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SOLVING EXTERIOR NEUMANN BOUNDARY VALUE PROBLEMS FOR BELTRAMI FIELDS THROUGH THE BELTRAMI SYSTEM

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ABSTRACT. The Neumann boundary value problem for Beltrami fields $\nabla \times u = ku$ with constant k is studied in exterior domains. The problem is approached by considering the extended Beltrami system for which the unique solvability of the Dirichlet problem and of the Neumann problem are shown with a boundary integral equation method. The boundary integral equations for the Neumann problem of the Beltrami fields and their solvability follow from the Beltrami system results.

1. Introduction. A vector field $u : \mathbf{R}^3 \to \mathbf{C}^3$ is a Beltrami field if

$$\nabla \times u = \lambda u, \tag{1}$$

where the proportionality factor λ, in general, is a space dependent scalar function. Beltrami fields appear in plasma physics, electromagnetics and fluid mechanics [**2, 9, 16, 18, 19, 24, 29, 31**]. In some sense, Beltrami fields are in between the scalar acoustic fields and the vector valued electromagnetic fields. In this paper, we study the right-handed Beltrami fields with constant $\lambda = k > 0$. In this case, the Beltrami fields satisfy the Helmholtz equation with the wave number k. The left-handed case, $k < 0$, is similar.

The Neumann boundary value of a Beltrami field is the normal component of the field, which corresponds to the flow through the boundary. The solvability of the interior Neumann boundary value problem for Beltrami fields has been studied and solved in [**1, 3, 10, 12, 13, 14, 21, 30**], also with nonconstant proportionality factors

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λ. In the exterior domain case, it is important to have a suitable radiation condition. In [**3**], the authors show that there are no exterior solutions in certain weighted Sobolev spaces. In [**1**], the authors give a radiation condition that is based on Bohren's decomposition of the electric and magnetic fields to left-handed and right-handed Beltrami fields, and, also, the corresponding representation formula for Beltrami fields in an exterior domain is given. Note that the Neumann boundary value problem for a single Beltrami field differs from the boundary value problems arising from Bohren's decomposition in which one has a pair of Beltrami fields with different handnesses that are tied together on the boundary. In [**11**], the Beltrami fields with a nonconstant proportionality factor are studied on the plane by reducing the equation (1) to a Vekua equation. In [**23**] the spectrum of the curl operator is studied in exterior domains. As a consequence the "limiting absorbtion principle" for the Beltrami fields is given, which implies solvability for the Neumann boundary value problem. In [**28**], the unique solvability for the exterior Neumann problem of Beltrami fields was achieved constructively with an Ansatz based integral equation approach when the wave number is not a Neumann eigenvalue of the interior problem and the topology of the obstacle is trivial. The restriction of wave numbers is needed in the existence part of the proof. In [**28**], the results have been applied to the inverse obstacle scattering problem.

In this paper, we prove the unique solvability of the exterior Neumann boundary value problem for Beltrami fields with a boundary integral equation method for all $k > 0$ with no exceptional points. Only the finiteness of the genus of each component is assumed about the topology. We approach the problem through the extended Beltrami system, or shortly, the Beltrami system, in which a divergence type equation together with a scalar unknown function is added to the original Beltrami equation (1), see (6). The Beltrami system is a natural bridge between the Beltrami fields and the Helmholtz equation. The concept is similar as in [**20, 22, 25**] for the Maxwell equations, see also [**21**]. The significant property of the Beltrami system is that it is possible to set the exterior boundary value problems of Neumann's type and of Dirichlet's type. Note that for the Beltrami fields it is not possible to fix the tangential component, see discussion at the end of Section 4. In the Beltrami system case, the uniqueness of solutions can be proven for both exterior boundary value problems. The existence of the Neumann problem follows under a suitable dual system from the uniqueness of the Dirichlet problem, and vice versa. The result for the Beltrami fields is a consequence of the Beltrami system case.

This paper is organized as follows. In Section 2, we write down the definitions for the Beltrami system. The question of a correct radiation condition is studied in Section 3. In Section 4 the representation formulae for the Beltrami system are given. We set the boundary value problems for the Beltrami system in Section 5. We show the uniqueness in Sections 6–7 for the boundary value problems of the Beltrami system. The solvability of the boundary value problems are obtained in Section 8 by considering a suitable dual system on the spaces of boundary data and the uniqueness of the dual problem.

Throughout the article $\Omega \subset \mathbf{R}^3$ is an open bounded set with a connected exterior domain $\Omega^s = \mathbf{R}^3 \setminus \overline{\Omega}$ and with a smooth boundary $\partial\Omega$. Assume that Ω consists of components Ω_j, $j = 1, \dots, J$.

2. The Beltrami system. Consider a Beltrami field u solving

$$\nabla \times u = ku, \tag{2}$$

where $k > 0$ is constant. Since $k \neq 0$, we see by taking the divergence from the equation (2) that

$$\nabla \cdot u = 0. \tag{3}$$

Hence, a Beltrami field u solves also the Helmholtz equation,

$$-\Delta u = (\nabla\times)^2 u - \nabla\nabla \cdot u = k^2 u.$$

The (extended) Beltrami system is

$$A(\nabla)U - kU = 0, \quad U : \mathbf{R}^3 \to \mathbf{C}^4, \tag{4}$$

where

$$A(p) = \begin{bmatrix} p\times & -p \\ p\cdot & 0 \end{bmatrix} = \begin{pmatrix} 0 & -p_3 & p_2 & -p_1 \\ p_3 & 0 & -p_1 & -p_2 \\ -p_2 & p_1 & 0 & -p_3 \\ p_1 & p_2 & p_3 & 0 \end{pmatrix} \tag{5}$$

for $p = (p_1, p_2, p_3)$. Note that

$$A(p)^T = -A(p), \quad A(p)^T A(p) = \begin{bmatrix} -(p\times)^2 + pp\cdot & 0 \\ 0 & p\cdot p \end{bmatrix} = p \cdot pI.$$

By writing

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix}, \quad u : \mathbf{R}^3 \to \mathbf{C}^3, \quad \phi : \mathbf{R}^3 \to \mathbf{C},$$

the Beltrami system (4) takes the form

$$\begin{cases} \nabla \times u - \nabla\phi = ku \\ \nabla \cdot u = k\phi. \end{cases} \tag{6}$$

The following relation between the Beltrami system and the Beltrami fields is immediate.

Lemma 2.1. *If*

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix}, \quad u : \mathbf{R}^3 \to \mathbf{C}^3, \quad \phi : \mathbf{R}^3 \to \mathbf{C},$$

solves Beltrami system (4)*, then u is a Beltrami field solving* (2) *if and only if $\phi = 0$.*

One advantage of the Beltrami system is that it factors the Helmholtz operator, and hence the fundamental solution for the Beltrami system can be easily derived from the fundamental solution of the Helmholtz equation. Let

$$\Phi(x) = \frac{1}{4\pi}\frac{e^{ik|x|}}{|x|}$$

be the fundamental solution of the Helmholtz equation,

$$-(\Delta + k^2)\Phi = \delta.$$

By the factorization

$$(A(\nabla) - kI)(A(\nabla) + kI) = -(\Delta + k^2)I, \tag{7}$$

the matrix

$$G = (A(\nabla) + kI)(\Phi I) = A(\nabla\Phi) + k\Phi I$$

is the fundamental solution of the Beltrami system,

$$(A(\nabla) - kI)G = \delta I. \tag{8}$$

Denote

$$G_z = A(\nabla\Phi_z) + k\Phi_z I,$$

where $\Phi_z(x) = \Phi(x - z)$. Since

$$A(\nabla_z\Phi_x(z)) + k\Phi_x(z)I = -A(\nabla_x\Phi_z(x)) + k\Phi_z(x)I,$$

we have the reciprocity relation

$$G_x(z) = G_z(x)^T \tag{9}$$

3. Radiation condition. To get the representation formula and to have hope for the unique solvability of any boundary value problem in the exterior domain, we need to have a correct radiation condition. By the correct radiation condition we mean that the exterior problems are uniquely solvable. If the radiation condition is too restrictive, then there will be no non-trivial fields in the exterior domain. If, in contrast, the radiation condition is too loose, then the uniqueness cannot hold. We need the radiation conditions for the Beltrami fields and for the extended Beltrami system.

A radiation condition can be obtained directly from the differential equation. Since the components of Beltrami fields, and also the components of the Beltrami system solutions, satisfy the Helmholtz equation, it is natural to try to obtain the radiation condition starting from the Sommerfeld radiation condition, which is

$$\widehat{x} \cdot \nabla v(x) - ikv(x) = o\left(\frac{1}{|x|}\right), \quad \widehat{x} = \frac{x}{|x|}, \tag{10}$$

uniformly in all directions as $|x| \to \infty$ where v solves the Helmholtz equation.

Recall, the solution v of the Helmholtz equation is radiating, if (10) holds. A radiating v can be written as

$$v(x) = \Phi(x) v_\infty(\widehat{x}) + o\left(\frac{1}{|x|}\right), \tag{11}$$

and v_∞ is called the far field (-pattern) of v, [**7**].

