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A CONVERGENCE ANALYSIS OF THE MIDPOINT RULE FOR FIRST KIND VOLTERRA INTEGRAL EQUATIONS WITH NOISY DATA

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ABSTRACT. In this paper we study convergence of the midpoint rule for linear Volterra integral equations of the first kind as a regularization method. We prove convergence and convergence rates as the noise level tends to zero. Numerical tests for an application to a thermoacoustic inverse problem illustrate performance of the method.

1. Introduction. In this paper we consider linear Volterra integral equations of the first kind

$$\int_a^x \mathbf{k}\,(x,\xi)q(\xi)\,d\xi = f(x) \quad x \in (a,b), \tag{1}$$

and their regularization by application of the midpoint quadrature rule to discretize the integral in (1). Regularization is necessary due to the fact that for a smooth kernel $\mathbf{k}$, (1) is ill-posed in the sense that its solution q (as an $L^{p/(p-1)}$ function) does not depend continuously on the data f (as an L^p function). Usually the data is not given exactly but only a noisy version f^δ is available, so that the lack of stability becomes crucial. We will here assume that we have an estimate δ of the noise level with respect to the L^∞ norm

$$\|f - f^\delta\|_{L^\infty} \le \delta. \tag{2}$$

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The degree of ill-posedness of (1) is given by the smoothing properties of the integral operator

$$T : v \longmapsto \int_0^{\cdot} \mathbf{k}\,(\cdot,\xi)v(\xi)\,d\xi, \tag{3}$$

cf. [**10**]. If

$$|\mathbf{k}\,(x,x)| \geq \gamma > 0 \quad \text{for all } x \in (a,b), \tag{4}$$

and $\mathbf{k}$ is differentiable with respect to its first argument, (1) can be written as a (well-posed, cf., e.g., [**3, 9**]) second kind Volterra integral equation

$$\mathbf{k}\,(x,x)q(x) + \int_0^x \partial_1 \mathbf{k}\,(x,\xi)q(\xi)\,d\xi = f_x(x) \tag{5}$$

containing first derivatives of the data f, so we can regard (3) as a *one-smoothing* operator which implies that (1) is as ill-posed as one numerical differentiation.

Combining collocation with the midpoint rule for approximating the integral in (1) yields

$$h\sum_{j=1}^{k} \mathbf{k}\,(x_k, x_{j-1/2})q^\delta_{j-1/2} = f^\delta(x_k), \quad k = 1,\dots,N, \tag{6}$$

where

$$h := \frac{b-a}{N}, \quad x_j = a + jh, \quad x_{j-1/2} = a + \left(j - \frac{1}{2}\right)h$$

for $q^\delta_{j-1/2} \approx q((j-(1/2))h)$, $j = 1,\dots,N$. Using the point values from (6), we define q^δ_h as the piecewise constant interpolate

$$q^\delta_h \in Q_h, \qquad q^\delta_h(x) = q^\delta_{j-1/2} \quad \text{for } x \in (x_{j-1}, x_j], \tag{7}$$

where Q_h is the space of piecewise constant functions with breakpoints x_k, $k = 1,\dots,N$. This leads to a convergent method for exact data, cf. [**11**]. For the convergence analysis of further quadrature rule based solution methods for Volterra integral equations of the first kind, we refer to [**5, 10, 11**], and the references therein.

The aim of this paper is to provide an analysis of (6) in the situation of noisy data (2) including convergence (rates) and a priori as well as a posteriori regularization parameter choices as $\delta \to 0$. Moreover, we will apply this method to an inverse thermoacoustic problem in the context of combustion noise.

2. Convergence analysis of the midpoint rule with noisy data. For noisy data, the stepsize h plays the role of a regularization parameter and has to be appropriately chosen. The following theorem gives an a priori rule for that purpose.

