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ACCELERATION TECHNIQUES BY POST-PROCESSING OF NUMERICAL SOLUTIONS OF THE HAMMERSTEIN EQUATION

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ABSTRACT. In this paper, several acceleration techniques for numerical solutions of the Hammerstein equation by post-processing are discussed. The paper is motivated by the results reported in papers [**7, 8**]. Results in these papers are concerned with certain post acceleration techniques for numerical solutions of the second kind Fredholm integral equation. Techniques consist of interpolation post-processing and extrapolation. Post-processed solutions are shown to exhibit better accuracy. We propose in this paper to generalize the results in [**7, 8**] to nonlinear integral equations of the Hammerstein type. An extrapolation technique for the Galerkin solution of Hammerstein equation is also obtained. This result appears new even in the setting of the linear Fredholm equation.

1. Introduction. In this paper, we investigate a number of post-processing techniques which can be used to enhance the accuracy of numerical solutions of nonlinear integral equations of the Hammerstein type. Post-processing techniques discussed here can be classified into two groups, one based upon an interpolation and another based upon an extrapolation. Motivation of this paper originates in a recent paper [**8**] in which similar results were obtained for linear integral equations of the Fredholm type. One of the goals of this paper is to extend the results in [**8**] to a class of nonlinear equations. Accuracy enhancing post-processing techniques by iterative methods are well documented in terms of linear [**1**] as well as nonlinear integral equations [**4, 5**]. The iterative method, when applied to the collocation method as well as to the Galerkin method, double the order of the accuracy of a

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numerical solution, provided that the solution and the kernel of integral equation are sufficiently smooth. Iterative method is an excellent method to improve the accuracy of a numerical solution. We note that, in order to double the rate of convergence of numerical approximation by the collocation or the Galerkin method, it is necessary that the order of basis functions must be doubled. This results in the solution process which involves a much larger system of linear or nonlinear equations. The iterative method, on the other hand, presents an accuracy enhancement technique which avoids a larger system and thus is computationally more efficient.

We present in this paper a class of post-processing techniques based upon an interpolation and an extrapolation. The cost of implementing these techniques turns out to be less than that of the iterative methods. In Section 2 we study a global superconvergence property of the interpolation post-processing technique for the collocation method for Hammerstein equations. The case for weakly singular Hammerstein equations is also included in this section. After the original version of this manuscript was completed, the paper by Huang and Zhang [**3**] was brought to the authors' attention which proved the same superconvergence result for the post-processing collocation method. Because of this discovery, we do not present our original results in Section 2, but refer to [**3**] for details for one-dimensional problems. Instead, we present, in Section 2, a numerical example exhibiting that the same superconvergence of the post-processed collocation method can be obtained for two-dimensional Hammerstein equations. The numerical experiment conducted on the two-dimensional Hammerstein equation reveals that there exist a number of interesting and important issues which must be addressed for a successful implementation of the post-processing technique via interpolation for multi-dimensional integral equations. They will be discussed in a future paper. In Section 2 we also briefly mention a post processing technique by interpolation to enhance the order of accuracy of a numerical solution of one-dimensional weakly singular Hammerstein equation. A numerical example using constant basis functions is reported in [**3**]. We include in Section 2 two additional numerical examples for weakly singular Hammerstein equations, this time using linear basis functions. We consider both types of weak singularity which are logarithmic and algebraic.

An extrapolation technique for the iterated collocation solution of the Hammerstein equation is discussed in Section 3. This provides an alternative way in which a post-processing technique can be used to improve the order of the convergence of numerical solution. Results in Section 3 play a critical role in establishing the global extrapolation method presented in Section 4.

The final two sections, Sections 5 and 6, are concerned with superconvergence of the Galerkin method by the post-processing techniques. In Section 5 we apply the interpolation technique explored in Section 2 and by Huang and Zhang [**3**] to the Galerkin method. We achieve a superconvergence result but the rate of acceleration is not as great as that of the collocation case of Section 2. In this connection, the reader is also referred to a recent paper by Huang and Xie, [**2**], which uses different interpolation operators under different regularity conditions to obtain a superconvergence of a higher order. Our result in Section 5 is consistent with a similar result for the Volterra equation reported in [**8**].

An extrapolation of the iterated Galerkin method is treated in the final section, Section 6. Results in Section 6 appear new even in the setting of the linear Fredholm equation.

In this paper, we consider the following Hammerstein equation:

$$u(t) - \int_0^1 k(t,s)\psi(s,u(s))\,ds = f(t), \quad t \in I \equiv [0,1], \tag{1.1}$$

where k, f and ψ are known functions and u is the function to be determined. We assume throughout the paper, unless otherwise stated, the following conditions on k, f and ψ.

1. $\lim_{t\to\tau} \|k_t - k_\tau\|_{0,\infty} = 0$, $\tau \in [0,1]$, (see (1.4) below for the definition of $\|\cdot\|_{0,\infty}$);

2. $\sup_t \int_0^1 |k(t,s)|\,ds < \infty$;

3. $f \in C[0,1]$;

4. $\psi(s,x)$ is continuous in $s \in [0,1]$ and Lipschitz continuous in $x \in (-\infty,\infty)$, i.e., there exists a constant $C_1 > 0$ for which

$$|\psi(s,x_1) - \psi(s,x_2)| \le C_1|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty,\infty); \tag{1.2}$$

5. The partial derivative $\psi^{(0,1)}$ of ψ with respect to the second variable exists and is Lipschitz continuous, i.e., there exists a constant $C_2 > 0$ such that

(1.3)
$$|\psi^{(0,1)}(s, x_1) - \psi^{(0,1)}(s, x_2)| \leq C_2|x_1 - x_2|, \quad \text{for all } x_1, x_2 \in (-\infty, \infty);$$

6. for $u \in C[0,1]$, $\psi(\cdot, u(\cdot))$, $\psi^{(0,1)}(\cdot, u(\cdot)) \in C[0,1]$.

Moreover, we introduce two norms

$$\|v\|_{m,\infty} \equiv \max_{0 \leq i \leq m} \left\{ \|v^{(i)}\|_\infty \right\} \tag{1.4}$$

and

$$\|v\|_{m,2} \equiv \left\{ \int_I \sum_{i=0}^{m} (v^{(i)}(x))^2 dx \right\}^{1/2}, \tag{1.5}$$

where m is a nonnegative integer.

2. Global superconvergence for Hammerstein equation by collocation method. As was stated in the introduction, a supercovergence of the collocation method via interpolation for the one-dimensional Hammerstein equation was recently obtained in [**3**]. Hence, the reader is referred to [**3**] for details. The purpose of this section is to demonstrate that the theory pertaining to the superconvergence, for most part, can be carried over to multi-variable Hammerstein equations. Through the numerical experiment reported below, interesting observations arose. For example, it is found that locations of the interpolation points are critical in achieving superconvergence for multi-variable equations. Additional issues relating to the post-processing of multi-variable integral equations will be discussed in a future paper. In order to better explain Example 2.4 below of a two-dimension Hammerstein equation, we make a brief review of the post-processing of the Hammerstein collocation method in the one-dimensional case. Let T^h be a partition of I:

$$0 = t_0 < t_1 < \cdots < t_N = 1$$

and $e_i \equiv [t_i, t_{i+1}]$, $h_i = t_{i+1} - t_i$ and $h = \max_i h_i$. We denote by S^h the space of piecewise polynomials of degree $\leq r$, i.e.,

$$S^h = \left\{ v \in L^2(I) : v|_{e_i} \in P_r, 0 \leq i \leq N-1 \right\},$$

where P_r denotes the space of all polynomials of degree $\leq r$. Let $B = B_{r+1}$ consist of zeroes of $r+1$ degree Legendre polynomials located in $[-1,1]$. Define $\Phi_i : [-1,1] \to e_i$, $i = 0, \ldots, N-1$, by

$$\Phi_i(t) = \frac{1+t}{2} t_{i+1} + \frac{1-t}{2} t_i, \quad t \in [-1,1],$$

and

$$A = \bigcup_{i=0}^{N-1} \Phi_i(B),$$

so that A contains the collocation points. The collocation approximation $u^h \in S^h$ is obtained under the assumption that the residual

$$R^h(t) \equiv u^h(t) - \int_0^1 k(t,s)\psi(s, u^h(s))\, ds - f(t)$$

disappears under the interpolation projection i_h^r of $C(I)$ onto S^h. Here $i_h^r : C(I) \to S^h$ is defined by

$$i_h^r u|_{e_i} \in P_r, \qquad i_h^r u(t) = u(t), \quad \text{for } t \in \Phi_i(B).$$