Lemma 3.1. *Assume v is a radiating solution of the Helmholtz equation in an exterior domain Ω^s. Then $\partial_j v$ is also a radiating solution with far field*

$$(\partial_j v)_\infty(\widehat{x}) = ik\widehat{x}_j v_\infty(\widehat{x}). \tag{12}$$

Proof. A single layer potential

$$S\phi(x) = \int_{\partial\Omega} \Phi_y(x)\phi(y)\, dS(y),$$

has the far field

$$(S\phi)_\infty(\widehat{x}) = \int_{\partial\Omega} e^{-ik\hat{x}\cdot y}\phi(y)\, dS(y). \tag{13}$$

Now,

$$(\partial_j \Phi_y)_\infty(\widehat{x}) = ik\widehat{x}_j e^{-ik\hat{x}\cdot y},$$

and so

$$(\partial_j S\phi)_\infty(\widehat{x}) = \int_{\partial\Omega} (\partial_j \Phi_y)_\infty(\widehat{x})\phi(y)\, dS(y) = ik\widehat{x}_j (S\phi)_\infty(\widehat{x}).$$

Thus, the claim holds for the single layer potentials. But, any radiating solution can be represented as a single layer potential: Choose $R > 0$ such that k^2 is not Dirichlet's eigenvalue of the ball $B = B(0, R)$ and that $\overline{\Omega} \subset B$. Then

$$v = S\phi,$$

where the single layer is of the boundary ∂B, and

$$\phi = S_{\partial B}^{-1} v|_{|x|=R}.$$

Note that $S_{\partial B}$ is invertible since k^2 is not Dirichlet's eigenvalue. □

Lemma 3.2. *Let*

$$P = P(D) = \sum_{j=1}^{3} P^j \partial_j$$

be a linear first-order differential operator with constant (matrix) coefficients P^j, $j = 1, \ldots, 3$. Assume that v solves the Helmholtz equation in an exterior domain and satisfies the Sommerfeld radiation condition. If v solves the equation

$$v = P(D)v,$$

then

$$v(x) - ikP(\widehat{x})v(x) = o\left(\frac{1}{|x|}\right)$$

uniformly in $\widehat{x}$ as $|x| \to \infty$.

Proof. First, by the previous lemma,

$$0 = [P(D)v - v]_\infty(\widehat{x}) = ikP(\widehat{x})v_\infty(\widehat{x}) - v_\infty(\widehat{x}).$$

Hence, as $|x| \to \infty$,

$$v(x) - ikP(\widehat{x})v(x) = \Phi(x)\left(v_\infty(\widehat{x}) - ikP(\widehat{x})v_\infty(\widehat{x})\right) + o\left(\frac{1}{|x|}\right) = o\left(\frac{1}{|x|}\right). \square$$

Assume that u is a Beltrami field and that each component of u satisfies the Sommerfeld radiation condition (10). The application of Lemma 3.2 with

$$P(D) = \frac{1}{k}\nabla\times$$

gives a radiation condition

$$u - i\widehat{x} \times u = o\left(\frac{1}{|x|}\right), \quad \widehat{x} = \frac{x}{|x|}, \tag{14}$$

uniformly as $|x| \to \infty$. Similarly, if U solves the Beltrami system and each component of U satisfies the Sommerfeld radiation condition (10), then by letting

$$P(D) = \frac{1}{k} A(\nabla)$$

in Lemma 3.2, we get that U satisfies a radiation condition

$$U(x) - iA(\widehat{x})U(x) = o\left(\frac{1}{|x|}\right) \tag{15}$$

uniformly in all directions as $|x| \to \infty$. We will see in the following that the conditions (14) and (15) are the correct radiation conditions. It turns out also that (14) and (15) are equivalent with the Sommerfeld radiation condition for each component.

In [**1**], the radiation condition for Beltrami fields is given by (14). With this behavior at infinity it is possible to get the representation formula in an exterior domain starting from the interior formula with the standard trick of enlarging balls.

4. Representation formulae. We derive the representation formulae for the Beltrami system both in the interior domain Ω and in the exterior domain Ω^s. The corresponding formulae for the Beltrami fields follow then from these.

In the proof, we are integrating against distributions f on Ω for which $\operatorname{singsupp}(f) \cap \partial\Omega = \varnothing$, by which we mean that we are testing f as

$$\int_\Omega f\, dx := \langle \psi f, \chi_\Omega \rangle + \langle f, (1-\psi)\chi_\Omega \rangle ,$$

where $\psi \in C^\infty(\mathbf{R}^3)$ is a cutting function with $\psi = 0$ in a neighborhood of $\operatorname{singsupp}(f)$ and $\psi = 1$ in a neighborhood of $\partial\Omega$. Recall also the integration by parts formula

$$\int_\Omega (L(\nabla)A)^T B\, dx = \int_{\partial\Omega} (L(n)A)^T B\, dS - \int_\Omega A^T L(\nabla)^T B\, dx,$$

for linear operators

$$L(\nabla) = \sum_j L^j \partial_j,$$

which follows from the scalar integration by parts formula.

Theorem 4.1. *Let $U \in C^1(\overline{\Omega})^4$ satisfy*

$$A(\nabla)U = kU$$

in Ω. *Then*

$$\int_{\partial\Omega} G_y(x)A(n(y))U(y)\,dS(y) = \begin{cases} -U(x) & x \in \Omega, \\ 0 & x \in \Omega^s. \end{cases} \tag{16}$$

Proof. We integrate against δ_x to get

$$\left.\begin{array}{rr} x \in \Omega: & U(x) \\ x \in \Omega^s: & 0 \end{array}\right\} = \int_{\Omega} \left[(A(\nabla_y) - kI)G_x(y)\right]^T U(y)\,dy$$
$$= \int_{\partial\Omega} [A(n)G_x(y)]^T U(y)\,dS(y) - \int_{\Omega} G_x(y)^T [A(\nabla_y)^T + kI]U(y)\,dy$$
$$= -\int_{\partial\Omega} G_y(x)A(n)U(y)\,dS(y)$$

by reciprocity (9). □

We can rewrite the integral of the representation formula with the single layer operator S as

$$\int_{\partial\Omega} G_y(x)A(n)U(y)\,dS(y) = (A(\nabla) + kI)S[A(n)U](x), \quad x \in \mathbf{R}^3 \setminus \partial\Omega. \tag{17}$$

Note that

$$A(n)U = \begin{bmatrix} n \times u - n\phi \\ n \cdot u \end{bmatrix}, \quad U = \begin{bmatrix} u \\ \phi \end{bmatrix}.$$

Corollary 4.2. *Let $u \in C^1(\overline{\Omega})^3$ be a Beltrami field in* Ω. *Then*

$$\left.\begin{array}{r} x \in \Omega: -u(x) \\ x \in \Omega^s: 0 \end{array}\right\} = \nabla \times S(n \times u)(x) + \frac{1}{k}(\nabla\times)^2 S(n \times u)(x). \tag{18}$$

Proof. We apply Theorem 4.1 for

$$U = \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

The lower equation of (16) gives the relation

$$\nabla \cdot S(n \times u) + kS(n \cdot u) = 0.$$

When we substitute this into the upper equation of (16), it gives

$$\left.\begin{array}{r} x \in \Omega : -u(x) \\ x \in \Omega^s : 0 \end{array}\right\}$$
$$\begin{aligned} &= \nabla \times S(n \times u)(x) - \nabla S(n \cdot u) + kS(n \times u) \\ &= \nabla \times S(n \times u)(x) + \frac{1}{k}\nabla\nabla \cdot S(n \times u) \\ &\quad - \frac{1}{k}\Delta S(n \times u) \\ &= \nabla \times S(n \times u)(x) + \frac{1}{k}(\nabla\times)^2 S(n \times u)(x). \qquad \square \end{aligned}$$

The following lemma gives a connection between the boundary values and the far fields of radiating solutions.

Lemma 4.3. i) *Let* $U \in C^1(\overline{\Omega^s})^4$ *solve the Beltrami system in the exterior domain* Ω^s *with the radiation condition* (15). *Then*

$$\lim_{R\to\infty} \int_{\partial B(0,R)} |U|^2 \, dS = i \int_{\partial\Omega} (A(n)U)^T \overline{U} \, dS. \tag{19}$$

ii) *Let* $u \in C^1(\overline{\Omega^s})^3$ *be a Beltrami field in the exterior domain* Ω^s *that satisfies the radiation condition* (14). *Then*

$$\lim_{R\to\infty} \int_{\partial B(0,R)} |u|^2 dS = i \int_{\partial\Omega} (n \times u)^T \overline{u} \, dS. \tag{20}$$

Proof. i) Let $R > 0$ be large such that $\overline{\Omega} \subset B(0, R)$. It holds, since $A(\widehat{x})^T A(\widehat{x}) = I$,

$$\begin{aligned}\int_{\partial B(0,R)} &|(I - iA(\widehat{x}))U(x)|^2 dS \\ &= 2\int_{\partial B(0,R)} \left(|U|^2 - i(A(\widehat{x})U)^T\overline{U}\right)\, dS \\ &\qquad = 2\int_{\partial B(0,R)} |U|^2 dS - 2i\int_{\partial\Omega} (A(n)U)^T\overline{U}\, dS.\end{aligned}$$

The left-hand side tends to zero by the radiation condition (15) as $R \to \infty$.

ii) Apply i) with

$$U = \begin{bmatrix} u \\ 0 \end{bmatrix}. \qquad \square$$

Remark. The proof above also holds for weak solutions $U \in H^1_{\mathrm{loc}}(\Omega^s)^4$ of the Beltrami system that satisfy the radiation condition since it is just an integration by parts argument. Recall that $H^1_{\mathrm{loc}}(\Omega^s)$ consists of such functions u such that $\phi u \in H^1(\Omega^s)$ for all compactly supported smooth functions ϕ.