Theorem 1. *Let* $\mathbf{k} \in C([a,b]^2)$, $\partial_1 \mathbf{k} \in C([a,b]^2)$, (4) *and* (2) *hold. Moreover, let* q_h^δ *be defined by* (6), (7), *and let* $N_* = N_*(\delta)$, $h_* = (b-a)/N_*$ *be chosen according to*

$$N_* \longrightarrow \infty \quad \text{and} \quad N_*\delta \longrightarrow 0 \qquad \text{as } \delta \to 0. \tag{8}$$

Then

(a) *If* $q \in L^1(a,b)$

$$\sum_{k=1}^{N_*} \left| \int_{x_{k-1}}^{x_k} [q - q_{h_*}^\delta](\xi)\, d\xi \right| \longrightarrow 0 \quad \text{as } \delta \to 0$$

and

$$\|q - q_{h_*}^\delta\|_{L^1(a,b)} \longrightarrow 0 \quad \text{as } \delta \to 0. \tag{9}$$

(b) *If* $q \in L^\infty(a,b)$,

$$\max_{k\in\{1,\dots,N_*\}} \frac{1}{h_*} \left| \int_{x_{k-1}}^{x_k} [q - q_{h_*}^\delta](\xi)\, d\xi \right| \longrightarrow 0 \quad \text{as } \delta \to 0$$

and if $q \in C(a,b)$,

$$\|q - q_{h_*}^\delta\|_{L^\infty(a,b)} \longrightarrow 0 \quad \text{as } \delta \to 0. \tag{10}$$

Proof. Denoting by P_h the projection onto the space Q_h of piecewise constant functions with breakpoints x_k, $k = 1, \dots, N$

$$P_h g(x) = g(x_{j-1/2}) \text{ for } x \in (x_{j-1}, x_j], \quad j = 1, \dots, N$$

we can rewrite (6) as

$$\int_a^{x_k} (P_h \mathbf{k}(x_k, \cdot))(\xi) q_h^\delta(\xi)\, d\xi = f^\delta(x_k) \tag{11}$$

for $k = 1, \dots, N$. Subtracting (11) from the collocation of (1) at the breakpoints

$$\int_a^{x_k} \left(\mathbf{k}(x_k, \xi) q(\xi) - (P_h \mathbf{k}(x_k, \cdot))(\xi) q_h^\delta(\xi) \right) d\xi = f(x_k) - f^\delta(x_k)$$

i.e.,

$$\begin{aligned} &\int_a^{x_k} P_h \mathbf{k}(x_k, \cdot))(\xi)[q - q_h^\delta](\xi)\, d\xi \\ &\qquad = -\int_a^{x_k} [\mathbf{k}(x_k, \cdot) - (P_h \mathbf{k}(x_k, \cdot)](\xi)] q(\xi)\, d\xi + f(x_k) - f^\delta(x_k) \end{aligned}$$

and taking the difference quotient yields a discrete analog of (5) for the error $q - q_h^\delta$

$$\begin{aligned} &\mathbf{k}(x_k, x_{k-1/2}) \frac{1}{h} \int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi \\ &\qquad + \sum_{j=1}^{k-1} \frac{\mathbf{k}(x_k, x_{j-1/2}) - \mathbf{k}(x_{k-1}, x_{j-1/2})}{h} \int_{x_{j-1}}^{x_j} [q - q_h^\delta](\xi)\, d\xi \\ &\quad = r_k^1 + r_k^2 + r_k^3 := r_k \end{aligned} \tag{12}$$

with

$$r_k^1 = -\frac{1}{h} \int_{x_{k-1}}^{x_k} ([I - P_h]\mathbf{k}(x_k, \cdot))(\xi) q(\xi)\, d\xi \tag{13}$$

$$r_k^2 = -\int_a^{x_{k-1}} [I - P_h] \frac{\mathbf{k}(x_k, \cdot) - \mathbf{k}(x_{k-1}, \cdot)}{h}(\xi) q(\xi)\, d\xi \tag{14}$$

$$r_k^3 = \frac{(f - f^\delta)(x_k) - (f - f^\delta)(x_{k-1})}{h},$$

where we have used the fact that

$$P_h \mathbf{k}(x_k, \cdot)) = \mathbf{k}(x_k, x_{j-1/2}) \text{ on } (x_{j-1}, x_j].$$

