of the systems of integral equations are unknown, one thinks of their approximate solutions as being exact for $m = 512$.

**Example 1.** Consider the system

$$(D_j f_j)(\tau)+\sum_{k=1}^{2}\int_{-1}^{1}h_{jk}(\tau,t)f_k(t)v^{\alpha_k,\beta_k}(t)\,dt=G_j(\tau),\quad j=1,2, \tag{8.1}$$

with

$$\alpha_1=\frac14,\quad \beta_1=-\frac14,\qquad \alpha_2=\frac12,\quad \beta_2=-\frac12,$$

$$a_1=\cos(\pi\alpha_1),\qquad a_2=\cos(\pi\alpha_2),$$
$$b_1=-\sin(\pi\alpha_1),\qquad b_2=-\sin(\pi\alpha_2),$$

$$(h_{jk}(\tau,t))_{j,k=1,2}=\begin{pmatrix}\frac{\sqrt2 t^3\tau^2}{2}+i\frac{t\tau^2}{2} & \frac{\sqrt2(t-\tau)^2}{\pi}+it^2\tau e^\tau\\ \frac{\sqrt2(t-\tau)^3}{\pi}+i\tau^2t^4 & t^2e^\tau+i\frac{(t+\tau)^4}{\pi}\end{pmatrix},$$

and

$$\mathbf{G}(\tau)=\begin{pmatrix}G_1(\tau)\\ G_2(\tau)\end{pmatrix},$$

where

$$G_1(\tau) = \left(\frac{768 - 11\pi\sqrt{2} + 32\pi}{128\sqrt{2}}\right)\tau^2 + \left(3\sqrt{2} - 2\pi e^{\tau}\right)\tau + \frac{\sqrt{2}+3}{\sqrt{2}} + i\left[\left(\frac{512 - 11\pi\sqrt{2} - 8\pi}{64\sqrt{2}}\right)\tau^2 + \left(\frac{8\sqrt{2} + 3\pi e^{\tau}}{2}\right)\tau + 2(1+\sqrt{2})\right]$$

and

$$G_2(\tau) = -4\tau^4 + 7\tau^3 - \left(\frac{1632 + 31\pi\sqrt{2}}{128}\right)\tau^2 + \frac{39}{8}\tau + \frac{85 + 96\pi e^{\tau}}{64} + i\left[3\tau^4 - 8\tau^3 + \left(\frac{1920 + 31\sqrt{2}\pi}{256}\right)\tau^2 - \frac{27}{4}\tau + \frac{153 + 64\pi e^{\tau}}{32}\right].$$

The exact solution is

$$\mathbf{f}(\tau) = \begin{pmatrix} 1 + i2 \\ 3 + i4 \end{pmatrix}.$$

Solving the above system using our method with $m = 3$ we get an approximation of the exact solution with 14 exact decimal digits. In Table 1 we show that the sequence of the condition numbers of matrices of the solved linear systems is convergent and, by virtue of (5.4), they can be considered to be approximations of the condition number of operator $\mathbf{D} + \mathbf{K}$ related to system (8.1).

TABLE 1.

| $m$ | cond $(\mathbf{A}_m)$ |
|---|---|
| 4 | **11.9**1667299603004 |
| 8 | **11.92684144437**3887 |
| 16 | **11.92684144374379** |

**Example 2.** Now we consider the system of integral equations

$$(D_j f_j)(\tau) + \sum_{k=1}^{3} \int_{-1}^{1} h_{jk}(\tau,t) f_k(t) v^{\alpha_k,\beta_k}(t)\,dt = G_j(\tau), \quad j = 1,2,3,$$

with

$$\begin{gathered} \alpha_1 = \frac{1}{3},\ \beta_1 = -\frac{1}{3}, \quad \alpha_2 = \frac{1}{2},\ \beta_2 = -\frac{1}{2}, \quad \alpha_3 = \frac{1}{5},\ \beta_3 = -\frac{1}{5}, \\ a_1 = \cos(\pi\alpha_1), \qquad a_2 = \cos(\pi\alpha_2), \qquad a_3 = \cos(\pi\alpha_3), \\ b_1 = -\sin(\pi\alpha_1), \quad b_2 = -\sin(\pi\alpha_2), \quad b_3 = -\sin(\pi\alpha_3), \end{gathered}$$