Equivalently,

$$u^h(t) - \int_0^1 k(t,s)\psi(s, u^h(s))\, ds = f(t), \quad \text{for all } t \in A. \tag{2.1}$$

To describe equations (1.1) and (2.1) in operator form, we let

$$K\Psi(u)(t) \equiv \int_0^1 k(t,s)\psi(s, u(s))\, ds, \quad t \in I,$$

and

$$\Psi(u)(s) = \psi(s, u(s)).$$

Then (1.1) and (2.1) can be written, respectively, as

$$u - K\Psi(u) = f, \tag{2.2}$$

and

$$u^h - i_h^r K\Psi(u^h) = i_h^r f. \tag{2.3}$$

An estimate on the size of $u^h - i_h^r u$, called a *superclose* identity, plays a vital role in establishing the superconvergence. It can be shown that, with

$$g(t, s, i_h^r u(s), u^h(s), \theta) \equiv k(t, s)\psi^{(0,1)}(s, i_h^r u(s) + \theta(u^h(s) - i_h^r u(s))),$$

where $0 < \theta < 1$ and

$$G_h u(s) \equiv \int_0^1 g(t, s, i_h^r u(s), u^h(s), \theta)u(s)\, ds,$$

assuming that 1 is not an eigenvalue of the operator G_h,

$$u^h - i_h^r u = (I - i_h^r G_h)^{-1} i_h^r K[\Psi(i_h^r u) - \Psi(u)]. \tag{2.4}$$

An estimate on the right side of (2.4) was investigated in [**5**] and, in the present setting, it is proved that

$$(I - i_h^r G_h)^{-1} i_h^r K[\Psi(i_h^r u) - \Psi(u)] = O(h^{2r+2}),$$

which gives the *superclose* identity

$$u^h - i_h^r u = O(h^{2r+2}). \tag{2.5}$$

See also [**3**] for additional discussions.

The remaining analysis for obtaining global superconvergence by interpolation post-processing technique is the same as that for the linear case. First, we obtain a collocation solution u^h over the partition T^h where it is assumed that the total number N of intervals is even. u^h is then interpolated at the collocation points over two consecutive

intervals $e_i \cup e_{i+1}$ by a polynomial of degree $2r+1$. In this connection, we define an interpolation operator I_{2h}^{2r+1} as follows:

$$I_{2h}^{2r+1}u\Big|_{e_i \cup e_{i+1}} \in P_{2r+1}, \quad i = 0, 2, \ldots N-2,$$

and

$$I_{2h}^{2r+1}u(t) = u(t), \quad t \in \Psi_i(B) \cup \Psi_{i+1}(B).$$

An application of the *superclose* identity (2.5) and the triangle inequality yields the estimate for the post-processed collocation solution via interpolation,

$$\|I_{2h}^{2r+1}u^h - u\|_{0,\infty} = O(h^{2r+2}). \tag{2.6}$$

We note that a similar estimate to (2.6) under the L^2 norm is also valid.

Example 2.1 below confirms the estimate described in (2.6). See [**3**] for additional examples. Throughout all numerical experiments reported below, we choose the piecewise linear polynomials to be the basis functions, i.e., $r = 1$. Moreover, corresponding nonlinear systems are solved by two different iteration schemes. The original Newton-Raphson iteration method is used first, followed by another approach, the quasi-Newton method which is obtained by

$$c_{k+1} = c_k - J^{-1}(c_{k'})F(c_k),$$

where c_k is unknown coefficient to be determined, k denotes the iteration step and k' is fixed. That is, the Jacobian is fixed throughout the iterations in the quasi-Newton method. A stopping criteria is taken within 14 digit accuracy, i.e., tolerance $\varepsilon < 10^{-14}$. The computer programs are run on a personal computer with 2.0GHz CPU and 4GB memory.

TABLE 1. The computational results of Example 2.1 by using collocation and interpolation techniques.

N	Collocation		Newton		quasi-Newton		Interpolation	
	e_h	R_h	NI	CT	NI	CT	$\overline{e}_h$	$\overline{R}_h$
2	7.5734e-2		5	0.12	7	0.09	9.0912e-2	
4	2.3643e-2	1.68	5	0.19	8	0.15	4.4949e-3	4.34
8	6.2950e-3	1.91	5	0.67	8	0.58	3.3793e-4	3.73
16	1.5982e-3	1.98	5	2.54	8	2.00	2.2050e-5	3.94
32	4.0108e-4	1.99	5	9.77	8	7.50	1.3928e-6	3.98
64	1.0037e-4	2.00	6	46.19	8	29.47	8.7283e-8	4.00
128	2.5098e-5	2.00	6	183.00	9	124.41	5.4588e-9	4.00

Numerical Example 2.1. Consider the equation

$$u(t) - \int_0^1 \sin(\pi(s+t))u^2(s)\,ds = f(t), \quad t \in [0,1]$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \sin(\pi t)$. The results are presented in Table 1. Notice that we defined NI to be the total number of iterations, and CT is the computing time of solving the collocation method of each scheme.

$$e_h = \|u - u^h\|_{0,\infty}, \qquad R_h = \log_2\left(\frac{e_h}{e_{h/2}}\right),$$
$$\overline{e}_h = ||u - I_{2h}^{2r+1}u^h||_{0,\infty}, \quad \overline{R}_h = \log_2\left(\frac{\overline{e}_h}{\overline{e}_{h/2}}\right).$$

We can see from Table 1 that, although the quasi-Newton method requires more iterations than Newton's method, the computing time of the quasi-Newton method is less. This difference is more pronounced in the Galerkin-based computation which will be presented in Sections 5 and 6.

In the case of Hammerstein equations with weakly singular kernels, the superconvergence result of (2.6) by the interpolation post-processing technique can also be obtained. Here, the kernel is assumed

to be of the type

$$k(t,s) = k^*\left(\frac{t}{s}\right)\frac{1}{s},$$

where, with $\sigma = t/s$ and D^ℓ denoting the differential operator of order ℓ,

$$\int_0^1 \sigma^\ell |D^\ell k^*(\sigma)| \frac{d\sigma}{\sigma} \leq \begin{cases} C_0 < 1 & \ell = 0, \\ C < \infty & \ell \geq 1. \end{cases}$$

It is known that the optimal order of convergence of the collocation method for weakly singular Hammerstein equations can be obtained by use of a graded mesh, see, e.g., [**1, 4**]. For example, if the spline of degree r is used in computation, one may select a partition T_h, $0 = t_0 < t_1 < \cdots < t_N = 1$, with $t_i = (i/N)^q$ and $q \geq r+1$, to preserve the optimal order of convergence. To attain a similar superconvergence result for the numerical solution of weakly singular equations by the interpolation post-processing technique, we simple select the partition by defining $t_i = (i/N)^q$, $q \geq 2r+2$, and perform post-processing by the interpolation described above over the intervals beginning at t_1. Note that a selection of $t_1 = N^{-q}$, $q \geq 2r+2$, guarantees the size of the first interval $[0, t_1]$ is small enough so that the approximation error from this interval is consistent with the errors from subsequent intervals despite the fact that the solution may not be differentiable over $[0, t_1]$. A numerical experiment is reported in [**3**] demonstrating the effectiveness of this approach using a constant basis. In what follows, we present two additional examples.