Since (4) and continuity of $\mathbf{k}$ by the Heine-Cantor theorem imply that

$$|\mathbf{k}(x_k, x_{k-1/2})| \geq \frac{\gamma}{2} > 0 \quad k = 1, \dots N, \tag{15}$$

for all N sufficiently large, this yields a recursive inequality of the form

$$\zeta_k \leq A \sum_{i=1}^{k-1} \zeta_i + B_k, \quad k = 1, \dots, N, \tag{16}$$

to which Theorem 7.1 in [**11**] applies, which is here written in a somewhat modified form

$$\zeta_k \leq (1+A)^{k-1} \max_{j \in \{1,\dots,k\}} B_j, \quad k = 1, \dots, N \tag{17}$$

that can be checked by induction as well. Here,

$$\begin{aligned}
\zeta_k &= \frac{1}{h} \int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi, \\
A &= \frac{2h}{\gamma} \|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)} \\
&\geq \max_{j,k \in \{1,\dots,N\}} \frac{2}{\gamma} |\mathbf{k}(x_k, x_{j-1/2}) - \mathbf{k}(x_{k-1}, x_{j-1/2})|, \\
B_k &= \frac{2|r_k|}{\gamma}.
\end{aligned} \tag{18}$$

Similarly, in place of the l_∞ type estimate (17) one can derive an l_p type estimate

$$|\vec{\zeta}|_{l^p} \leq |\vec{M}^p|_{l^{P/(P-1)}} |\vec{B}|_{l^P} \tag{19}$$

for $p \in [1, \infty)$, $P \in [1, \infty]$,

$$M_j^p = \prod_{i=j+1}^{N} (1 + A(i-1)^{(p-1)/p}),$$

by simple induction. Application of (17) and of (19) with $p = P = 1$ to (16) with (18) yields

$$\begin{aligned}(20)\quad &\max_{k\in\{1,\dots,N\}} \frac{1}{h}\left|\int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi\right| \\ &\leq \frac{2}{\gamma}\left(1 + \frac{2}{\gamma}\|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)} \frac{b-a}{N}\right)^N \max_{k\in\{1,\dots,N\}} |r_k| \\ &\leq \frac{2}{\gamma} \exp\big((2(b-a)/\gamma)\|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)}\big) \max_{k\in\{1,\dots,N\}} |r_k|\end{aligned}$$

and

$$\begin{aligned}(21)\quad &\sum_{k=1}^N \left|\int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi\right| \\ &\leq \frac{2}{\gamma} \exp\big((2(b-a)/\gamma)\|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)}\big)\; h\sum_{k=1}^N |r_k|,\end{aligned}$$

respectively.

Convergence of the right hand sides in (20), (21) can be seen as follows: First of all,

$$\begin{aligned}(22)\quad &\max_{k\in\{1,\dots,N\}} |r_k^1| \\ &\leq \|q\|_{L^\infty(a,b)} \max_{k\in\{1,\dots,N\}} \frac{1}{h}\int_{x_{k-1}}^{x_k} |\mathbf{k}(x_k,\xi) - \mathbf{k}(x_k, x_{k-1/2})|\, d\xi \\ &\leq \|q\|_{L^\infty(a,b)} \max_{k\in\{1,\dots,N\}} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\leq h/2} |\mathbf{k}(x_k,\xi) - \mathbf{k}(x_k,\zeta)| \\ &\longrightarrow 0 \quad \text{as } N\to\infty,\end{aligned}$$

since $\mathbf{k}$ as a continuous function on the compact set $[a,b]^2$ is uniformly continuous.