Theorem 4.4. *Let $U \in C^1(\overline{\Omega^s})^4$ solve the Beltrami system in the exterior domain Ω^s with the radiation condition* (15)*. Then*

$$\int_{\partial\Omega} G_y(x)A(n(y))U(y)\, dS(y) = \begin{cases} 0 & x \in \Omega, \\ U(x) & x \in \Omega^s. \end{cases} \tag{21}$$

Proof. First, we study the far field of the fundamental solution matrix. Since

$$G_z(x) = A(\nabla\Phi_z) + k\Phi_z I,$$

we get

$$G_z(x) = \Phi(x)G_{z,\infty}(\widehat{x}) + o\left(\frac{1}{|x|}\right),$$

where

$$G_{z,\infty}(\widehat{x}) = k(iA(\widehat{x}) + I)\Phi_{z,\infty}(\widehat{x}) \tag{22}$$

by (12). Now

$$(I - iA(\widehat{x}))G_{z,\infty}(\widehat{x}) = 0,$$

and so

$$(I - iA(\widehat{x}))G_z(x) = o\left(\frac{1}{|x|}\right).$$

Let $x \in \mathbf{R}^3 \setminus \partial\Omega$, and let $R > |x|$ be so large that $\overline{\Omega} \subset B(0,R)$. Set

$$\Omega_R = \Omega^s \cap B(0,R),$$

and denote by n_R the unit outer normal of $\partial\Omega_R$. Here n is the unit outer normal of $\partial\Omega$. By the representation formula of interior solutions applied to Ω_R,

$$\left.\begin{array}{r} x \in \Omega : 0 \\ x \in \Omega^s : U(x) \end{array}\right\} = -\int_{\partial\Omega_R} G_y(x)A(n_R)U(y)\,dS(y)$$

$$= \int_{\partial\Omega} G_y(x)A(n)U(y)\,dS(y)$$

$$- \int_{\partial B(0,R)} G_y(x)A(\widehat{y})U(y)\,dS(y).$$

The second integral tends to zero as R grows: By (9) and (22), the integrand is

$$\begin{aligned} G_y(x)A(\widehat{y})U(y) &= G_x(y)^T A(\widehat{y})U(y) \\ &= ik\Phi(y)\Phi_{x,\infty}(\widehat{y})(I - iA(\widehat{y}))U(y) + o\left(\frac{1}{R}\right)U(y) \\ &= o\left(\frac{1}{R^2}\right) + o\left(\frac{1}{R}\right)U(y) \end{aligned}$$

by the radiation condition (15). Hence,

$$\left|\int_{\partial B(0,R)} G_y(x)A(\widehat{y})U(y)\,dS(y)\right|$$

$$\leq o(1) + \left(\int_{\partial B(0,R)} |U|^2 dS\right)^{1/2} o(1),$$

and

$$\int_{\partial B(0,R)} |U|^2 dS$$

is bounded as $R \to \infty$ by the previous Lemma 4.3. □

Corollary 4.5. *Let $u \in C^1(\overline{\Omega^s})^3$ be a Beltrami field in the exterior domain Ω^s that satisfies the radiation condition* (14). *Then*

$$\left.\begin{array}{r} x \in \Omega : 0 \\ x \in \Omega^s : u(x) \end{array}\right\} = \nabla \times S(n \times u)(x) + \frac{1}{k}(\nabla\times)^2 S(n \times u)(x). \tag{23}$$

Proof. Now

$$U = \begin{bmatrix} u \\ 0 \end{bmatrix}$$

solves the Beltrami system in the exterior domain and satisfies the radiation condition (15), so we can apply Theorem 4.4. The normal component $n \cdot u$ can be eliminated in the same way as in the interior case. □

Earlier, in Section 3, we saw that the Sommerfeld radiation condition (10) implies the radiation condition (15) for the Beltrami system and (14) for the Beltrami fields. Now, we assumed that the radiation condition (15), or (14), holds, and then we got a representation in terms of the radiating fundamental solution Φ of the Helmholtz equation. Hence, the radiation conditions (15) and (14) are equivalent with the Sommerfeld radiation condition.

From the representation formula (23) it follows that the tangential component alone determines the Beltrami field u, and hence, fixing the tangential component would lead to an overdetermined problem. If one tries to ask the tangential boundary condition

$$n \times u^s + n \times u^i = 0$$

for a scattering phenomenon, where u^i is a Beltrami field in $\mathbf{R}^3$ and u^s satisfies the radiation condition (14), then in the exterior domain

$$\begin{aligned} u^s(x) &= \nabla \times S(n \times u^s)(x) + \frac{1}{k}(\nabla\times)^2 S(n \times u^s)(x) \\ &= -\nabla \times S(n \times u^i)(x) - \frac{1}{k}(\nabla\times)^2 S(n \times u^i)(x) \\ &= 0, \end{aligned}$$

and $u^s \equiv 0$. But then, on the boundary,

$$n \times u^i = -n \times u^s = 0,$$

and so also $u^i \equiv 0$ by the representation formula for interior solutions. Hence, the zero field is the only "scattering field" for the obstacles with tangential boundary conditions, that is, there are no such obstacles.

5. Boundary value problems in exterior domains. The exterior Neumann boundary value problem of Beltrami fields is to find a solution $u \in H^1_{\mathrm{loc}}(\Omega^s)^3$ for

$$\begin{cases} \nabla \times u = ku & \text{in } \Omega^s, \\ n \cdot u|^+_{\partial\Omega} = g, \end{cases} \tag{24}$$

where u is a radiating solution satisfying (14) and $g \in H^{1/2}(\partial\Omega)$. We solve this problem by studying boundary value problems for the Beltrami system.

Let

$$B^s(\partial\Omega) = \begin{matrix} H^s(\partial\Omega)^3 \\ \times \\ H^s(\partial\Omega) \end{matrix}, \tag{25}$$

$s \in \mathbf{R}$, be the space of the boundary values. It can be decomposed as

$$B^s(\partial\Omega) = B^s_N(\partial\Omega) \oplus B^s_D(\partial\Omega), \tag{26}$$

where

$$B^s_N(\partial\Omega) = P_N B^s(\partial\Omega), \qquad B^s_D(\partial\Omega) = P_D B^s(\partial\Omega), \tag{27}$$

and P_N, P_D are projection operators

$$(28) \qquad P_N = P_N(n) = \begin{bmatrix} nn\cdot & 0 \\ 0 & 1 \end{bmatrix}, \quad P_D = P_D(n) = \begin{bmatrix} -(n\times)^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

We call the space $B^s_N(\partial\Omega)$ as the space of Neumann's boundary values, and $B^s_D(\partial\Omega)$ as the space of Dirichlet's boundary values.

Definition 5.1. The exterior Neumann boundary value problem for the Beltrami system is to find a solution $U \in H^1_{\mathrm{loc}}(\Omega^s)^4$ for

$$(29) \qquad \begin{cases} A(\nabla)U - kU = 0 & \text{in } \Omega^s, \\ P_N(n)U|^+_{\partial\Omega} = G, \end{cases}$$

where U satisfies the radiation condition (15) and $G \in B^{1/2}_N(\partial\Omega)$. □

Definition 5.2. The exterior Dirichlet boundary value problem for the Beltrami system is to find a solution $U \in H^1_{\mathrm{loc}}(\Omega^s)^4$ for

$$(30) \qquad \begin{cases} A(\nabla)U - kU = 0 & \text{in } \Omega^s, \\ P_D(n)U|^+_{\partial\Omega} = F, \end{cases}$$

where U satisfies the radiation condition (15) and $F \in B^{1/2}_D(\partial\Omega)$. □

In both boundary value problems for the Beltrami system (29)–(30), the number of unknown components is four and the boundary condition fixes two components, which is natural in view of decomposition (26). In the Neumann problem of Beltrami fields (24), the number of the unknown is three but the Neumann condition has only one component, which is peculiar. The explanation is that in the Beltrami field case the divergence equation is still there implicitly with a zero boundary condition. A similar thing occurs also with the Maxwell equations.

6. Uniqueness for the Neumann problem. The uniqueness proof of an exterior boundary value problem is usually based on a formula like (20). The zero boundary condition implies that the right-hand side vanishes, and then an application of the Rellich's lemma gives that the zero field is the only solution. This approach can not be directly applied. However, the space of the boundary values of the Beltrami fields with vanishing Neumann boundary data can be characterized in terms of the Hodge decomposition.