Numerical Example 2.2. Consider the equation

$$u(t) - \int_0^1 \log|s-t| u^2(s)\, ds = f(t), \quad t \in [0,1]$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$. The results are presented in Table 2.

TABLE 2. The computational results of Example 2.2 by using collocation and interpolation techniques.

N	Collocation		Interpolation	
	e_h	R_h	$\overline{e}_h$	$\overline{R}_h$
4	1.006500e-2		8.492456e-3	
8	2.606709e-3	1.9490	7.007576e-4	3.5992
16	6.518285e-4	1.9997	5.297109e-4	3.7256
32	1.628305e-4	2.0011	3.722810e-6	3.8307
64	4.069555e-5	2.0004	2.388927e-7	3.9620
128	1.017298e-5	2.0001	1.431229e-8	4.0610

TABLE 3. The computational results of Example 2.3 by using collocation and interpolation techniques.

N	Collocation		Interpolation	
	e_h	R_h	$\overline{e}_h$	$\overline{R}_h$
4	3.202584e-1		4.562162e-1	
8	8.081319e-3	5.3085	7.209956e-3	5.9836
16	1.035501e-3	2.9643	5.693164e-4	3.6627
32	1.897584e-4	2.4481	4.070014e-5	3.8061
64	4.280083e-5	2.1485	3.377003e-6	3.5912
128	1.038697e-5	2.0429	2.746804e-7	3.6199

Numerical Example 2.3. Consider the equation

$$u(t) - \int_0^1 \frac{1}{\sqrt{|s-t|}} u^2(s)\, ds = f(t), \quad t \in [0,1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = t^2$. The results are presented in Table 3.

Now we are ready to exhibit an example of multi-variable Hammerstein equations to demonstrate that the post-processing technique

based upon the interpolation can be applied to a multi-variable Hammerstein equation. For $\overline{t}, \overline{s} \in R^2$, we consider

$$u(\overline{t}) - \int_0^1 \int_0^1 k(\overline{t}, \overline{s}) \psi(\overline{s}, u(\overline{s}))\, d\overline{s} = f(\overline{t}), \quad \overline{t}, \overline{s} \in I \times I. \tag{2.7}$$

We use the tensor product of S^h with itself, $S^h \otimes S^h$, as our approximating space. Note that $\cup_{h>0}(S^h \otimes S^h)$ is essentially dense in $C(I \times I)$, see [**6**]. With $\overline{i}_h^r \equiv i_h^r \otimes i_h^r$, $\overline{i}_h^r : C(I \times I) \to S^h \otimes S^h$ satisfies

$$\overline{i}_h^r u|_{e_i \times e_j} \in P_r \otimes P_r, \quad \overline{i}_h^r u(\overline{t}) = u(\overline{t}), \text{ for } \overline{t} \in \Phi_i(B) \times \Phi_j(B).$$

The collocation method is to solve

$$u^h(\overline{t}) - \int_0^1 \int_0^1 k(\overline{t}, \overline{s}) \psi(\overline{s}, u^h(\overline{s}))\, d\overline{s} = f(\overline{t}), \quad \overline{t} \in \cup_{0 \le i,j \le N-1} \Phi_i(B) \times \Phi_j(B). \tag{2.8}$$

Once $u^h(\overline{t})$ is obtained, one may interpolate its values at the collocation points over four squares $e_i \cup e_{i+1} \times e_j \cup e_{j+1}$ by the two-dimensional polynomial in the form

$$\begin{aligned} u^h(t, t') = {} & a_1 + a_2 t' + a_3 t + a_4 t\, t' + a_5 t'^2 2 + a_6 t^2 + a_7 t\, t'^2 \\ & + a_8 t^2\, t' + a_9 t'^3 + a_{10} t^3. \end{aligned} \tag{2.9}$$

Here, recall that linear splines are used to discretize the solution in each direction. To double the order of accuracy, we require a polynomial of degree 3 in two variables t, t' as indicated in (2.9). Our numerical experiment indicates that the location of these ten interpolation points influences much in achieving the superconvergence of a desired accuracy. More discussions on the post-processing technique for multi-variable Hammerstein equations will be made in a future paper.

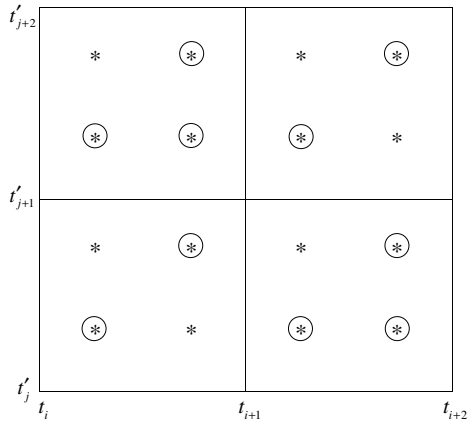

FIGURE 1. Location of ten interpolating points.

TABLE 4. The computational results of Example 2.4 by using collocation and interpolation techniques.

(N,N)	NU	Collocation		Interpolation	
		e_h	R_h	$\overline{e}_h$	$\overline{R}_h$
(2,2)	16	2.604955e-1		5.303465e-2	
(4,4)	64	7.078941e-2	1.8797	4.782085e-3	3.4712
(8,8)	256	1.845359e-2	1.9396	3.604376e-4	3.7298
(16,16)	1024	4.710457e-3	1.9700	2.412651e-5	3.9011
(32,32)	4096	1.189914e-3	1.9850	1.376186e-6	4.1319

Numerical Example 2.4. Consider the following two-dimensional equation

$$u(t,t') - \int_0^1 \int_0^1 (t-s)(t'-s')(s+s'+u(s,s'))^2 ds\, ds' = f(t,t'),$$
$$(t,t') \in I \times I$$

where f is chosen so that the exact solution is $u(t,t') = \exp(t+t')$. Ten points are selected from four contiguous squares $e_i \cup e_{i+1} \times e_j \cup e_{j+1}$,

$i, j = 0, 2, \ldots, 2N-2$, and they are circled in Figure 1, or $(t^*_{i,1}, t'^*_{j,1})$, $(t^*_{i+1,1}, t'^*_{j,1})$, $(t^*_{i+1,2}, t'^*_{j,1})$, $(t^*_{i,2}, t'^*_{j,2})$, $(t^*_{i+1,2}, t'^*_{j,2})$, $(t^*_{i,1}, t'^*_{j+1,1})$, $(t^*_{i,2}, t'^*_{j+1,1})$, $(t^*_{i+1,1}, t'^*_{j+1,1})$, $(t^*_{i,2}, t'^*_{j+1,2})$ and $(t^*_{i+1,2}, t'^*_{j+1,2})$.

In Table 4, NU is the total number of unknown coefficients to be determined. With the ten interpolation points described, our numerical experiment confirms the same superconvergence as the one-dimensional problem in Example 2.1.

3. Extrapolation of iterated collocation solution for Hammerstein equations. In this section, we generalize the result obtained in [**7**] concerning an extrapolation technique for the iterated collocation method. The iterated collocation solution u^h_{it} for the Hammerstein equation is defined as follows:

For a collocation solution u^h of (2.1),

$$u^h_{it}(t) = f(t) + \int_0^1 k(t,s)\psi(s, u^h(s))\, ds,$$

or, in operator form,

$$u^h_{it} = f + K\Psi(u^h). \tag{3.1}$$

From equation (3.1),

$$i^r_h u^h_{it} = i^r_h f + i^r_h K\Psi(u^h), \tag{3.2}$$

and (2.3) and (3.2) yield

$$i^r_h u^h_{it} = u^h.$$

Thus, (3.1) becomes

$$u^h_{it} = f + K\Psi(i^r_h u^h_{it}). \tag{3.3}$$

It is proved in [**4**] that if $f \in C^{2r+2}(I)$ and $k \in C^{2r+2}(I \times I)$, then

$$\left\| u - u^h_{it} \right\|_{0,\infty} = O\left(h^{2r+2}\right).$$

Since $\|u - u^h\|_{0,\infty} = O(h^{r+1})$, the iterated solution converges at a rate twice as fast as the collocation solution.