Moreover, with $j(\xi)$ such that $\xi \in (x_{j(\xi)-1}, x_{j(\xi)}]$,

(23)
$$\begin{aligned}
&\max_{k\in\{1,\dots,N\}} |r_k^2| \\
&\le \|q\|_{L^1(a,b)} \max_{k\in\{1,\dots,N\}} \sup_{\xi\in(a,b)} \left| [I-P_h] \frac{\mathbf{k}(x_k,\cdot)-\mathbf{k}(x_{k-1},\cdot)}{h}(\xi) \right| \\
&= \|q\|_{L^1(a,b)} \max_{k\in\{1,\dots,N\}} \sup_{\xi\in(a,b)} \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} (\partial_1 \mathbf{k}(x,\xi) \right. \\
&\qquad\qquad \left. - \partial_1 \mathbf{k}(x, x_{j(\xi)-1/2}))\, dx \right| \\
&\le \|q\|_{L^1(a,b)} \sup_{x\in(a,b)} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\le h/2} |\partial_1 \mathbf{k}(x,\xi) - \partial_1 \mathbf{k}(x,\zeta)| \\
&\longrightarrow 0 \quad \text{as } N \to \infty,
\end{aligned}$$

since $\partial_1 \mathbf{k}$ is uniformly continuous on the compact set $[a,b]^2$.

For the respective l_1 norms we get

(24)
$$\begin{aligned}
h\sum_{k=1}^N |r_k^1| &\le \sum_{k=1}^N \int_{x_{k-1}}^{x_k} |\mathbf{k}(x_k,\xi) - \mathbf{k}(x_k, x_{k-1/2}))q(\xi)|\, d\xi \\
&\le \sum_{k=1}^N \int_{x_{k-1}}^{x_k} |q(\xi)| \max_{k\in\{1,\dots,N\}} |\mathbf{k}(x_k,\xi) \\
&\qquad\qquad - \mathbf{k}(x_k, x_{k-1/2}))|\, dx\, d\xi \\
&\le \|q\|_{L^1(a,b)} \max_{k\in\{1,\dots,N\}} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\le h/2} |\mathbf{k}(x_k,\xi) - \mathbf{k}(x_k,\zeta))| \\
&\longrightarrow 0 \quad \text{as } N \to \infty,
\end{aligned}$$

and

(25)
$$\begin{aligned}
h\sum_{k=1}^N |r_k^2| &\le hN \max_{k\in\{1,\dots,N\}} |r_k^2| \\
&\le (b-a)\|q\|_{L^1(a,b)} \\
&\quad \times \sup_{x\in(a,b)} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\le h/2} |\partial_1 \mathbf{k}(x,\xi) - \partial_1 \mathbf{k}(x,\zeta)|
\end{aligned}$$

(26)
$$\longrightarrow 0 \quad \text{as } N \to \infty.$$

Finally,

$$\max_{k\in\{1,\dots,N_*\}} |r_k^3| \le \frac{2\delta}{h_*} = \frac{2}{b-a} N_* \delta \longrightarrow 0 \quad \text{as } \delta \to 0, \tag{27}$$

and

$$h_* \sum_{k=1}^{N_*} |r_k^3| \le 2N_*\delta \longrightarrow 0 \quad \text{as } \delta \to 0, \tag{28}$$

by the a priori choice (8).

To see (9), (10), note that on the other hand the left hand sides in (20), (21) can be estimated from below by

$$\begin{aligned}
(29)\quad &\max_{k\in\{1,\dots,N\}} \frac{1}{h}\left| \int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi \right| \\
&\ge \max_{k\in\{1,\dots,N\}} \frac{1}{h}\left| \int_{x_{k-1}}^{x_k} [\tilde{q}_h - q_h^\delta](\xi)\, d\xi \right| \\
&\quad - \max_{k\in\{1,\dots,N\}} \frac{1}{h}\left| \int_{x_{k-1}}^{x_k} [q - \tilde{q}_h](\xi)\, d\xi \right| \\
&= \max_{k\in\{1,\dots,N\}} |\tilde{q}_{k-1/2} - q_{k-1/2}^\delta| - \max_{k\in\{1,\dots,N\}} \frac{1}{h}\left| \int_{x_{k-1}}^{x_k} [q - \tilde{q}_h](\xi)\, d\xi \right| \\
&\ge \|\tilde{q}_h - q_h^\delta\|_{L^\infty(a,b)} - \|q - \tilde{q}_h\|_{L^\infty(a,b)} \\
&\ge \|q - q_h^\delta\|_{L^\infty(a,b)} - 2\|q - \tilde{q}_h\|_{L^\infty(a,b)} \quad \text{for all } \tilde{q}_h \in Q_h,
\end{aligned}$$