$$\begin{aligned} &(h_{jk}(\tau,t))_{j,k=1,2,3} \\ &= \begin{pmatrix} |\tau-t|^{7/2} + ie^{t-\tau} & \tau t\cos(\tau+t) + ie^{\tau}\sin(t) & \tau t^2\sin(\tau-t) + i\tau e^{\tau+t} \\ |\tau-t|^{9/2} + i\sin(\tau+t) & \tau^3 e^t + i(\tau^2+t^2) & (\tau+t)\cos(\tau-t) + i\tau^3 e^{\tau-t} \\ (\tau t)^2 + i\cos(\tau)\sin(t) & \tau^3 t^4 + it^2\cos^2(\tau) & |\sin(\tau-t)|^{9/2} + i\cos(\tau+t) \end{pmatrix}, \end{aligned}$$

and

$$\mathbf{G}(\tau) = \begin{pmatrix} |\tau|^{11/2} \\ \tau\cos(\tau) \\ \tau^2 e^{\tau} \end{pmatrix},$$

whose exact solution is unknown. In this case not all the kernels and right-hand sides are very smooth so that, according to (5.5), we need to increase $m$ to take some exact digits in the approximation of the solutions of the system. In Table 2 we show the condition numbers of the matrices $\mathbf{A}_m$, and in Tables 3–5 we show the values of the approximate solutions computed in the point 0.5.

TABLE 2.

| $m$ | cond $(\mathbf{A_m})$ |
|---|---|
| 8 | **49.37**306654177218 |
| 16 | **49.37280**039226505 |
| 32 | **49.3728031**8970958 |
| 64 | **49.37280347**482547 |
| 128 | **49.37280348**988397 |
| 256 | **49.37280349059**117 |

TABLE 3.

| $m$ | $f_{m,1}(0.5)$ | | |
|---|---|---|---|
| 8 | **0.320**8187252362181 | $+\ i$ | **0.4447**419102329236 |
| 16 | **0.32061**34683482964 | $+\ i$ | **0.44476**77809862061 |
| 32 | **0.32061544**39095880 | $+\ i$ | **0.4447686**331026605 |
| 64 | **0.320615440**6932612 | $+\ i$ | **0.444768667**6428222 |
| 128 | **0.3206154401**649992 | $+\ i$ | **0.4447686691**262363 |
| 256 | **0.32061544015 8**3543 | $+\ i$ | **0.44476866919**11085 |

TABLE 4.

| $m$ | $f_{m,2}(0.5)$ | | |
|---|---|---|---|
| 8 | **-0.603**0621160766333 | $+\ i$ | **1.53**2017462701152 |
| 16 | **-0.60311**62916284238 | $+\ i$ | **1.531993**295914450 |
| 32 | **-0.603117**0690580403 | $+\ i$ | **1.5319934**57171011 |
| 64 | **-0.60311710**45004728 | $+\ i$ | **1.53199347**5674232 |
| 128 | **-0.603117106**0711339 | $+\ i$ | **1.531993476**779080 |
| 256 | **-0.60311710614**05157 | $+\ i$ | **1.53199347683**4780 |

TABLE 5.

| $m$ | $f_{m,3}(0.5)$ | | |
|---|---|---|---|
| 8 | **0.621**8708551225152 | $+\ i$ | **0.371**5244490699837 |
| 16 | **0.6219006**414075582 | $+\ i$ | **0.371492**2734386401 |
| 32 | **0.6219006**424168296 | $+\ i$ | **0.37149209**84177940 |
| 64 | **0.621900641**9218478 | $+\ i$ | **0.371492094**3196969 |
| 128 | **0.6219006418**865786 | $+\ i$ | **0.3714920941**846753 |
| 256 | **0.621900641884**6876 | $+\ i$ | **0.37149209417 9**3784 |

Now we apply the proposed method to some Cauchy singular integral equations considered in Section 6.