We define the surface differential operators, and obtain the Hodge decomposition, on the space of compactly supported distributions $\mathcal{E}'(\partial\Omega)$. The mapping properties between the Sobolev spaces $H^s(\partial\Omega)$ follow then from the degree of their symbols, [**26, 27**]. For different definitions, see e.g., [**4, 5**].

Denote

$$\mathcal{E}(\partial\Omega) = C^\infty(\partial\Omega).$$

Define

$$T\mathcal{E}'(\partial\Omega) = \{u \in \mathcal{E}'(\partial\Omega)^3 \mid n \cdot u = 0\},$$
$$T\mathcal{E}(\partial\Omega) = \{u \in \mathcal{E}(\partial\Omega)^3 \mid n \cdot u = 0\}.$$

Since n is smooth on $\partial\Omega$, $n \cdot u$ is a well-defined distribution for $u \in \mathcal{E}'(\partial\Omega)^3$,

$$\langle n \cdot u, \phi\rangle = \langle u, \phi n\rangle, \quad \phi \in \mathcal{E}(\partial\Omega).$$

Also, define

$$\mathcal{E}_0'(\partial\Omega) = \{u \in \mathcal{E}'(\partial\Omega) \mid \langle u, \chi_j\rangle = 0, \quad j = 1, \dots, J\},$$
$$\mathcal{E}_0(\partial\Omega) = \Big\{u \in \mathcal{E}(\partial\Omega) \mid \int_{\partial\Omega_j} u\, dS = 0\Big\},$$

where χ_j are the characteristic functions of the components $\partial\Omega_j$ of $\partial\Omega$, $j = 1, \dots, J$.

For $\phi \in \mathcal{E}(\partial\Omega)$, the surface gradient is

$$\nabla_T \phi = -(n\times)^2 \nabla\widetilde{\phi}|_{\partial\Omega},$$

where $\widetilde{\phi}$ is an extension of ϕ to the neighborhood of $\partial\Omega$. The surface divergence $\mathrm{Div}\,(\psi)$ for $\psi \in T\mathcal{E}(\partial\Omega)$ is defined by

$$\int_{\partial\Omega} \mathrm{Div}\,(\psi)\phi\, dS = -\int_{\partial\Omega} \psi \cdot \nabla_T \phi\, dS, \quad \phi \in \mathcal{E}(\partial\Omega),$$

and the surface curl by

$$\int_{\partial\Omega} \mathrm{Curl}\,(\psi)\phi\, dS = -\int_{\partial\Omega} \psi \cdot n \times \nabla_T \phi\, dS, \quad \phi \in \mathcal{E}(\partial\Omega).$$

Now,

$$\mathrm{Curl}\,(\psi) = -\mathrm{Div}\,(n \times \psi),$$

and

$$\mathrm{Curl}\,(\nabla_T \phi) \equiv 0, \quad \mathrm{Div}\,(n \times \nabla_T \phi) \equiv 0.$$

For $u \in \mathcal{E}'(\partial\Omega)$, the surface gradient is defined by

$$\langle \nabla_T u, \psi \rangle = -\langle u, Div(\psi) \rangle, \quad \psi \in T\mathcal{E}(\partial\Omega).$$

The surface divergence for $f \in T\mathcal{E}'(\partial\Omega)$ is defined by

$$\langle \mathrm{Div}\,(f), \phi \rangle = -\langle f, \nabla_T \phi \rangle, \quad \phi \in \mathcal{E}(\partial\Omega),$$

and the surface curl by

$$\langle \mathrm{Curl}\,(f), \phi \rangle = -\langle f, n \times \nabla_T \phi \rangle, \quad \phi \in \mathcal{E}(\partial\Omega).$$

The surface Laplacian is

$$\Delta_T = \mathrm{Div}\,\nabla_T.$$

If $\partial\Omega$ is connected, then Δ_T is invertible in $\mathcal{E}_0'(\partial\Omega)$: The principal symbol of $-\Delta_T$ is $|\xi|^2$, and hence, $-\Delta_T$ is a Fredholm operator $H^1(\partial\Omega) \to H^{-1}(\partial\Omega)$ with index zero. Also, $\Delta_T : H_0^1(\partial\Omega) \to H_0^{-1}(\partial\Omega)$ is injective, and hence invertible. Particularly, Δ_T is invertible in $\mathcal{E}_0(\partial\Omega)$. Denote $G = \Delta_T^{-1}$ in $\mathcal{E}_0(\partial\Omega)$. Now G can be extended to $\mathcal{E}_0'(\partial\Omega)$ by

$$\langle Gu, \phi \rangle = \langle u, G\phi \rangle,$$

and $G\Delta_T = \Delta_T G = I$ on $\mathcal{E}_0'(\partial\Omega)$.

Let $H_0(\partial\Omega)$ be the space of tangential harmonic vector fields,

$$H_0(\partial\Omega) = \{f \in T\mathcal{E}'(\partial\Omega) \mid \mathrm{Div}\,(f) = 0 = \mathrm{Curl}\,(f)\}. \tag{31}$$

Every $g \in T\mathcal{E}'(\partial\Omega)$ has a unique Hodge decomposition

$$g = \nabla_T u + f + n \times \nabla_T v, \tag{32}$$

where $u, v \in \mathcal{E}_0'(\partial\Omega)$ are

$$\begin{cases} u = G\mathrm{Div}\,(g), \\ v = G\mathrm{Curl}\,(g), \end{cases}$$

and

$$f = g - \nabla_T u - n \times \nabla_T v \in H_0(\partial\Omega).$$

Lemma 6.1. i) *For each* $f \in H_0(\partial\Omega)$ *there exists at most one* $\nabla_T \phi$ *such that*

$$f + \nabla_T \phi = -(n\times)^2 u|^+_{\partial\Omega} \tag{33}$$

for some radiating Beltrami field u *in the exterior domain* Ω^s*.*

ii) *Let* $N_0(\partial\Omega)$ *be the space of tangential vector fields*

$$f + \nabla_T \phi, \qquad f \in H_0(\partial\Omega), \tag{34}$$

for which there exists a radiating Beltrami field u *satisfying* (33)*. Then*

$$N_0(\partial\Omega) = \{-(n\times)^2 u \mid u \text{ is a radiating Beltrami field}, \; n\cdot u|_{\partial\Omega} = 0\}. \tag{35}$$

Proof. i) Let $f \in H_0(\partial\Omega)$. Let u_1 and u_2 be radiating Beltrami fields with

$$-(n\times)^2 u_j = f + \nabla_T \phi_j, \quad j = 1, 2.$$

Now $u = u_1 - u_2$ is a radiating Beltrami field with

$$-(n\times)^2 u = \nabla_T \psi, \quad \psi = \phi_1 - \phi_2.$$

By (20), it holds that

$$\lim_{R\to\infty} \int_{\partial B(0,R)} |u|^2 dS = i \int_{\partial\Omega} (n \times \nabla_T \psi) \cdot \overline{\nabla_T \psi}\, dS = 0.$$

Now $u = 0$ by Rellich's Lemma [**7**], because u is a radiating solution of the Helmholtz equation. Especially,

$$0 = -(n\times)^2 u = \nabla_T \phi_1 - \nabla_T \phi_2,$$

which proves the claim.

ii) Note that

$$\mathrm{Curl}\,(-(n\times)^2 u|^+_{\partial\Omega}) = n\cdot(\nabla\times u) = kn\cdot u$$

for a Beltrami field u. If $h\in N_0(\partial\Omega)$, then there is a Beltrami field u with

$$-(n\times)^2 u|_{\partial\Omega} = h,$$

and

$$n\cdot u = \frac{1}{k}\mathrm{Curl}\,(h) = 0.$$

If u is a Beltrami field with $n\cdot u|^+_{\partial\Omega} = 0$, then

$$\mathrm{Curl}\,(u|_{\partial\Omega}) = kn\cdot u = 0,$$

and so the Hodge decomposition for $u|_{\partial\Omega}$ implies that $u|_{\partial\Omega}\in N_0(\partial\Omega)$. □

Theorem 6.2. *If*

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix} \in H^1_{\mathrm{loc}}(\Omega^s)^4$$

is a solution of the exterior Neumann boundary value problem (29) *with*

$$P_N(n)U|^+_{\partial\Omega} = 0$$

and

$$\int_{\partial\Omega} u\cdot\overline{h}\,dS = 0 \tag{36}$$

for every $h\in N_0(\partial\Omega)$, *then* $U = 0$.