The following theorem, which generalizes Theorem 1 of [**7**], establishes the basic fact which underlines the extrapolation technique for the numerical solution of Hammerstein equation.

Theorem 3.1. *Assume that* $k \in C^{r+3}([0,1] \times [0,1])$ *and the solution* u *of* (1.1) *satisfies* $u \in C^{2r+4}[0,1]$. *Also assume that* 1 *is not an eigenvalue of the linear operator* $(K\Psi)'(u)$. *Then there exists a* $b \in C([0,1] \times [0,1])$, *independent of the partition, such that*

$$u(t) - u_{it}^h(t) = \sum_{k=1}^{N} h_k^{2r+2} \int_{e_k} b(t,s)\, ds + O\left(h^{2r+4}\right), \quad t \in [0,1].$$

Proof. From (2.2) and (3.3),

$$\begin{aligned} u - u_{it}^h &= K\Psi(u) - K\Psi(i_h^r u_{it}^h) \\ &= K\Psi(u) - K\Psi(i_h^r u) + K\Psi(i_h^r u) - K\Psi(i_h^r u_{it}^h). \end{aligned} \tag{3.4}$$

Now, recall from the previous section,

$$G_h u(s) \equiv \int_0^1 g(t, s, i_h^r u(s), i_h^r u_{it}^h(s), \theta) u(s)\, ds,$$

where g is also defined in Section 2. Then

$$K\Psi(i_h^r u) - K\Psi(i_h^r u_{it}^h) = G_h i_h^r (u - u_{it}^h).$$

Equation (3.4) becomes

$$u - u_{it}^h = K\Psi(u) - K\Psi(i_h^r u_{it}^h) + G_h i_h^r (u - u_{it}^h). \tag{3.5}$$

Arguing as in [**4**] and using assumptions 2, 5 and 6, we can show that $\{G_h i_h^r\}$ is a family of collectively compact operators and $G_h i_h^r \to G \equiv (K\Psi)'(u)$ pointwise as $h \to 0$. Since G is compact and $(I - G)^{-1}$ exists by assumption, from a theory of compact operators (see, e.g., [**1**]), $(I - G_h i_h^r)^{-1}$ exists and is uniformly bounded. This shows that

$$\|u - u_{it}^h\|_{0,\infty} \le C \|K\Psi(u) - K\Psi(i_h^r u)\|_{0,\infty} = O(h^{2r+2}), \tag{3.6}$$

which establishes the superconvergence of the iterated collocation solution [4]. For our present purposes, we require the following. Since

$$(3.7)\quad (I-G_h i_h^r)^{-1} = (I-G_h)^{-1} - (I-G_h i_h^r)^{-1} G_h (I-i_h^r)(I-G_h)^{-1},$$

using (3.5) and (3.7), we obtain

(3.8)

$$\begin{aligned} u - u_{it}^h &= (I - G_h i_h^r)^{-1}\{K\Psi(u) - K\Psi(i_h^r u)\} \\ &= (I - G_h)^{-1} w^h - (I - G_h i_h^r)^{-1} G_h (I - i_h^r)(I - G_h)^{-1} w^h \\ &= v^h - (I - G_h i_h^r)^{-1} G_h (I - i_h^r) v^h, \end{aligned}$$

where

$$w^h \equiv K\Psi(u) - K\Psi(i_h^r u)$$

and

$$v^h \equiv (I - G_h)^{-1} w^h = (I - G_h)^{-1} K[\Psi(u) - \Psi(i_h^r u)].$$

Let $L \equiv (I - G_h)^{-1} K$ where L is an integral operator with a kernel $l^*(s,t)$ with the same smoothness properties as $k(s,t)$. Then

$$v^h(t) = L[\Psi(u) - \Psi(i_h^r u)] = \int_0^1 l^*(s,t)[\psi(s,u(s)) - \psi(s, i_h^r u(s))]\, ds.$$

Using the mean value theorem, when $0 < \theta < 1$, we have

$$\begin{aligned} v^h(t) &= \int_0^1 l^*(s,t) \frac{\partial \psi}{\partial u}\left(s, (u + \theta(i_h^r u - u))(s)\right)(u - i_h^r u)(s)\, ds \\ &= \int_0^1 l(s,t)(u - i_h^r u)(s)\, ds = \sum_{k=1}^{N} \int_{E_k} l(s,t)(u - i_h^r u)(s)\, ds \end{aligned}$$

where $l(s,t) \equiv l^*(s,t)(\partial\psi/\partial u)(s, (u + \theta(i_h^r u - u))(s))$.

By applying the results of Lemma 3 of [7] to each subinterval E_k (and noting that the change of scale introduce a factor $(h_k/2)^j$ for the jth derivative), we obtain

$$\begin{aligned} v^h(t) = \sum_{k=1}^{N} \left(\frac{h_k}{2}\right)^{2r+2} \sum_{\substack{i=r+1 \\ j+1=2r+2}}^{2r+2} c_{ji} \int_{E_k} D_s^j l\,(s,t)\, D^i u(s)\, ds \\ + O\left(h^{2r+4}\right) \left\| D^{2r+4} u \right\|_{0,\infty} \end{aligned}$$

where D_s denotes the partial derivative with respect to s. The result may be rewritten as

$$v^h(t) = \sum_{k=1}^{N} h_k^{2r+2} \int_{E_k} b(s,t)ds + O(h^{2r+4}), \tag{3.9}$$

where

$$b(s,t) = 2^{-(2r+2)} \sum_{\substack{i=r+1 \\ j+1=2r+2}}^{2r+2} c_{ji} D_s^j l\,(s,t)\, D^i u(s)$$

Also,

$$\begin{aligned} \|(I - G_h i_h^r)^{-1} G_h (I - i_h^r) v^h\|_{0,\infty} &\le C \|G_h (I - i_h^r) v^h\|_{0,\infty} \\ &\le C \|(I - i_h^r) v^h\|_{0,\infty} \\ &\le C h^2 \|v^h\|_{2,\infty} = O(h^{2r+4}). \end{aligned} \tag{3.10}$$

Here, the first two derivatives of v^h are taken using (3.9); and, arguing in the same way, we see that $\|v^h\|_{2,\infty} = O(h^{2r+2})$. Equations (3.8)–(3.10) give the desired result. □

Theorem 3.1 lends naturally to an extrapolation of the iterated collocation method for Hammerstein equation. Let $T^{h/2}$ be a partition of I:

$$0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-1/2} < t_N = 1$$

where

$$t_{k-1/2} = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \ldots, N.$$

Let $u^{h/2}$ and $u_{it}^{h/2}$ denote the collocation and iterated collocation approximation for the Hammerstein equation with respect to this new partition. Theorem 3.1 yields

$$u(t) - u_{it}^{h/2}(t) = 2^{-(2r+2)} \sum_{k=1}^{N} h_k^{2r+2} \int_{e_k} b(t,s)\, ds + O(h^{2r+4}).$$

Richardson extrapolation gives a new approximation

$$\overline{u}_{it}^{h/2}(t) \equiv \frac{2^{2r+2} u_{it}^{h/2} - u_{it}^{h}(t)}{2^{2r+2} - 1}.$$