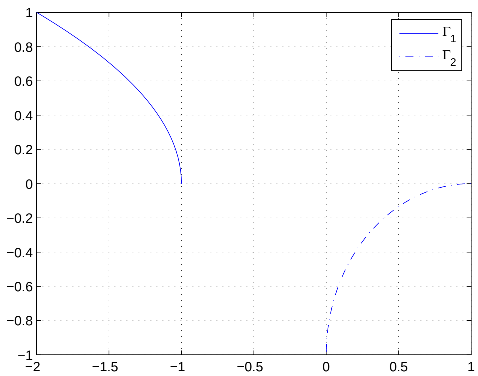


FIGURE 1.

**Example 3.** Let the equation

$$\frac{\sqrt{3}}{2}u(x) - \frac{1}{\pi}\int_{\Gamma} \frac{\cos(i(x-y)+(\pi/3))}{y-x} u(y)\,dy = ie^{ix}, \quad x \in \Gamma, \tag{8.2}$$

be given with $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1$ is an arc of the parabola having axes coincident with the $x$-axis, vertex in the point $-1$ and passing through point $-2+i$ and $\Gamma_2$ is the arc of the circle centered in $1-i$ and of radius 1 (see Figure 1).

The exact solution of equation (8.2) is unknown.

Following the procedure shown in Section 6 with the following parametrizations of arcs $\Gamma_1$ and $\Gamma_2$:

$$\sigma_1 : t \in [-1,1] \longrightarrow \sigma_1(t) = -\left(\frac{t^2+2t+5}{4}\right) + i\left(\frac{t+1}{2}\right) \in \Gamma_1$$

and

$$\sigma_2 : t \in [-1,1] \longrightarrow \sigma_2(t) = (1-i) + e^{i\{\pi[(t+1)/4]+(\pi/2)\}} \in \Gamma_2,$$

respectively, we transform (8.2) into the system of integral equations

$$(D_j f_j)(\tau) + \sum_{k=1}^{2}\int_{-1}^{1} h_{jk}(\tau,t) f_k(t) v^{\alpha_k,\beta_k}(t)\,dt = G_j(\tau), \quad j = 1,2,$$

with

$$\begin{aligned}
&\alpha_1 = \frac{1}{6}, \quad \beta_1 = -\frac{1}{6}, \qquad \alpha_2 = \frac{1}{6}, \quad \beta_2 = -\frac{1}{6},\\
&a_1 = \cos(\pi\alpha_1),\ a_2 = \cos(\pi\alpha_2), \quad b_1 = -\sin(\pi\alpha_1),\ b_2 = -\sin(\pi\alpha_2),
\end{aligned}$$

$$\begin{aligned}
h_{11}(\tau,t) &= \frac{2-2i+\tau+t-4(1-i+t)\sin((\pi/6)+(1/4)i(\tau-t)(2-2i+\tau+t))}{2\pi(t-\tau)(2-2i+\tau+t)},\\
h_{12}(\tau,t) &= \frac{ie^{(1/4)i\pi(1+t)}\sin\Big(e^{(1/4)i\pi(1+t)}-(\pi/6)-(1/4)i(9-6i+\tau(2-2i+\tau))\Big)}{4e^{(1/4)i\pi(1+t)}-i(9-6i+\tau(2-2i+\tau))},\\
h_{21}(\tau,t) &= \frac{2(1-i+t)\sin\Big(((3/2)+(9/4)i)-e^{(1/4)i\pi(1+\tau)}-(\pi/6)+(1/4)it(2-2i+t)\Big)}{\pi\Big(9-6i+4ie^{(1/4)i\pi(1+\tau)}+t(2-2i+t)\Big)},\\
h_{22}(\tau,t) &= \frac{2\Big(e^{(1/4)i\pi(1+\tau)}-e^{(1/4)i\pi(1+t)}\Big)}{4\pi(t-\tau)\Big(e^{(1/4)i\pi(1+\tau)}-e^{(1/4)i\pi(1+t)}\Big)},\\
&\quad + \frac{ie^{(1/4)i\pi(1+t)}\pi(t-\tau)\sin\Big(e^{(1/4)i\pi(1+\tau)}-e^{(1/4)i\pi(1+t)}+(\pi/6)\Big)}{4\pi(t-\tau)\Big(e^{(1/4)i\pi(1+\tau)}-e^{(1/4)i\pi(1+t)}\Big)},
\end{aligned}$$