Proof. Let

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix}$$

be a radiating solution of the exterior Neumann boundary value problem of the Beltrami system (29). The radiation condition (15) implies that ϕ satisfies the Sommerfeld radiation condition. Hence, ϕ is a radiating solution of the Helmholtz equation with a vanishing boundary value, and so

$$\phi = 0$$

in Ω^s, see [**7**]. This means that u is a Beltrami field with

$$n \cdot u|_{\partial\Omega} = 0,$$

and so $u|_{\partial\Omega} \in N_0(\partial\Omega)$ by the previous Lemma 6.1. The condition (36) implies $u|_{\partial\Omega} = 0$. Hence, $u = 0$ by the representation formula (23). □

Corollary 6.3. *The exterior Neumann boundary value problem of the Beltrami system* (29) *has at most one radiating solution provided condition* (36) *holds.*

Corollary 6.4. *The exterior Neumann boundary value problem of Beltrami fields* (24) *has at most one radiating solution provided the condition* (36) *holds.*

7. Uniqueness for the Dirichlet problem. The exterior Dirichlet problem of the Beltrami system is not uniquely solvable. Namely, if ϕ is a radiating solution of the Helmholtz equation with

$$\phi|_{\partial\Omega_j} = c_j \text{ (constant)}$$

on each component $\partial\Omega_j$ of $\partial\Omega$, then

$$U = \begin{bmatrix} -\frac{1}{k}\nabla\phi \\ \phi \end{bmatrix}$$

is a radiating solution of the Beltrami system with

$$P_D(n)U = 0.$$

Next, we show that this is the only non-uniqueness.

Theorem 7.1. *If $U \in H^1_{\mathrm{loc}}(\Omega^s)^4$ solves the exterior Dirichlet boundary value problem* (30) *with a zero boundary value*

$$P_D(n)U|^+_{\partial\Omega} = 0,$$

and

$$\int_{\partial\Omega_j} \phi\, dS = 0, \qquad U = \begin{bmatrix} u \\ \phi \end{bmatrix}, \tag{37}$$

for each component $\partial\Omega_j$ of $\partial\Omega$, then $U = 0$.

Proof. Let

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix}$$

satisfy the conditions of the theorem. Because $P_D(n)U = 0$, formula (19) gives

$$\lim_R \int_{\partial B_R} |U|^2 dS = i \int_{\partial\Omega} \begin{bmatrix} -n\phi \\ n \cdot u \end{bmatrix} \cdot \begin{bmatrix} \overline{u} \\ \overline{\phi} \end{bmatrix} dS = 2\,\mathrm{Im}\left(\int_{\partial\Omega} \phi n \cdot \overline{u}\, dS \right).$$

By the first equation of the Beltrami system, and the boundary condition,

$$n \cdot u = \frac{1}{k}\left(-\mathrm{Div}\,(n \times u) - n \cdot \nabla\phi\right) = -\frac{1}{k}\partial_n \phi.$$

Also, by integrating by parts

$$\mathrm{Im}\left(\int_{\partial\Omega} \phi \partial_n \overline{\phi}\, dS \right) = -\frac{k}{4\pi} \int_{S^2} |\phi_\infty|^2 dS,$$

see (3.10) of [**6**] for the details. Hence,

$$\lim_R \int_{\partial B_R} |U|^2 dS = \frac{2}{4\pi} \int_{S^2} |\phi_\infty|^2 dS. \tag{38}$$

On the other hand,

$$\lim_R \int_{\partial B_R} |U|^2 dS = \frac{1}{4\pi} \int_{S^2} \left(|\xi \times u_\infty|^2 + 2|\phi_\infty|^2\right) dS(\xi), \tag{39}$$

because

$$i\xi \cdot u_\infty = \phi_\infty$$

by the radiation condition (15). By (38) and (39),

$$\xi \times u_\infty(\xi) = 0.$$

Now,

$$v = \nabla \times u$$

is a radiating solution for the Helmholtz equation and

$$v_\infty = ik\xi \times u_\infty = 0,$$

and hence,

$$\nabla \times u = v = 0$$

by Rellich's lemma. This means, because U solves the Beltrami system, that

$$ku = -\nabla \phi,$$

and the boundary condition implies

$$n \times \nabla \phi = -kn \times u = 0.$$

Hence,

$$\phi|_{\partial \Omega_j} = c_j$$

is constant on each component $\partial \Omega_j$. By the integral condition (37), every

$$c_j = 0.$$

Hence,

$$\phi|_{\partial \Omega} = 0,$$

and because ϕ is a radiating solution for the Helmholtz equation, $\phi = 0$, [**6**]. But then also

$$u = -\frac{1}{k} \nabla \phi = 0.$$

8. Existence. The existence of solutions for the boundary value problems is shown by proving that a certain boundary integral equation has a solution. To show this we use the relation

$$\overline{R(T)} = \operatorname{Ker}(T^*)^\perp \tag{40}$$

between the range of a boundary integral operator T and the kernel of its adjoint T^*, [**17**]. Here, the perpendicularity and the adjointness are defined with a suitable dual system. If T has a closed range, then the relation (40) gives the range of T.

We define a dual system, see [**15**], by extending the nondegenerate bilinear form

$$\langle F, G\rangle_A = \int_{\partial\Omega} A(n)F\cdot G\,dS, \quad F, G\in L^2(\partial\Omega)^4, \tag{41}$$

to $B^{-1/2}(\partial\Omega)\times B^{1/2}(\partial\Omega)$. When restricted onto the Neumann boundary values this duality induces a duality

$$\begin{gathered}\langle\,\cdot,\cdot\,\rangle_N : B_N^{-1/2}(\partial\Omega)\times B_N^{1/2}(\partial\Omega)\to \mathbf{C}\\ \left\langle \begin{bmatrix}\phi_1 n\\ \phi_2\end{bmatrix}, \begin{bmatrix}\psi_1 n\\ \psi_2\end{bmatrix}\right\rangle_N = \langle\phi_1,\psi_2\rangle - \langle\phi_2,\psi_1\rangle\,.\end{gathered}\tag{42}$$

On the Dirichlet boundary values the form $\langle\,\cdot,\cdot\,\rangle_A$ induces a duality

$$\begin{gathered}\langle\,\cdot,\cdot\,\rangle_D : B_D^{-1/2}(\partial\Omega)\times B_D^{1/2}(\partial\Omega)\longrightarrow \mathbf{C}\\ \left\langle \begin{bmatrix}\gamma\\ 0\end{bmatrix}, \begin{bmatrix}\eta\\ 0\end{bmatrix}\right\rangle_D = \langle(n\times\gamma),\eta\rangle\,.\end{gathered}\tag{43}$$

Note that

$$\langle F, G\rangle_A = \langle P_N F, P_N G\rangle_N + \langle P_D F, P_D G\rangle_D\,.$$

If U and V solve the Beltrami system in Ω, then

$$\begin{aligned}\left\langle U|_{\partial\Omega}^-, V|_{\partial\Omega}^-\right\rangle_A &= \int_{\partial\Omega} A(n)U\cdot V\,dS\\ &= \int_\Omega A(\nabla)U\cdot V\,dx - \int_\Omega U\cdot A(\nabla)V\,dx = 0,\end{aligned}$$

and so in this case,

$$\left\langle P_N U|_{\partial\Omega}^-, P_N V|_{\partial\Omega}^-\right\rangle_N + \left\langle P_D U|_{\partial\Omega}^-, P_D V|_{\partial\Omega}^-\right\rangle_D = 0. \tag{44}$$

Next, we define the boundary integral operators that arise from the operator of the representation formula integral (17),

$$T := (A(\nabla)+kI)SA(n).$$

If

$$U = TF, \quad F \in B_D^{1/2}(\partial\Omega), \tag{45}$$
$$V = TG, \quad G \in B_N^{1/2}(\partial\Omega),$$

then $U, V \in H^1_{\mathrm{loc}}(\Omega^s)^4$ and $U, V \in H^1(\Omega)^4$ solve the Beltrami system, and the following jump relations can be obtained from the jump relations of the single layer operator and its derivatives, see [**17**],

$$P_D U|^{\pm}_{\partial\Omega} = T_{DD}F \pm \frac{1}{2}F, \qquad P_N U|^{\pm}_{\partial\Omega} = T^{+}_{ND}F, \tag{46}$$

and

$$P_D V|^{\pm}_{\partial\Omega} = T^{+}_{DN}G, \qquad P_N V|^{\pm}_{\partial\Omega} = T_{NN}G \pm \frac{1}{2}G, \tag{47}$$

where

$$T^{+}_{ND}F = P_N(TF)|_{\partial\Omega} = \begin{bmatrix} [-\mathrm{Div}\,(n \times Sn \times f) + kn \cdot Sn \times f]n \\ S\mathrm{Div}\,(n \times f) \end{bmatrix} \tag{48}$$

and

$$T_{DD}F = \begin{bmatrix} -(n\times)^2[\nabla \times S(n \times f) + kS(n \times f)] \\ 0 \end{bmatrix} \tag{49}$$

for

$$F = \begin{bmatrix} f \\ 0 \end{bmatrix} \in B_D^{1/2}(\partial\Omega),$$

and where

$$T^{+}_{DN}G = \begin{bmatrix} -(n\times)^2[S(n \times \nabla_T \phi) - \nabla Sg - kS(n\phi)] \\ 0 \end{bmatrix} \tag{50}$$

and

$$T_{NN}G = \begin{bmatrix} [\mathrm{Div}\,(n \times S(n\phi)) - \partial_n Sg - kn \cdot S(n\phi)]n \\ D\phi + kSg \end{bmatrix} \tag{51}$$

for

$$G = \begin{bmatrix} gn \\ \phi \end{bmatrix} \in B_N^{1/2}(\partial\Omega).$$

Here D is the double layer operator

$$D\phi(x) = \int_{\partial\Omega} \frac{\partial \Phi_x}{\partial n(y)}(y)\phi(y)\,dS(y).$$

In computing the expressions we have applied the differential formulae

$$\nabla \times S(n\phi) = -S(n \times \nabla_T \phi), \qquad \nabla \cdot S(n \times f) = S(\mathrm{Div}\,(n \times f)). \tag{52}$$

The ranges of operators $T^+_{DN}, T^+_{ND}, T_{NN}, T_{DD}$ are in $B^{1/2}(\partial\Omega)$. The $+$ sign in T^+_{DN}, T^+_{ND} is indicating the degree $+1/2$ of the space. For duality reasons, we need to also define the operators T^-_{DN} and T^-_{ND} on $B^{-1/2}(\partial\Omega)$.