It is straightforward that

$$u(t) - \overline{u}_{it}^{h/2}(t) = O(h^{2r+4}). \tag{3.11}$$

Numerical Example 3.1. Consider the equation

$$u(t) - \int_0^1 \exp(\pi(s-t))u^2(s)\,ds = f(t), \quad t \in [0,1]$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \cos(t)$. The results are presented in Table 5. Notice that we defined

$$\widehat{e}_h = \|u - u_{it}^h\|_{0,\infty}, \quad \widehat{R}_h \quad = \log_2\left(\frac{\widehat{e}_h}{\widehat{e}_{h/2}}\right)$$

$$\widehat{\overline{e}}_h = \|u - \overline{u}_{it}^{h/2}\|_{0,\infty}, \text{ and } \widehat{\overline{R}}_h = \log_2\left(\frac{\widehat{\overline{e}}_h}{\widehat{\overline{e}}_{h/2}}\right).$$

4. Global extrapolation for Hammerstein equations. Theorem 3.1 once again plays a critical role in establishing another method of improving the accuracy of numerical solution of the Hammerstein equation. Here, we examine a global extrapolation method for the Hammerstein equations. From (3.9),

$$\begin{aligned} v^h(t) &= \sum_{k=1}^{N} h_k^{2r+2} \int_{E_k} b(s,t)\,ds + O\left(h^{2r+4}\right) \\ &= h^{2r+2} \sum_{k=1}^{N} \left(\frac{h_k}{h}\right)^{2r+2} \int_{E_k} b(s,t)\,ds + O\left(h^{2r+4}\right) \\ &= h^{2r+2} w(t) + O(h^{2r+4}), \end{aligned}$$

where

$$w(t) \equiv \sum_{k=1}^{N} \left(\frac{h_k}{h}\right)^{2r+2} \int_{E_k} b(s,t)\,ds.$$

Equivalently,

$$(I - G_h)^{-1} G_h (u - i_h^r u)(t) = h^{2r+2} w(t) + O\left(h^{2r+4}\right). \tag{4.1}$$

TABLE 5. The computational results of Example 3.1 by using collocation and post-processing techniques.

N	Collocation		Newton		quasi-Newton		Post-Processing			
	e_h	R_h	NI	CT	NI	CT	$\widehat{e}_h$	$\widehat{R}_h$	$\widehat{\widehat{e}}_h$	$\widehat{\widehat{R}}_h$
2	1.1667e-2		5	0.11	10	0.08	1.2547e-3			
4	2.7077e-3	2.11	5	0.19	10	0.19	7.1920e-5	4.12	1.7389e-5	
8	6.5885e-4	2.04	5	0.67	10	0.64	4.4079e-6	4.03	2.3326e-7	6.22
16	1.6330e-4	2.01	5	2.48	10	2.29	2.7419e-7	4.01	3.4813e-9	6.07
32	4.7258e-5	1.79	5	9.64	10	8.58	1.7117e-8	4.00	5.3746e-11	6.02
64	1.0175e-5	2.21	5	38.17	10	33.26	1.0695e-9	4.00	8.4259e-13	5.99
128	2.5433e-6	2.00	5	151.04	10	131.60	6.6836e-11	4.00	1.7541e-14	5.59

Applying (2.4) and using the fact that

$$(I - i_h^r G_h)^{-1} = (I - G_h)^{-1} - (I - i_h^r G_h)^{-1}(I - i_h^r)G_h(I - G_h)^{-1},$$

we get

(4.2)

$$\begin{aligned} u^h - i_h^r u &= (I - G_h)^{-1} i_h^r G_h (i_h^r u - u) \\ &\quad - (I - i_h^r G_h)^{-1}(I - i_h^r)G_h(I - G_h)^{-1} i_h^r G_h (i_h^r u - u) \\ &= (I - G_h)^{-1} G_h (i_h^r u - u) + (I - G_h)^{-1}(I - i_h^r)G_h(u - i_h^r u) \\ &\quad - (I - i_h^r G_h)^{-1}(I - i_h^r)G_h(I - G_h)^{-1} i_h^r G_h (i_h^r u - u) \end{aligned}$$

By virtue of the fact that $u - i_h^r u = 0$ on Gaussian points and arguing similarly to [**8**], it can be shown that

$$(I - G_h)^{-1}(I - i_h^r)G_h(u - i_h^r u) = O(h^{3r+3}), \tag{4.3}$$

and

$$(I - i_h^r G_h)^{-1}(I - i_h^r)G_h(I - G_h)^{-1} i_h^r G_h (i_h^r u - u) = O\left(h^{3r+3}\right). \tag{4.4}$$

Equations (4.1)–(4.4) yield the *superclose* identity with asymptotic error terms.

$$u^h(t) - i_h^r u(t) = \begin{cases} h^{2r+2} w(t) + O(h^{2r+4}) & r \geq 1, \\ h^2 w(t) + O(h^3) & r = 0. \end{cases} \tag{4.5}$$

Equations in (4.5) naturally lead to the following extrapolation method. The theory follows in exactly the same way as the one given in [**8**]. We include it for completeness. Let N be the number of elements of T^h, and assume that it is a multiple of 3. Define an interpolation operator I_{3h}^{2r+3} mapping into a space of polynomials of degree $2r+3$, $r \geq 1$, as follows:

$$I_{3h}^{2r+3} u|_{e_{i-1} \cup e_i \cup e_{i+1}} \in P_{2r+3}, \quad i = 3\ell + 1, \ell = 0, 1, \ldots, \frac{N}{3} - 1,$$
$$I_{3h}^{2r+3} u(t) = u(t), \qquad t \in \Phi_{i-1}(B) \cup \Phi_{i+1}(B) \cup \{s_i^0, s_i^r\}$$

where $\Phi_i(B) = \{s_i^0, \ldots, s_i^r\}$. Using (4.5) and

$$I_{3h}^{2r+3} i_h^r = I_{3h}^{2r+3},$$

and arguing as in [**8**], we obtain

$$I_{3h}^{2r+3} u^h - u = h^{2r+2} w + O\left(h^{2r+4}\right). \tag{4.6}$$

Equation (4.6) leads naturally to a global extrapolation method for the solution of the Hammerstein equation. In order to implement the global extrapolation, let $S^{h/2}$ be the space of piecewise polynomials of degree less than or equal to r with partition points

$$T^{h/2}: 0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-(1/2)} < t_N = 1,$$

where

$$t_{i-(1/2)} = \frac{t_{i-1} + t_i}{2}, \quad i = 1, \ldots, N.$$

Denote the collocation approximation and interpolation operator of degree $2r+3$ with respect to the partition $T^{h/2}$ by $u^{h/2}$ and $I_{3h/2}^{2r+3}$ so that

$$I_{3h/2}^{2r+3} u^{h/2}(t) - u(t) = \left(\frac{h}{2}\right)^{2r+2} w(t) + O\left(h^{2r+4}\right).$$

The standard Richardson extrapolation gives an approximation with higher order of accuracy, namely

$$I_{3h/2}^{2r+3} \overline{u}^{h/2}(t) - u(t) = O\left(h^{2r+4}\right),$$

where

$$I_{3h/2}^{2r+3}\overline{u}^{h/2}(t) \equiv \frac{2^{2r+2}I_{3h/2}^{2r+3}u^{h/2}(t) - I_{3h}^{2r+3}u^{h}(t)}{2^{2r+2}-1}.$$

Numerical Example 4.1. Consider the equation

$$u(t) - \int_0^1 \exp(s-t)u^2(s)\,ds = f(t), \quad t \in [0,1],$$

where $f(t)$ is chosen so that the exact solution is $u(t) = \exp(t)$. The results are presented in Table 6. Notice that we defined

$$\widetilde{e}_h = \|u - I_{3h}^{2r+3}u\|_{0,\infty}, \qquad \widetilde{R}_h = \log_2\left(\frac{\widetilde{e}_h}{\widetilde{e}_{h/2}}\right)$$

$$\widehat{\widetilde{e}}_h = \|u - I_{3h}^{2r+3}\overline{u}^{h/2}\|_{0,\infty}, \quad \text{and} \quad \widehat{\widetilde{R}}_h = \log_2\left(\frac{\widehat{\widetilde{e}}_h}{\widehat{\widetilde{e}}_{h/2}}\right).$$

5. Global superconvergence for Hammerstein equations by the Galerkin method. In this section, we examine global superconvergence of the post-processed Galerkin method by interpolation. In other words, we apply the technique in Section 2 to the Galerkin method. We denote by P_h the orthogonal projection of $L^2(I)$ onto S^h. More precisely,

$$(u - P_h u, v) = 0, \quad \text{for all } v \in S^h. \tag{5.1}$$

Then the Galerkin method in solving (2.5) can be written as

$$u^h - P_h K\Psi(u^h) = P_h f, \quad u^h \in S^h. \tag{5.2}$$

The weak forms of (2.2) and (5.2) are

$$(u, v) - (K\Psi(u), v) = (f, v), \quad \text{for all } v \in L^2(I), \tag{5.3}$$

and

$$(u^h, v) - (K\Psi(u^h), v) = (f, v), \quad \text{for all } v \in S^h. \tag{5.4}$$

TABLE 6. The computational results of Example 4.1 by using collocation and post-processing techniques.