$$\begin{aligned}
(30)\quad &\sum_{k=1}^{N} \left| \int_{x_{k-1}}^{x_k} [q - q_h^\delta](\xi)\, d\xi \right| \\
&\qquad \ge \sum_{k=1}^{N} \left(\left| \int_{x_{k-1}}^{x_k} [\tilde{q}_h - q_h^\delta](\xi)\, d\xi \right| - \left| \int_{x_{k-1}}^{x_k} [q - \tilde{q}_h](\xi)\, d\xi \right| \right) \\
&\qquad = \|\tilde{q}_h - q_h^\delta\|_{L^1(a,b)} - \sum_{k=1}^{N} \left| \int_{x_{k-1}}^{x_k} [q - \tilde{q}_h](\xi)\, d\xi \right| \\
&\qquad \ge \|q - q_h^\delta\|_{L^1(a,b)} - 2\|q - \tilde{q}_h\|_{L^1(a,b)}. \quad \text{for all } \tilde{q}_h \in Q_h,
\end{aligned}$$

where $\tilde{q}_{k-1/2} = (\tilde{q}_h)|_{(x_{k-1},x_k)}$.

Convergence of $\inf_{\tilde q_h \in Q_h} \|q - \tilde q_h\|_{L^p}$ to zero as $h \to 0$ follows from denseness of piecewise constant functions with regular (such as equidistant, as assumed here) breakpoints in $L^p(a,b)$ for all $p \in [1,\infty]$. For $p \in [1,\infty)$ this follows from the fact that the Haar system is a basis of L^p. For $p = \infty$ we have

$$\begin{aligned}\inf_{\tilde q_h \in Q_h} &\|\tilde q_h - q\|_{L^\infty(a,b)} \\ &\le \max_{k \in \{1,\dots,N\}} \sup_{\sigma \in [x_{k-1},x_k]} \left| \frac{1}{h}\int_{x_{k-1}}^{x_k} q(\xi)\,d\xi - q(\sigma)\right| \\ &\le \sup_{|\xi-\sigma|\le h} |q(\xi) - q(\sigma)| \longrightarrow 0 \quad \text{as } h \to 0\end{aligned}$$

by uniform continuity of q on the compact interval $[a,b]$. □

Remark 1. Note that for obtaining L^1 convergence according to Theorem 1 (a), an L^1 type noise estimate $h\sum_{k=1}^N |(f - f^\delta)(x_k)| \le \delta$ for all $N \in \mathbf{N}$ in place of (2) suffices.

A well-known and well-analyzed a posteriori rule for choosing the regularization parameter is the discrepancy principle, whose straightforward application in the present situation would lead to the choice

$$h_* = \max\{h = \beta^{-l}(b-a) \mid \|Tq_h^\delta - f^\delta\|_{L^\infty(a,b)} \le \tau\delta, \quad l \in \mathbf{N}_0\},$$

where $\beta > 1$ and $\tau > 1$ are a priori fixed constants. However, in the analysis below it turns out that one should rather consider the differentiated version of the integral equation and stop refinement according to

$$\begin{aligned}(31)\quad h_* = \max\{h = \beta^{-l}(b-a) \mid & \\ \max_{k\in\{1,\dots,N_*\}} |[(Tq_h^\delta)(x_k) - (Tq_h^\delta)(x_{k-1})] &- [f^\delta(x_k) - f^\delta(x_{k-1})]| \\ &\le 2\tau\delta,\ l \in \mathbf{N}_0\},\end{aligned}$$

where $N_* = (b-a)/h_*$.