If $U \in L^2(\Omega)^4$ solves the Beltrami system, we define $U|^{\pm}_{\partial\Omega} \in B^{-1/2}(\partial\Omega)$ by

$$\begin{aligned}\left\langle U|^{\pm}_{\partial\Omega}, V\right\rangle_A = \mp \int_{\Omega^\pm} U \cdot (A(\nabla) - kI)\,(\eta A(n) V)\,dx,\\ V \in B^{1/2}(\partial\Omega),\end{aligned} \tag{53}$$

where $\eta : H^{1/2}(\partial\Omega) \to H^1_{\mathrm{loc}}(\Omega^\pm)$ is a right inverse of the trace operator. If $F \in B^{-1/2}_D(\partial\Omega)$ and $G \in B^{-1/2}_N(\partial\Omega)$, then $TF, TG \in L^2_{\mathrm{loc}}(\mathbf{R}^3)^4$ satisfy the Beltrami system, and we can define

$$T^-_{ND}F \in B^{-1/2}_N(\partial\Omega), \qquad T^-_{DN}G \in B^{-1/2}_D(\partial\Omega),$$

by

$$\begin{aligned}\left\langle T^-_{ND}F, P_N W\right\rangle_A &= \langle (TF)|_{\partial\Omega}, P_N W\rangle_A\,,\\ \left\langle T^-_{DN}G, P_D W\right\rangle_A &= \langle (TG)|_{\partial\Omega}, P_D W\rangle_A\,,\end{aligned} \quad W \in B^{1/2}(\partial\Omega). \tag{54}$$

The definitions (48)–(50) and (54) agree,

$$T^-_{ND}|_{B^{1/2}_D(\partial\Omega)} = T^+_{ND}, \qquad T^-_{DN}|_{B^{1/2}_N(\partial\Omega)} = T^+_{DN},$$

which can be seen by integrating by parts with smooth functions that are dense.

Lemma 8.1. *The adjoints with respect to the duality* $\langle\cdot,\cdot\rangle_A$ *are*

$$T_{ND}^{+*} = -T_{DN}^{-}, \qquad T_{DN}^{+*} = -T_{ND}^{-},$$
$$T_{ND}^{-*} = -T_{DN}^{+}, \qquad T_{DN}^{-*} = -T_{ND}^{+}. \tag{55}$$

Proof. By the density of smooth functions, the claim follows, if we show that

$$\langle G, T_{ND}F\rangle_A = -\langle T_{DN}G, F\rangle_A, \quad \langle F, T_{DN}G\rangle_A = -\langle T_{ND}F, G\rangle_A,$$

for smooth

$$F = \begin{bmatrix} f \\ 0 \end{bmatrix} \in B_D(\partial\Omega), \qquad G = \begin{bmatrix} ng \\ \phi \end{bmatrix} \in B_N(\partial\Omega)$$

in which case the $\pm$ operators agree.

Now,

$$\langle G, T_{ND}F\rangle_A = \int_{\partial\Omega} A(n)G \cdot \left(A(\nabla)SA(n)F + kSA(n)F\right)\, dS.$$

By (52),

$$\begin{aligned}
\int_{\partial\Omega} & A(n)G \cdot A(\nabla)SA(n)F\, dS \\
&= \int_{\partial\Omega} \begin{bmatrix} -\phi n \\ g \end{bmatrix} \cdot \begin{bmatrix} -\mathrm{Div}\,(n \times S(n \times f))n \\ S\mathrm{Div}\,(n \times f) \end{bmatrix} \\
&= \int_{\partial\Omega} (-n \times S(n \times \nabla_T \phi) + n \times \nabla_T Sg) \cdot f\, dS \\
&= \int_{\partial\Omega} (n \times (\nabla \times S(n\phi) + \nabla_T Sg)) \cdot f\, dS \\
&= -\int_{\partial\Omega} A(n)A(\nabla)SA(n)G \cdot F\, dS.
\end{aligned}$$

Also,

$$\int_{\partial\Omega} A(n)G \cdot SA(n)F\, dS = -\int_{\partial\Omega} A(n)SA(n)G \cdot F\, dS.$$

Hence,

$$\begin{aligned}\langle G, T_{ND}F\rangle_A &= -\int_{\partial\Omega} A(n)\,(A(\nabla)SA(n)G + kSA(n)G)\cdot F\,dS\\ &= -\langle T_{DN}G, F\rangle_A\,.\end{aligned}$$

Similarly,

$$\langle F, T_{DN}G\rangle_A = \int_{\partial\Omega} A(n)F\cdot(A(\nabla)SA(n)G + kSA(n)G)\;dS,$$

where, by (52),

$$\begin{aligned}&\int_{\partial\Omega} A(n)F\cdot A(\nabla)SA(n)G\,dS\\ &\qquad= \int_{\partial\Omega}(n\times f)\cdot(S(n\times\nabla_T\phi) - \nabla Sg)\;dS\\ &\qquad= \int_{\partial\Omega}(\mathrm{Div}\,(n\times S(n\times f))\phi + S(\mathrm{Div}\,(n\times f))g)\;dS\\ &\qquad= -\int_{\partial\Omega} A(n)A(\nabla)SA(n)F\cdot G\,dS.\end{aligned}$$

Hence,

$$\begin{aligned}\langle F, T_{DN}G\rangle_A &= -\int_{\partial\Omega} A(n)\,(A(\nabla)SA(n)F + kSA(n)F)\cdot G\,dS\\ &= -\langle T_{ND}F, G\rangle_A\,. \qquad \square\end{aligned}$$

A similar computation as in the previous lemma shows that

$$\begin{aligned}\langle T_{NN}G, H\rangle_N &= -\langle G, T_{NN}H\rangle_N\,,\\ \langle T_{DD}F, E\rangle_D &= -\langle F, T_{DD}E\rangle_D\,,\end{aligned}\tag{56}$$

for every $G, H \in B^{1/2}_N(\partial\Omega)$ and $F, E \in B^{1/2}_D(\partial\Omega)$.

We collect the needed Fredholm type properties of $T^{\pm}_{ND}, T^{\pm}_{DN}$ in the next lemma.

Lemma 8.2. *Let $H_0(\partial\Omega)$ be finite dimensional. The ranges of operators*

$$\begin{aligned} T_{ND}^{+} &: B_D^{1/2} \to B_N^{1/2}(\partial\Omega), \quad T_{ND}^{-} : B_D^{-1/2}(\partial\Omega) \to B_N^{-1/2}(\partial\Omega), \\ T_{DN}^{+} &: B_N^{1/2}(\partial\Omega) \to B_D^{1/2}(\partial\Omega), \quad T_{DN}^{-} : B_N^{-1/2}(\partial\Omega) \to B_D^{-1/2}(\partial\Omega), \end{aligned}$$

are closed, and

$$\begin{aligned} R(T_{DN}^{+}) &= B_D^{1/2}(\partial\Omega) \cap R(T_{DN}^{-}), \\ R(T_{ND}^{+}) &= B_N^{1/2}(\partial\Omega) \cap R(T_{ND}^{-}). \end{aligned}$$

Proof. To prove the claim for T_{ND}^{-}, define

$$T'_{ND} = \begin{bmatrix} n\cdot & 0 \\ 0 & 1 \end{bmatrix} T_{ND}^{-} B(\nabla), \quad B(\nabla) = \begin{bmatrix} \nabla_T & n \times \nabla_T \\ 0 & 0 \end{bmatrix},$$

on $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$. Consider T'_{ND} as a pseudodifferential operator on $\mathcal{E}'(\partial\Omega)^2$. The expression for operator T'_{ND} is

$$T'_{ND} = \begin{bmatrix} -\mathrm{Div}\ n \times Sn \times \nabla_T + kn \cdot Sn \times \nabla_T & \mathrm{Div}\ n \times S\nabla_T - kn \cdot S\nabla_T \\ 0 & -S\mathrm{Div}\ \nabla_T \end{bmatrix},$$

so, the principal symbol of T'_{ND} is, see [**8**, Lemma 4.5],

$$\sigma_{pr}(T'_{ND}) = \begin{bmatrix} |\xi| & 0 \\ 0 & |\xi| \end{bmatrix}.$$

By Gårding's inequality T'_{ND} is a coercive pseudodifferential operator of degree 1, and

$$T'_{ND} : \begin{matrix} H^{1/2}(\partial\Omega) \\ \times \\ H^{1/2}(\partial\Omega) \end{matrix} \longrightarrow \begin{matrix} H^{-1/2}(\partial\Omega) \\ \times \\ H^{-1/2}(\partial\Omega) \end{matrix},$$

is a Fredholm operator with index zero [**17**]. Hence, T'_{ND} has a parametrix P'_{ND} of degree -1 so that

$$P'_{ND} T'_{ND} = I + \Psi,$$

where Ψ is of degree -1, [**26**]. Particularly, the range of T'_{ND} is closed. By the Hodge decomposition (32), the range of T^-_{ND} is

$$\begin{aligned} R(T^-_{ND}) &= T^-_{ND} B_D^{-1/2}(\partial\Omega) \\ &= T^-_{ND}\left(B(\nabla) \begin{bmatrix} H^{1/2}(\partial\Omega) \\ \times \\ H^{1/2}(\partial\Omega) \end{bmatrix} \oplus H_0(\partial\Omega)\right) \\ &= \begin{bmatrix} n & 0 \\ 0 & 1 \end{bmatrix} R(T'_{ND}) + T^-_{ND} H_0(\partial\Omega), \end{aligned}$$

which is closed, because $T^-_{ND}H_0(\partial\Omega)$ is finite dimensional.