N	Collocation		Newton		quasi-Newton		Post-Processing			
	e_h	R_h	NI	CT	NI	CT	$\widetilde{e}_h$	$\widetilde{R}_h$	$\widehat{\widetilde{e}}_h$	$\widehat{\widetilde{R}}_h$
3	2.2399e-2		6	0.20	13	0.14	4.2392e-4			
6	5.9444e-3	1.91	6	0.44	13	0.45	2.7526e-5	3.94	4.8714e-7	
12	1.5295e-3	1.96	6	1.70	14	1.65	1.7389e-6	3.98	7.7383e-9	5.98
24	3.8781e-4	1.98	6	6.38	14	6.08	1.0906e-7	3.99	1.2189e-10	5.99
48	9.7635e-5	1.99	6	24.82	14	23.23	6.8247e-9	4.00	1.9062e-12	6.00
96	2.4494e-5	1.99	6	98.93	14	91.31	4.2677e-10	4.00	3.2196e-14	5.89

Using (5.3) and (5.4) along with (5.1), it is obtained that

$$(u^h - P_h u, v) - (K\Psi(u^h) - K\Psi(P_h u), v) = (K\Psi(P_h u) - K\Psi(u), v), \quad \text{for all } v \in S^h,$$

which can be further reduced to

(5.5)
$$(u^h - P_h u, v) - ((K\Psi)'(\xi)(u^h - P_h u), v) = (K\Psi(P_h u) - K\Psi(u), v), \quad \text{for all } v \in S^h,$$

where $\xi = \theta u^h + (1-\theta)P_h u$ for some $\theta \in (0,1)$. Standing conditions 4–6 described in Section 2 guarantee that $(K\Psi)'(\xi)$ is a compact linear operator. Since $P_h \to I$ pointwise as $h \to 0$, a standard argument shows that $(I - P_h(K\Psi)'(\xi))^{-1}$ exists for sufficiently small h. Using the strong form of (5.5) and its rearrangement of terms, we see with $\xi_1 = \theta P_h u + (1-\theta)u$ that

(5.6)
$$\begin{aligned} u^h - P_h u &= (I - P_h(K\Psi)'(\xi))^{-1}[P_h(K\Psi)'(\xi_1)(P_h u - u)] \\ &= (I - K\Psi'(\xi))^{-1} K\Psi'(\xi_1)(P_h u - u) \\ &\quad + (I - K\Psi'(\xi_1))^{-1}(I - P_h)K\Psi'(\xi_1)(u - P_h u) \\ &\quad - (I - P_h K\Psi'(\xi_1))^{-1}(I - P_h)K\Psi'(\xi_1)(I - K\Psi'(\xi_1))^{-1} \\ &\quad \times P_h K\Psi'(\xi_1)(P_h u - u). \end{aligned}$$

Conditions 4–6 once again guarantee that $K\Psi'(\xi_1)$ is a compact linear operator, and we assume that it is in the form

$$K\Psi'(\xi_1)u(t) = \int_I k^*(t,s)u(s)\,ds,$$

with $k^* \in C^{r+2}(I \times I)$. Since, for each $t \in I$,

$$\int_I P_h(s)k^*(t,s)(P_hu-u)(s)\,ds = 0,$$

we obtain

$$\begin{aligned} K\Psi'(\xi_1)(P_hu-u) &= \sum_{i=0}^{N-1}\int_{e_i} k^*(t,s)(P_hu-u)(s)\,ds \\ &= \sum_{i=0}^{N-1}\int_{e_i}(I-P_h(s))k^*(t,s)(P_hu-u)(s)\,ds \\ &= O(h^{r+2})\|u\|_{r+1,\infty}. \end{aligned}$$

Hence,

$$\|(I-K\Psi'(\xi))^{-1}K\Psi'(\xi_1)(P_hu-u)\|_{0,q} = O(h^{r+2})\|u\|_{r+1,q}, \tag{5.7}$$

where $q = 2, \infty$. For the second and third terms in (5.6), we proceed as follows:

$$\begin{aligned} &\|(I-K\Psi'(\xi_1))^{-1}(I-P_h)K\Psi'(\xi_1)(u-P_hu)\|_{0,q} \\ &\qquad \le C\|(I-P_h)K\Psi'(\xi_1)(u-P_hu)\|_{0,q} \\ &\qquad \le Ch\|k^*(t,s)\|_{1,q}\|u-P_hu\|_{0,q} \le Ch^{r+2}\|u\|_{r+1,q} \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &\|(I-P_hK\Psi'(\xi_1))^{-1}(I-P_h)K\Psi'(\xi_1)(I-K\Psi'(\xi_1))^{-1} \\ &\qquad \times P_hK\Psi'(\xi_1)(P_hu-u)\|_{0,q} \\ &\qquad \le Ch\|(I-K\Psi'(\xi_1))^{-1}P_hK\Psi'(\xi_1)(P_hu-u)\|_{0,q} \\ &\qquad \le Ch\|P_hu-u\|_{0,q} \\ &\qquad \le Ch^{r+2}\|u\|_{r+1,q}. \end{aligned} \tag{5.9}$$

When (5.7), (5.8) and (5.9) are combined with (5.6), we obtain

$$\|u^h - P_hu\|_{0,q} = O\left(h^{r+2}\right)\|u\|_{r+1,\infty}. \tag{5.10}$$

In order to utilize (5.10) and obtain a global superconvergence of the Galerkin method by interpolation, it is necessary to define an interpolation operator I_{2h}^{r+1} as follows. In relation to the mesh T^{2h},

$$I_{2h}^{r+1}u|_{e_i\cup e_{i+1}} \in P_{r+1}, \quad i = 0, 2, \ldots, N-2,$$

such that

$$\int_{e_i} I_{2h}^{r+1}u\,ds = \int_{e_i} u\,ds, \qquad \int_{e_{i+1}} I_{2h}^{r+1}u\,ds = \int_{e_{i+1}} u\,ds$$

and

$$\int_{e_i\cup e_{i+1}} vI_{2h}^{r+1}u\,ds = \int_{e_i\cup e_{i+1}} vu\,ds, \quad \text{for all } v \in P_r(e_i \cup e_{i+1}).$$

Using

$$\begin{aligned} I_{2h}^{r+1}P_h &= I_{2h}^{r+1}, \\ \|I_{2h}^{r+1}v\|_{0,q} &\le C\|v\|_{0,q}, \quad \text{for all } v \in S^h, \\ \|I_{2h}^{r+1}v - v\|_{0,q} &\le Ch^{r+2}\|v\|_{r+2,q}, \end{aligned}$$

with $q = 2, \infty$. The global superconvergence of the Galerkin method by interpolation for the Hammerstein equation is now attained from

$$\begin{aligned} \|I_{2h}^{r+1}u^h - u\|_{0,q} &\le \|I_{2h}^{r+1}u - I_{2h}^{r+1}P_hu\|_{0,q} + \|I_{2h}^{r+1}P_hu - u\|_{0,q} \\ &\le C\|u^h - P_hu\|_{0,q} + \|I_{2h}^{r+1}u - u\|_{0,q} \\ &= O(h^{r+2})(\|u\|_{r+1,\infty} + \|u\|_{r+2,q}). \end{aligned}$$

Numerical Example 5.1. Consider the equation

$$u(t) - \int_0^1 stu^2(s)\,ds = f(t), \quad t \in [0,1]$$

TABLE 7. The computational results of Example 5.1 by using Galerkin and interpolation techniques.