To show well-definedness of $h_* > 0$ according to (31) for $\delta > 0$, we first of all prove a stability result.

Proposition 1. *Let* $\mathbf{k} \in C([a,b]^2)$, $\partial_1 \mathbf{k} \in C([a,b]^2)$, (4), $f \in C(a,b)$, *as well as* (2) *hold. Then*

(32)
$$\|q_h^\delta\|_{L^\infty(a,b)} = \max_{k\in\{1,\dots,N\}} |q_{k-1/2}| \leq \frac{C(\mathbf{k})}{h}\left(\sup_{|x-\xi|\leq h} |f(x)-f(\xi)| + \delta\right);$$

hence, for fixed $\delta > 0$,

$$\limsup_{h\to 0} \frac{h}{\delta}\|q_h^\delta\|_{L^\infty(a,b)} \leq C(\mathbf{k}), \tag{33}$$

where

$$C(\mathbf{k}) = \frac{4}{\gamma}\exp\left((2(b-a)/\gamma)\|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)}\right).$$

Proof. The finite difference version of (11) (i.e., subtracting (11) for subsequent indices and dividing by h) reads as

$$\begin{aligned} \mathbf{k}(x_k, x_{k-1/2})\frac{1}{h}\int_{x_{k-1}}^{x_k} q_h^\delta(\xi)\,d\xi \\ + \sum_{j=1}^{k-1} \frac{\mathbf{k}(x_k, x_{j-1/2}) - \mathbf{k}(x_{k-1}, x_{j-1/2})}{h}\int_{x_{j-1}}^{x_j} q_h^\delta(\xi)\,d\xi \\ = \frac{f^\delta(x_k) - f^\delta(x_{k-1})}{h}, \end{aligned} \tag{34}$$

Therewith, from (17) for

$$\begin{aligned} \zeta_k &= q_{k-1/2}^\delta, \\ A &= \frac{2h}{\gamma}\|\partial_1 \mathbf{k}\|_{L^\infty((a,b)^2)} \\ &\geq \max_{j,k\in\{1,\dots,N\}} \frac{2}{\gamma}|\mathbf{k}(x_k, x_{j-1/2}) - \mathbf{k}(x_{k-1}, x_{j-1/2})|, \\ B_k &= \frac{4}{\gamma h}\left(\sup_{|x-\xi|\leq h}|f(x)-f(\xi)| + \delta\right) \\ &\geq \frac{2}{\gamma}\left(\frac{f^\delta(x_k) - f^\delta(x_{k-1})}{h}\right), \end{aligned}$$

we get (32) in a similar manner as (20). $\square$

Proposition 2. *Let* $\mathbf{k} \in C([a,b]^2)$, $\partial_1 \mathbf{k} \in C([a,b]^2)$, (4), $f \in C(a,b)$, *as well as* (2) *hold. Then there exists an* $h_* > 0$ *satisfying* (31).

Proof. The assertion follows from the fact that by (6)

$$\begin{aligned}
&\frac{1}{\delta}|[(Tq_h^\delta)(x_k) - (Tq_h^\delta)(x_{k-1})] - [f^\delta(x_k) - f^\delta(x_{k-1})]| \\
&\qquad = \frac{h}{\delta}\bigg|\frac{1}{h}\int_{x_{k-1}}^{x_k} ([I - P_h]\mathbf{k}\,(x_k,\cdot))(\xi) q_h^\delta(\xi)\, d\xi \\
&\qquad\quad + \int_a^{x_{k-1}} [I - P_h] \frac{\mathbf{k}\,(x_k,\cdot) - \mathbf{k}\,(x_{k-1},\cdot)}{h}(\xi) q_h^\delta(\xi)\, d\xi\bigg| \\
&\qquad =: \frac{h}{\delta}(r_k^{1,h} + r_k^{2,h}),
\end{aligned}$$