Next, check the claims for T^+_{ND}.

Let $G \in R(T^+_{ND})$. Then $G \in B_N^{1/2}(\partial\Omega)$ and there is such an $F \in B_D^{1/2}(\partial\Omega)$ that

$$G = T^+_{ND}F = T^-_{ND}F.$$

Hence,

$$R(T^+_{ND}) \subset B_N^{1/2}(\partial\Omega) \cap R(T^-_{ND}).$$

Let $G \in B_N^{1/2}(\partial\Omega) \cap R(T^-_{ND})$, and assume

$$G = T^-_{ND}F$$

for $F \in B_D^{-1/2}(\partial\Omega)$. Represent F with the Hodge decomposition

$$F = B(\nabla)\phi + \widetilde{f},\ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix},\ \phi_j \in H^{1/2}(\partial\Omega),\ \widetilde{f} = \begin{bmatrix} f \\ 0 \end{bmatrix},\ f \in H_0(\partial\Omega).$$

Then

$$P'_{ND}T'_{ND}\phi = \phi + \Psi\phi,$$

or

$$\phi = P'_{ND}T'_{N'D}\phi - \Psi\phi = P'_{ND}\begin{bmatrix} n\cdot & 0 \\ 0 & 1 \end{bmatrix}(G - T^-_{ND}\widetilde{f}) - \Psi\phi \in \begin{matrix} H^{3/2}(\partial\Omega) \\ \times \\ H^{3/2}(\partial\Omega) \end{matrix},$$

because $G \in B_N^{1/2}(\partial\Omega)$, $T^-_{ND}\widetilde{f}$ is smooth, and the operators P'_{ND} and Ψ are of degree -1. Hence,

$$F = B(\nabla)\phi + \widetilde{f} \in B_D^{1/2}(\partial\Omega),$$

and

$$G = T^-_{ND} F = T^+_{ND} F \in R(T^+_{ND}).$$

This proves

$$B^{1/2}_N(\partial\Omega) \cap R(T^-_{ND}) \subset R(T^+_{ND}).$$

The range $R(T^+_{ND})$ is closed in $B^{1/2}_N(\partial\Omega)$, because

$$B^{1/2}_N(\partial\Omega) \cap R(T^-_{ND}) = R(T^+_{ND}),$$

and $R(T^-_{ND})$ is closed in $B^{-1/2}_N(\partial\Omega)$.

The argument for $R(T^-_{DN})$ and $R(T^+_{DN})$ is similar, but consider the operator

$$T'_{DN} = \begin{bmatrix} -(n\times)^2 & 0 \end{bmatrix} T^-_{DN} \begin{bmatrix} n \operatorname{Div} \\ \operatorname{Curl} \end{bmatrix} : TH^{1/2}(\partial\Omega) \longrightarrow TH^{-1/2}(\partial\Omega). \quad \square$$

Theorem 8.3. *Let $H_0(\partial\Omega)$ be finite dimensional. Let*

$$F = \begin{bmatrix} f \\ 0 \end{bmatrix} \in B^{1/2}_D(\partial\Omega)$$

satisfy

$$\int_{\partial\Omega} (n \times f) \cdot h \, dS = 0 \tag{57}$$

for all $h \in N_0(\partial\Omega)$. Then there exists a solution $G \in B^{1/2}_N(\partial\Omega)$ of

$$\begin{cases} T^+_{DN} G = \frac{1}{2} F - T_{DD} F, \\ \int_{\partial\Omega_j} \begin{pmatrix} 0 & 1 \end{pmatrix} (T_{NN} G + \frac{1}{2} G + T_{ND} F) \, dS = 0 \quad j = 1, \dots, J, \end{cases} \tag{58}$$

and

$$U = T(F + G) \in H^1_{\mathrm{loc}}(\Omega^s)^4 \tag{59}$$

is a radiating solution satisfying (37) *for the exterior Dirichlet boundary value problem for the Beltrami system* (29),

$$\begin{cases} A(\nabla)U - kU = 0 \quad \text{in } \Omega^s, \\ P_D(n) U|^+_{\partial\Omega} = F. \end{cases}$$

Proof. Let $F \in B_D^{1/2}(\partial\Omega)$ satisfy (57). We begin by proving that the first equation of (58) has a solution. It is enough to show that the equation

$$T_{DN}^{-}G = \frac{1}{2}F - T_{DD}F \tag{60}$$

has a solution $G \in B_N^{-1/2}(\partial\Omega)$. Namely, now the right hand side is in $B_D^{1/2}(\partial\Omega)$, so by Lemma 8.2, if (60) has a solution in $B_N^{-1/2}(\partial\Omega)$, then there also exists a solution $G \in B_N^{1/2}(\partial\Omega)$ of (58). Then U defined by (59) is a radiating solution for the Beltrami system in $H^1_{\mathrm{loc}}(\Omega^s)^4$ with Dirichlet's boundary value

$$P_D U|_{\partial\Omega}^{+} = T_{DD}F + \frac{1}{2}F + T_{DN}G = F.$$

To show the solvability for (60), we show that

$$\left\langle \frac{1}{2}F - T_{DD}F, E \right\rangle_D = 0$$

for every $E \in \mathrm{Ker}\,(T_{ND}^{+})$. This proves the claim since

$$R(T_{DN}^{-}) = \overline{R(T_{DN}^{-})} = \mathrm{Ker}\,(T_{DN}^{-*})^{\perp} = \mathrm{Ker}\,(T_{ND}^{+})^{\perp},$$

where the perpendicularity is with respect to the dual system. So, let $E \in \mathrm{Ker}\,(T_{ND}^{+})$, and define

$$V = TE.$$

Now in Ω^s, V is a radiating solution of the Beltrami system with

$$P_N V|_{\partial\Omega}^{+} = T_{ND}^{+}E = 0,$$

and so

$$V = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

in Ω^s with $v|_{\partial\Omega}^{+} \in N_0(\partial\Omega)$ by (35). In Ω, V solves the Beltrami system with boundary values

$$P_N V|_{\partial\Omega}^{-} = T_{ND}^{+}E = 0,$$

and

$$P_D V|_{\partial\Omega}^+ - P_D V|_{\partial\Omega}^- = \left(T_{DD}E + \frac{1}{2}E\right) - \left(T_{DD}E - \frac{1}{2}E\right) = E.$$

Let

$$W = TF.$$

Then, by (56), (44) and (46),

$$\begin{aligned}\left\langle \frac{1}{2}F - T_{DD}F, E \right\rangle_D &= \left\langle \frac{1}{2}F - T_{DD}F, P_D V|_{\partial\Omega}^+ - P_D V|_{\partial\Omega}^- \right\rangle_D \\ &= \left\langle F, \frac{1}{2}P_D V|_{\partial\Omega}^+ + T_{DD}P_D V|_{\partial\Omega}^+ \right\rangle_D \\ &\quad + \left\langle P_D W|_{\partial\Omega}^-, P_D V|_{\partial\Omega}^- \right\rangle_D \\ &= \left\langle F, P_D V|_{\partial\Omega}^+ \right\rangle_D + \left\langle P_N W|_{\partial\Omega}^-, P_N V|_{\partial\Omega}^- \right\rangle_N \\ &= \int_{\partial\Omega} (n \times f) \cdot v \, dS \\ &= 0\end{aligned}$$

because $v|_{\partial\Omega}^+ \in N_0(\partial\Omega)$. Hence, (60) is solvable.