N	Galerkin		Newton		quasi-Newton		Interpolation	
	e'_h	R'_h	NI	CT	NI	CT	$\overline{e'}_h$	$\overline{R'}_h$
2	1.6482e-2		5	0.11	7	0.06	2.1565e-3	
4	4.1484e-3	1.99	5	0.26	7	0.17	1.8942e-4	3.51
8	1.0398e-3	2.00	5	1.03	7	0.69	2.0170e-5	3.23
16	2.6013e-4	2.00	5	3.98	7	2.62	2.3970e-6	3.07
32	6.5044e-5	2.00	5	15.57	7	10.30	2.9560e-7	3.02
64	1.6262e-5	2.00	5	61.70	7	40.95	3.6823e-8	3.00
128	4.0655e-6	2.00	8	396.36	7	164.33	4.5988e-9	3.00

where $f(t)$ is chosen so that the exact solution is $u(t) = \exp(t)$. The results are presented in Table 7. Notice that we defined

$$e'_h = \|u - u_h\|_{0,2}, \qquad R'_h = \log_2\left(\frac{e'_h}{e'_{h/2}}\right)$$

$$\overline{e'}_h = \|u - I_{2h}^{r+1} u^h\|_{0,2} \qquad \overline{R'}_h = \log_2\left(\frac{\overline{e'}_h}{\overline{e'}_{h/2}}\right)$$

We point out a much shorter computational time with a quasi-Newton algorithm for the Galerkin method compared with the collocation methods in earlier sections.

6. Extrapolation of iterated Galerkin solutions for the Hammerstein equation. In this final section, we explore the extrapolation technique developed in Section 3 for the iterated collocation method for Hammerstein equations and extend it to further accelerate the rate of convergence of the iterated Galerkin method. The results reported in this section appear new even for linear Fredholm equations. The iterated Galerkin solution, $\overline{u}_{it}^h$, is obtained by

$$\overline{u}_{it}^h = f + K\Psi(u^h), \tag{6.1}$$

where u^h is the solution of the Galerkin method (see (5.2)),

$$u^h - P_h K\Psi(u^h) = P_h f, \quad u^h \in S^h,$$

and P_h is the orthogonal projection of $L^2(I)$ onto S^h. From (6.1),

$$P_h \overline{u}_{it}^h = P_h f + P_h K \Psi(u^h). \tag{6.2}$$

From (6.2) and (5.2), we see that

$$P_h \overline{u}_{it}^h = u^h. \tag{6.3}$$

It is shown in [**4**] that if $f \in C^{2r+2}(I)$ and $k \in C^{2r+2}(I)$, then

$$\|u - \overline{u}_{it}^h\|_{0,2} = O\left(h^{2r+2}\right).$$

In order to successfully complete the current extrapolation method, it is necessary to establish an asymptotic error expansion for the iterated Galerkin solution which is analogous to Theorem 3.1. A proof can be made similar to the proof of Theorem 3.1 but the interpolation projection i_h^r must be replaced by the orthogonal projection P_h.

Theorem 6.1. *Assume that $k \in C^{2r+4}([0,1] \times [0,1])$ and the solution u of (2.1) satisfies $u \in C^{2r+4}[0,1]$. Also assume that 1 is not an eigenvalue of the linear operator $(K\Psi)'(u)$. Then there exists a $b \in C([0,1] \times [0,1])$, independent of the partition, such that*

$$u(t) - \overline{u}_{it}^h(t) = \sum_{k=1}^{N} h_k^{2r+2} \int_{e_k} b(t,s)\, ds + O(h^{2r+4}), \quad t \in [0,1].$$

Theorem 6.1 is based upon the following lemma.

Lemma 6.2. *Assume that $\chi, z \in C^{r+4}[-1,1]$, and let P_h be the orthogonal projection of $L^2[-1,1]$ onto S_h. Then there exists a constant $c_{r+1\,r+1}$, independent of χ and z, such that*

$$\int_{-1}^{1} \chi(z - P_h z)\, ds = c_{r+1\,r+1} \int_{-1}^{1} D^{r+1}\chi D^{r+1} z\, ds + O(1) \sum_{i+j \geq 2r+4} \|D^j \chi\|_{0,2} \|D^i z\|_{0,2}.$$

Proof. Expand χ and z in the Maclaurin series to get

$$\chi(s) = \sum_{j=0}^{r+3} \frac{1}{j!} D^j \chi(0) \phi_j + O\left(D^{r+4}\chi\right),$$
$$z(s) = \sum_{i=0}^{r+3} \frac{1}{i!} D^i z(0) \phi_i + O\left(D^{r+4} z\right)$$

where $\phi_j(s) = s^j$. Since $P_h : L^2[-1,1] \to S^h$, $P_h \phi_j = \phi_j$ for $0 \le j \le r$, and thus

$$z - P_h z = \sum_{i=r+1}^{r+3} \frac{1}{i!} D^i z(0) (\phi_i - P_h \phi_i) + O\left(D^{r+4} z\right).$$

Also noting that

$$\int_{-1}^{1} \varphi(s)(z - P_h z)(s)\, ds = 0, \quad \text{for all } \varphi \in S^h,$$

$$\begin{aligned}
(6.4)\quad & \int_{-1}^{1} \chi(s)(z - P_h z)(s)\, ds \\
&= \sum_{j=r+1}^{r+3} \sum_{i=r+1}^{r+3} \frac{1}{i!j!} D^j \chi(0) D^i z(0) \int_{-1}^{1} \phi_j(s)(\phi_i - P_h \phi_i)(s)\, ds \\
&\quad + O(1)\Bigg(\|D^{r+4} z\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j \chi\|_{0,2} \\
&\qquad\qquad + \|D^{r+4}\chi\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j z\|_{0,2} \Bigg) \\
&= \sum_{j=r+1}^{r+3} \sum_{i=r+1}^{r+3} c_{ij} D^j \chi(0) D^i z(0) \\
&\quad + O(1)\Bigg(\|D^{r+4}\chi\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j z\|_{0,2} \\
&\qquad\qquad + \|D^{r+4}\chi\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j z\|_{0,2} \Bigg),
\end{aligned}$$

where $c_{ij} = 1/(i!j!)\int_{-1}^{1} \phi_j(s)(\phi_i - P_h\phi_i)(s)\,ds$. Note that, in the first term of the last expression, $c_{r+1\,r+2} = c_{r+2\,r+1} = 0$. To see this, note that, if $r+1$ is odd, then $\phi_{r+1} - P_h\phi_{r+1}$ is also odd, since, with $(u,v) \equiv \int_{-1}^{1} uv\,ds$,

$$P_h\phi_{r+1} = \sum_{i=0}^{r} b_i\phi_i, \quad \text{where } b_i = \frac{(\phi_{r+1},\phi_i)}{(\phi_i,\phi_i)},$$

and thus $b_i = 0$ whenever i is even for in this case $\phi_{r+1}\phi_i$ becomes an odd function. Hence, $P_h\phi_{r+1}$ is odd, and thus $\phi_{r+1} - P_h\phi_{r+1}$ is also odd. Under the assumption that $r+1$ is odd, ϕ_{r+2} an even function which in turn makes $\phi_{r+2}(\phi_{r+1} - P_h\phi_{r+1})$ odd, provides the result that $c_{r+1\,r+2} = 0$. $c_{r+2\,r+1} = 0$ is similar. Returning to (6.4),