$$\begin{aligned}
G_1(\tau) &= ie^{i\big((1/2)i(1+\tau)-(1/4)(1+\tau)^2-1\big)},\\
G_2(\tau) &= ie^{i\big((1-i)+e^{i((\pi/2)+(1/4)\pi(1+\tau))}\big)},
\end{aligned}$$

and we apply to it the proposed quadrature method. In Table 6 we show the values of the condition numbers of matrices $\mathbf{A}_m$. Moreover, denoting by $u_m$ the approximations of the solution $u$ of (8.2), in Tables 7 and 8 we show their values in the points $x_1 = -[(25)/(16)]+i(3/4) \in \Gamma_1$ and $x_2 = (1-[\sqrt{2+\sqrt{2}}/2])-i(1-[\sqrt{2-\sqrt{2}}/2]) \in \Gamma_2$, respectively. As one can see, solving a linear system of order 128, we get approximations with 15 exact decimal digits.

TABLE 6.

| $m$ | cond $(\mathbf{A}_m)$ |
|---|---|
| 8 | **3.04884191**3959566 |
| 16 | **3.04884194084305**5 |
| 32 | **3.048841940843052** |

TABLE 7.

| $m$ | $u_m(x_1)$ | | |
|---|---|---|---|
| 8 | **1.4969221**02477298 | - $i$ | **0.0320110**31307591 |
| 16 | **1.4969221007 2**5485 | - $i$ | **0.032011042414**615 |
| 32 | **1.49692210072467**7 | - $i$ | **0.032011042414120** |
| 64 | **1.496922100724676** | - $i$ | **0.032011042414120** |

TABLE 8.

| $m$ | $u_m(x_2)$ | | |
|---|---|---|---|
| 8 | **-0.4612132**128020694 | $+ i$ | **2.175802**671268618 |
| 16 | **-0.4612132**436793596 | $+ i$ | **2.175802**204035568 |
| 32 | **-0.461213231572191**5 | $+ i$ | **2.17580219221399**3 |
| 64 | **-0.4612132315721912** | $+ i$ | **2.17580219221399**0 |

**Example 4.** As the last example we take the singular integral equation

$$\frac{1}{\pi}\int_\Gamma \frac{e^{i(x-y+\pi)}}{y-x}u(y)\,dy = ix^2\cos^2 x, \quad x \in \Gamma,$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1$ is an arc of the parabola having an axis that is parallel to the $y$-axis, a vertex in the point $1+i$ and passing through the point $2+i3$, $\Gamma_2$ is the segment joining points $5/2+i(14/5)$ and $7/2$ and $\Gamma_3$ is the segment joining the points $4+i$ and $5+i2$ (see Figure 2). Also, in this case, one does not know the exact solution.

Considering the following parametrizations

$$\sigma_1 : t \in [-1,1] \longrightarrow \sigma_1(t) = \left(\frac{t+3}{2}\right) + i\left(\frac{t^2+2t+3}{2}\right) \in \Gamma_1,$$
$$\sigma_2 : t \in [-1,1] \longrightarrow \sigma_2(t) = \left(\frac{t+6}{2}\right) - i\left(\frac{7(t-1)}{5}\right) \in \Gamma_2$$

and

$$\sigma_3 : t \in [-1,1] \longrightarrow \sigma_3(t) = \left(\frac{t+9}{2}\right) + i\left(\frac{t+3}{2}\right) \in \Gamma_3$$

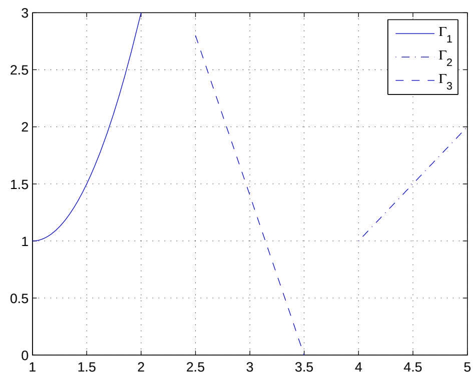


FIGURE 2.