Next, let G be a solution for the first equation (58), and let

$$V = \begin{bmatrix} -\frac{1}{k}\nabla\phi \\ \phi \end{bmatrix},$$

where ϕ is the radiating solution for the Helmholtz equation with

$$\phi|_{\partial\Omega_j} = c_j = \frac{1}{|\partial\Omega_j|} \int_{\partial\Omega_j} (\,0 \quad 1\,)\,(T_{NN}G + \frac{1}{2}G + T_{ND}^+ F)\, dS.$$

Denote

$$H = P_N V|_{\partial\Omega}^+.$$

Now, V is a radiating solution for the Beltrami system with vanishing Dirichlet's boundary values. Hence,

$$T_{DN}^+ H = P_D V|_{\partial\Omega}^+ = 0,$$

and

$$H = P_N V|^+_{\partial\Omega} = T_{NN} H + \frac{1}{2} H.$$

Now $G - H$ satisfies the first equation of (58) and also the second equation,

$$\int_{\partial\Omega_j} (\,0 \quad 1\,)\,(T_{NN}(G-H) + \frac{1}{2}(G-H) + T^+_{ND} F)\,dS$$
$$= \int_{\partial\Omega_j} (\,0 \quad 1\,)\,(T_{NN} G + \frac{1}{2} G + T^+_{ND} F)\,dS - \int_{\partial\Omega_j} c_j\,dS = 0.$$

Redefine

$$G := G - H,$$

which is a solution for (58). Now,

$$U = T(F+G)$$

is a radiating solution for the Beltrami system and

$$P_D U|^+_{\partial\Omega} = T_{DD} F + \frac{1}{2} F + T^+_{DN} G = F.$$

The second equation of (58) implies that the condition (37) is fulfilled. □

Theorem 8.4. *Let $H_0(\partial\Omega)$ be finite dimensional, and let*

$$P_0 : B^{1/2}_D(\partial\Omega) \longrightarrow B^{1/2}_D(\partial\Omega) \tag{61}$$

be the orthogonal projection onto $N_0(\partial\Omega) \times \{0\}$, see (34). Let $G \in B^{1/2}_N(\partial\Omega)$ be such that

$$\left\langle G, \begin{pmatrix} -\partial_n \phi n \\ k\phi \end{pmatrix} \right\rangle_N = 0 \tag{62}$$

when ϕ is a radiating solution of the Helmholtz equation in Ω^s and $\phi|^+_{\partial\Omega_j}$ is constant for each component $\partial\Omega_j$, $j = 1, \dots, J$. Then there exists an $F \in B^{1/2}_D(\partial\Omega)$ which satisfies

$$\begin{cases} T^+_{ND} F = \frac{1}{2} G - T_{NN} G, \\ P_0(T_{DD} F + \frac{1}{2} F + T^+_{DN} G) = 0, \end{cases} \tag{63}$$

and

$$U = T(F+G) \in H^1_{\mathrm{loc}}(\Omega^s)^4 \tag{64}$$

is a radiating solution satisfying (36) *for the exterior Neumann boundary value problem for the Beltrami system* (29),

$$\begin{cases} A(\nabla)U - kU = 0 & \text{in } \Omega^s \\ P_N(n)U|^+_{\partial\Omega} = G. \end{cases}$$

Proof. Begin by proving that the first equation in (62) has a solution. Let $H \in \operatorname{Ker}(T^+_{DN})$, and define

$$V = TH.$$

Then

$$P_D V|^\pm_{\partial\Omega} = T^+_{DN} H = 0,$$

and by the uniqueness proof for Dirichlet's problem, Theorem 7.1,

$$V|_{\Omega^s} = \begin{pmatrix} -\nabla\phi \\ k\phi \end{pmatrix},$$

where $\phi|_{\partial\Omega_j}$ is constant for each component $\partial\Omega_j$, $j = 1, \ldots, J$. So

$$P_N V|^+_{\partial\Omega} = \begin{pmatrix} -\partial_n \phi n \\ k\phi \end{pmatrix}$$

is that type of function to which condition (62) for G is applied. On the other hand,

$$P_N V|^\pm_{\partial\Omega} = T^+_{NN} H \pm \frac{1}{2} H,$$

and so

$$H = P_N V|^+_{\partial\Omega} - P_N V|^-_{\partial\Omega}.$$

The representation formula gives

$$V|_{\Omega^s} = T(V|^+_{\partial\Omega}) = T(P_N V|^+_{\partial\Omega}),$$

so we have

$$P_N V|^+_{\partial\Omega} = T_{NN} P_N V|^+_{\partial\Omega} + \frac{1}{2} P_N V|^+_{\partial\Omega}.$$

Let

$$W = TG.$$

Now, by (47), (44) and (56),

$$\begin{aligned}\left\langle T_{NN}G - \frac{1}{2}G, H\right\rangle_N &= \left\langle T_{NN}G - \frac{1}{2}G, P_N V|^+_{\partial\Omega}\right\rangle_N \\ &\quad - \left\langle P_N W|^-_{\partial\Omega}, P_N V|^-_{\partial\Omega}\right\rangle_N \\ &= \left\langle G, -T_{NN}P_N V|^+_{\partial\Omega} - \frac{1}{2}P_N V|^+_{\partial\Omega}\right\rangle_N \\ &\quad + \left\langle P_D W|^-_{\partial\Omega}, P_D V|^-_{\partial\Omega}\right\rangle_D \\ &= -\left\langle G, P_N V|^+_{\partial\Omega}\right\rangle_N \\ &= 0\end{aligned}$$

by condition (62). Hence,

$$\frac{1}{2}G - T_{NN}G \in \operatorname{Ker}(T^+_{DN})^\perp = R(T^-_{ND}),$$

and because $G \in B^{1/2}_N(\partial\Omega)$, there is, by Lemma 8.2, a solution $F \in B^{1/2}_D(\partial\Omega)$ for the first equation of (63).

Next, let F be a solution for the first equation of (63), and define

$$H = P_0(T_{DD}F + \frac{1}{2}F + T^+_{DN}G).$$

By the definition (34) there is a radiating V such that

$$V|^+_{\partial\Omega} = H,$$

and so

$$H = P_D V|^+_{\partial\Omega} = P_D(T(V|^+_{\partial\Omega}))|^+_{\partial\Omega} = T_{DD}H + \frac{1}{2}H,$$

and

$$T^+_{ND}H = 0.$$

Redefine

$$F := F - H,$$

which is a solution for both equations of (63). Now,

$$U = T(F + G)$$

is a radiating solution for the Beltrami system, and

$$P_N U|_{\partial\Omega}^{+} = T_{ND}^{+} F + T_{NN} G + \frac{1}{2} G = G.$$

The second equation in (63) implies that the condition (36) is fulfilled. □

Corollary 8.5. *Let $H_0(\partial\Omega)$ be finite dimensional, and let*

$$P_0 : T\partial\Omega \longrightarrow T\partial\Omega \tag{65}$$

be the orthogonal projection onto $N_0(\partial\Omega)$, see (34)*. Let $g \in H^{1/2}(\partial\Omega)$ satisfy*

$$\int_{\partial\Omega_j} g \, dS = 0, \tag{66}$$

for every component $\partial\Omega_j$, $j = 1, \dots, J$. Then there is a solution $f \in TH^{1/2}(\partial\Omega)$ for

$$\begin{cases} -\mathrm{Div}\,(n \times S(n \times f)) + kn \cdot S(n \times f) = \frac{1}{2} g + \partial_n S g \\ S\mathrm{Div}\,(n \times f) = -kSg, \\ P_0 \left(-(n\times)^2 [\nabla \times S(n \times f) + kS(n \times f)] \right) = P_0 \nabla_T S g, \end{cases} \tag{67}$$

and

$$u = \nabla \times S(n \times f) - \nabla S g + kS(n \times f) \in H^1_{\mathrm{loc}}(\Omega^s)^3 \tag{68}$$

is a radiating solution satisfying (36) *of the exterior Neumann boundary value problem for Beltrami fields* (24),

$$\begin{cases} \nabla \times u = ku \ \text{in} \ \Omega^s, \\ n \cdot u|_{\partial\Omega}^{+} = g. \end{cases}$$

Proof. Put

$$G = \begin{pmatrix} gn \\ 0 \end{pmatrix}.$$

Now G satisfies the condition (62), so there is a solution

$$F = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

for (63), which is (67) with the given G. Theorem 8.4 gives a solution

$$U = \begin{bmatrix} u \\ \phi \end{bmatrix}$$

of the exterior Neumann boundary value problem of the Beltrami system and u satisfies the condition (36). The vector field part of (64) is (68). Now, the scalar field $\phi \equiv 0$, because the exterior Dirichlet problem for the Helmholtz equation is unique. Hence, u is a radiating Beltrami field that satisfies

$$n \cdot u = g. \qquad \square$$

Remark. It is easy to see that F is the boundary value for some radiating U that solves the Beltrami system in the exterior domain if and only if

$$TF - \frac{1}{2}F = 0$$

on $\partial\Omega$. Hence, $N_0(\partial\Omega)$ can be expressed in practice by computing the solutions for

$$-(n\times)^2[\nabla \times S(n \times (f + \nabla_T \phi)) + kS(n \times (f + \nabla_T \phi))] - \frac{1}{2}(f + \nabla_T \phi) = 0,$$

where $f \in H_0(\partial\Omega)$. The space $H_0(\partial\Omega)$ can be expressed by computing a basis for

$$\text{Ker}\,(\text{Div}\,) \cap \text{Ker}\,(\text{Curl}).$$

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