$$\begin{aligned}
\int_{-1}^{1} \chi(s)(z - P_h z)(s)\,ds &= 2c_{r+1\ r+1} D^{r+1}\chi(0) D^{r+1} z(0) \\
&\quad + c_{r+1\,r+3} D^{r+3}\chi(0) D^{r+1} z(0) \\
&\quad + c_{r+3\,r+1} D^{r+1}\chi(0) D^{r+3} z(0) \\
&\quad + O(1)\Big(\|D^{r+4} z\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j \chi\|_{0,2} \\
&\qquad + \|D^{r+4}\chi\|_{0,2} \sum_{j=r+1}^{r+3} \|D^j z\|_{0,2} \Big) \\
&= c_{r+1\,r+1} \int_{-1}^{1} D^{r+1}\chi D^{r+1} z\,ds \\
&\quad + O(1) \sum_{i+j \geq 2r+4} \|D^i\chi\|_{0,2} \|D^j z\|_{0,2},
\end{aligned}$$

where the second order Maclaurin expansion was used in the last step. □

Proof of Theorem 6.1. Arguing exactly the same way between (3.4) and (3.7) with the interpolation projection i_h^r replaced by the orthogonal projection P_h, we obtain

$$u - \overline{u}_{it}^h = v^h - (I - G_h P_h)^{-1} G_h (I - P_h) v^h \tag{6.5}$$

where

$$G_h u(s) \equiv \int_0^1 g(t, s, P_h u(s), P_h \overline{u}_{it}^h(s), \theta) u(s)\, ds,$$

where g is defined in Section 2,

$$w^h \equiv K\Psi(u) - K\Psi(P_h u)$$

and

$$v^h \equiv (I - G_h)^{-1} w^h = (I - G_h)^{-1} K[\Psi(u) - \Psi(P_h u)].$$

Let $L \equiv (I - G_h)^{-1} K$ so that L is an integral operator with a kernel $l^*(s,t)$ with the same smoothness properties as $k(s,t)$. Then

$$v^h(t) = L[\Psi(u) - \Psi(P_h u)] = \int_0^1 l^*(s,t)[\psi(s, u(s)) - \psi(s, P_h u(s))]\, ds.$$

Using the mean value theorem as was done before in the proof of Theorem 3.1, we obtain

$$v^h(t) = \sum_{k=1}^N \int_{E_k} l(s,t)(u - P_h u)(s)\, ds$$

where $l(s,t) \equiv l^*(s,t)(\partial\psi/\partial u)(s, (u + \theta P_h u)(s))$.

By applying Lemma 6.2 to each subinterval E_k and noting that the change of scale introduces a factor $(h_k/2)^j$ for the jth derivative, we obtain

$$\begin{aligned} v^h(t) = \sum_{k=1}^N \left(\frac{h_k}{2}\right)^{2r+2} & c_{r+1\, r+1} \int_{E_k} D_s^{r+1} l\,(s,t)\, D^{r+1} u(s)\, ds \\ & + O\left(h^{2r+4}\right) \|u\|_{0,\infty}, \end{aligned} \tag{6.6}$$

where D_s denotes the partial derivative with respect to s. The result may be rewritten as

$$v^h(t) = \sum_{k=1}^N h_k^{2r+2} \int_{E_k} b(s,t)\, ds + O\left(h^{2r+4}\right), \tag{6.7}$$

where

$$b(s,t) = 2^{-(2r+2)} D_s^{r+1} l\,(s,t)\, D^{r+1} u(s).$$

We already know that $\|v^h\|_{0,2} = O(h^{2r+2})$. Also,

$$\begin{aligned} \|(I - G_h P_h)^{-1} G_h (I - P_h) v^h\|_{0,2} &\le C \|G_h (I - P_h) v^h\|_{0,2} \\ &\le C \|(I - P_h) v^h\|_{0,2} \\ &\le C h^2 \|v^h\|_{2,2} = O\left(h^{2r+4}\right). \end{aligned} \tag{6.8}$$

For (6.8), see the argument used in (3.10). Equations (6.5), (6.7) and (6.8) give the desired result. □

Theorem 6.1 engenders an extrapolation of the iterated Galerkin method for the Hammerstein equation. A process is the same with the extrapolation of the iterated collocation method, namely, we use the classical Richardson extrapolation technique. Let $T^{h/2}$ be a partition of I:

$$0 = t_0 < t_{1/2} < t_1 < t_{3/2} < \cdots < t_{N-1/2} < t_N = 1,$$

where

$$t_{k-1/2} = \frac{t_{k-1} + t_k}{2}, \quad k = 1, \ldots, N.$$

Let $u^{h/2}$ and $\overline{u}_{it}^{h/2}$ denote the Galerkin and iterated Galerkin approximations for the Hammerstein equation with respect to this new partition. Theorem 6.1 yields

$$u(t) - \overline{u}_{it}^{h/2}(t) = 2^{-(2r+2)} \sum_{k=1}^{N} h_k^{2r+2} \int_{e_k} b(t,s)\, ds + O\left(h^{2r+4}\right).$$

An extrapolation gives a new approximation

$$\overline{\overline{u}}_{it}^{h/2}(t) \equiv \frac{2^{2r+2} \overline{u}_{it}^{h/2} - \overline{u}_{it}^{h}(t)}{2^{2r+2} - 1}.$$

It is straightforward that

$$u(t) - \overline{\overline{u}}_{it}^{h/2}(t) = O\left(h^{2r+4}\right).$$

Numerical Example 6.1. Consider the equation

$$u(t) - \int_0^1 stu^2(s)\, ds = f(t), \quad t \in [0,1],$$

TABLE 8. The computational results of Example 6.1 using Galerkin and post-processing techniques.

N	Galerkin		Newton		quasi-Newton		Post-Processing			
	e'_h	R'_h	NI	CT	NI	CT	$\widehat{e}'_h$	$\widehat{R}'_h$	$\widehat{\overline{e}}'_h$	$\widehat{\overline{R}}'_h$
2	1.6482e-2		5	0.11	7	0.06	1.6450e-3			
4	4.1484e-3	1.99	5	0.26	7	0.17	1.0573e-4	3.96	5.3988e-6	
8	1.0398e-3	2.00	5	1.03	7	0.69	6.6550e-6	3.99	8.6269e-8	5.96
16	2.6013e-4	2.00	5	3.98	7	2.62	4.1667e-7	4.00	1.3552e-9	5.99
32	6.5044e-5	2.00	5	15.57	7	10.30	2.6053e-8	4.00	2.1203e-11	6.00
64	1.6262e-5	2.00	5	61.70	7	40.95	1.6285e-9	4.00	3.3129e-13	6.00
128	4.0655e-6	2.00	8	396.36	7	164.33	1.0179e-10	4.00	7.9936e-15	5.37

where $f(t)$ is chosen so that the exact solution is $u(t) = \exp(t)$. The results are presented in Table 8. Notice that we defined

$$\widehat{e}'_h = ||u - \overline{u}^h_{it}||_{0,2}, \qquad \widehat{R}'_h = \log_2\left(\frac{\widehat{e}'_h}{\widehat{e}'_{h/2}}\right)$$

$$\widehat{\overline{e}}'_h = ||u - \overline{\overline{u}}^{h/2}_{it}||_{0,2} \quad \text{and} \quad \widehat{\overline{R}}'_h = \log_2\left(\frac{\widehat{\overline{e}}'_h}{\widehat{\overline{e}}'_{h/2}}.\right)$$

As was the case with Example 5.1, we note that the quasi-Newton method results in less than half of the computing time than that of Newton's method.

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