of arcs $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, respectively, we solve the system

$$\left(D_j f_j\right)(\tau) + \sum_{k=1}^{3} \int_{-1}^{1} h_{jk}(\tau,t) f_k(t) v^{\alpha_k,\beta_k}(t)\, dt = G_j(\tau), \quad j=1,2,3,$$

where

$$h_{11}(\tau,t) = \frac{2-i+\tau+t+e^{-\frac{1}{2}(\tau-t)(2-i+\tau+t)}(i-2-2t)}{\pi(t-\tau)(2-i+\tau+t)},$$

$$h_{12}(\tau,t) = -\frac{(14+5i)e^{-(1/10)(1+15i+5\tau(2-i+\tau)+(14+5i)t)}}{\pi(1+15i+5\tau(2-i+\tau)+(14+5i)t)},$$

$$h_{13}(\tau,t) = \frac{(1+i)e^{-(1/2)(6i+\tau(2-i+\tau)-(1-i)t)}}{\pi(-6+(1+2i+i\tau)\tau-(1+i)t)},$$

$$h_{21}(\tau,t) = -\frac{5e^{(1/10)(1+15i+(14+5i)\tau+5t(2-i+t))}(2-i+2t)}{\pi(1+15i+(14+5i)\tau+5t(2-i+t))},$$

$$h_{22}(\tau,t) = \frac{1-e^{[(7/5)+(i/2)](\tau-t)}}{\pi(t-\tau)},$$

$$h_{23}(\tau,t) = -\frac{5(1+i)e^{(1/10)(1-15i+(14+5i)\tau+5t(1-i))}}{\pi(15+i-(5-14i)\tau+5t(1+i))},$$

$$h_{31}(\tau,t) = \frac{e^{(1/2)(6i-(1-i)\tau+t(2-i+t))}(i-2-2t)}{\pi(6i-(1-i)\tau+t(2-i+t))},$$

$$h_{32}(\tau,t) = \frac{(5-14i)e^{(i/10)(15+i+5(1+i)\tau-(5-14i)t)}}{\pi(15+i+5(1+i)\tau-(5-14i)t)},$$
$$h_{33}(\tau,t) = \frac{-1+e^{[-(1/2)+(i/2)](\tau-t)}}{\pi(\tau-t)},$$

$$G_1(\tau) = -\frac{i}{8}(3(1-i)+\tau(2-i+\tau))^2 \times (1+\cos(3(1+i)+(1+2i+i\tau)\tau)),$$
$$G_2(\tau) = -\frac{i}{100}(30i-14+(14+5i)\tau)^2\cos^2\left(\left(3+\frac{7}{5}i\right)+\left(\frac{1}{2}-\frac{7}{5}i\right)\tau\right)$$

and

$$G_3(\tau) = -\frac{1}{4}(6-3i+\tau)^2(1+\cos(9+3i+(1+i)\tau)).$$

In Table 9 we show the values of the condition numbers, and in Tables 10–12 we show the values of the approximate solutions $u_m$ in the points $x_1 = -[(25)/(16)] + i(3/4) \in \Gamma_1$, $x_2 = (13/4) + i(7/10) \in \Gamma_2$ and $x_3 = (19/4) + i(7/4) \in \Gamma_3$, respectively.

TABLE 9.

| $m$ | cond $(\mathbf{A_m})$ |
|---|---|
| 8 | **26.425**614167183726 |
| 16 | **26.42524780**9212281 |
| 32 | **26.42524786252493** |

TABLE 10.

| $m$ | $u_m(x_1)$ | | |
|---|---|---|---|
| 8 | **687**.2469651366328 | - $i$ | **649**.744464069650 |
| 16 | **688.9**514329720981 | - $i$ | **648.5**66296428546 |
| 32 | **688.9164**768918923 | - $i$ | **648.515709**396098 |
| 64 | **688.91648048888**37 | - $i$ | **648.51570292768**2 |
| 128 | **688.9164804888829** | - $i$ | **648.515702927681** |

TABLE 11.

| $m$ | $u_m(x_2)$ | | |
|---|---|---|---|
| 8 | **622**.652653218937 | $+\ i$ | **106**.4300154346315 |
| 16 | **622.70**0128323161 | $+\ i$ | **106.03**81793811991 |
| 32 | **622.70111**1024852 | $+\ i$ | **106.0310**498810687 |
| 64 | **622.70111083862**7 | $+\ i$ | **106.03105014421**89 |
| 128 | **622.701110838629** | $+\ i$ | **106.03105014421**87 |

TABLE 12.

| $m$ | $u_m(x_3)$ | | |
|---|---|---|---|
| 8 | **17.87**034933914506 | $+\ i$ | **231.899**57193126585 |
| 16 | **17.870837689**77788 | $+\ i$ | **231.89920684**791701 |
| 32 | **17.8708376896627**2 | $+\ i$ | **231.8992068476801**2 |
| 64 | **17.87083768966**258 | $+\ i$ | **231.8992068476800**6 |
| 128 | **17.87083768966**271 | $+\ i$ | **231.8992068476801**2 |

**Acknowledgments.** The authors are grateful to Prof. Giuseppe Mastroianni and Prof. Peter Junghanns for their helpful remarks and suggestions.

## REFERENCES

**1.** K.E. Atkinson, *The numerical solution of integral equations of the second kind*, Cambridge Mono. Appl. Comp. Math., Cambridge University Press, Cambridge 1997.

**2.** D. Berthold, W. Hoppe and B. Silbermann, *A fast algorithm for solving the generalized airfoil equation*, J. Comp. Appl. Math. **43** (1992), 185–219.

**3.** P. Junghanns and B. Silbermann, *Numerical analysis for one-dimensional Cauchy singular integral equations*, J. Comp. Appl. Math. **125** (2000), 395–421.

**4.** C. Laurita, *Condition numbers for singular integral equations in weighted $L^2$ spaces*, J. Comp. Appl. Math. **116** (2000), 23–40.

**5.** ———, *A quadrature method for Cauchy singular integral equations with index* $-1$, IMA J. Numer. Anal., to appear, doi:10.1093/imanum/drr032.

**6.** C. Laurita and G. Mastroianni, *Condition numbers in numerical methods for Fredholm integral equations of the second kind*, J. Integral Equations Appl. **14** (2002), 311–341.

**7.** G. Mastroianni and S. Prössdorf, *A quadrature method for Cauchy integral equations with weakly singular perturbation kernel*, J. Integral Equations Appl. **4** (1992), 205–228.

**8.** ———, *Some nodes matrices appearing in the numerical analysis for singular integral equations*, BIT **34** (1994), 120–128.

**9.** G. Mastroianni and M.G. Russo, *Lagrange interpolation in weighted Besov spaces*, Constr. Approx. **15** (1999), 257–289.

**10.** S.G. Mikhlin and S. Prössdorf, *Singular integral operators*, Akademie-Verlag, Berlin, 1986.

**11.** N.I. Muskhelishvili, *Singular integral equations. Boundary problems of function theory and their application to mathematical physics*, Dover Publications, Inc., New York, 1992, translated from the second (1946) Russian edition and with a preface by J.R.M. Radok; corrected reprint of the 1953 English translation.

**12.** S. Prössdorf and B. Silbermann, *Numerical analysis for integral and related operator equations*, Akademie-Verlag, Berlin, 1991 and Birkhäuser Verlag, Basel, 1991.

**13.** A.G. Timan, *Theory of approximation of functions of a real variable*, Dover, New York, 1994.

**14.** N.P. Vekua, *Systems of singular integral equations*, J.H. Ferziger, ed., P. Noordhoff, Ltd., Groningen, 1967, translated from the Russian by A.G. Gibbs and G.M. Simmons.

Department of Mathematics and Computer Sciences, University of Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy
**Email address: mariacarmela.debonis@unibas.it**

Department of Mathematics and Computer Sciences, University of Basilicata, Via dell'Ateneo Lucano 10, 85100 Potenza, Italy
**Email address: concetta.laurita@unibas.it**