whereas in (22), (23),

$$\begin{aligned}
\max_{k\in\{1,\dots,N\}} \frac{h}{\delta}|r_k^{1,h}| &\le \frac{h}{\delta}\|q_h^\delta\|_{L^\infty(a,b)} \\
&\quad \times \max_{k\in\{1,\dots,N\}} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\le h/2} |\mathbf{k}\,(x_k,\xi) - \mathbf{k}\,(x_k,\zeta))| \\
&\qquad\qquad \longrightarrow 0 \quad \text{as } N \to \infty,
\end{aligned}$$

and

$$\begin{aligned}
\max_{k\in\{1,\dots,N\}} \frac{h}{\delta}|r_k^{2,h}| &\le \frac{h}{\delta}\|q_h^\delta\|_{L^1(a,b)} \\
&\quad \sup_{x\in(a,b)} \sup_{\xi\in(a,b)} \sup_{|\xi-\zeta|\le h/2} |\partial_1\mathbf{k}\,(x,\xi) - \partial_1\mathbf{k}\,(x,\zeta)| \\
&\qquad\qquad \longrightarrow 0 \quad \text{as } N \to \infty
\end{aligned}$$

by (33) and uniform continuity of $\mathbf{k}$, $\partial_1\mathbf{k}$. □

Additionally, we need enhanced estimates of the expressions r_k^1, r_k^2 according to (13), (14) under somewhat stronger smoothness assumptions on $\mathbf{k}$. The following lemma will also provide further estimates that will be required for proving convergence rates later on. Here, for some bivariate function $v : (a,b)^2 \to \mathbf{R}$, we use the notations

$$\begin{aligned}
v \in L^\infty(a,b;L^P(a,b)) :&\Longleftrightarrow \|v\|_{L^\infty(a,b;L^P(a,b))} \\
&:= \sup_{x\in(a,b)} \|v(x,\cdot)\|_{L^P(a,b)} < \infty
\end{aligned}$$

$$\begin{aligned} v \in L^1(a,b;L^P(a,b)) :\Longleftrightarrow \|v\|_{L^1(a,b;L^P(a,b))} \\ := \int_a^b \|v(x,\cdot)\|_{L^P(a,b)}\,dx < \infty \\ v^*(x,\xi) := v(\xi,x), \quad x,\xi \in (a,b). \end{aligned}$$

Lemma 1. (a) *Let* $P \in [1,\infty]$, $\partial_2\mathbf{k} \in L^\infty(a,b;L^P(a,b))$, $(\partial_1\partial_2\mathbf{k})^* \in L^1(a,b;L^P(a,b))$, $q \in L^\infty(a,b)$. *Then*

$$\max_{k\in\{1,\dots,N\}} |r_k^1| \le \frac{1}{2}\|\partial_2\mathbf{k}\|_{L^\infty(a,b;L^P(a,b))}\|q\|_{L^\infty(a,b)} h^{1-1/P} \tag{35}$$

$$\max_{k\in\{1,\dots,N\}} |r_k^2| \le \frac{1}{2}\|(\partial_1\partial_2\mathbf{k})^*\|_{L^1(a,b;L^P(a,b))}\|q\|_{L^\infty(a,b)} h^{1-1/P}. \tag{36}$$

(b) *Let* $P,R,Q \in [1,\infty]$, $\partial_2\mathbf{k} \in L^\infty(a,b;L^R(a,b))$, $\partial_2^2\mathbf{k} \in L^\infty(a,b;L^P(a,b))$, $\partial_1\partial_2\mathbf{k} \in L^1(a,b;L^R(a,b))$, $\partial_1\partial_2^2\mathbf{k} \in L^1(a,b;L^P(a,b))$, $q \in L^\infty(a,b)$, $q' \in L^Q(a,b)$. *Then*

$$\begin{aligned} \max_{k\in\{1,\dots,N\}} |r_k^1| \le\ & C^1(P)\|\partial_2^2\mathbf{k}\|_{L^\infty(a,b;L^P(a,b))}\|q\|_{L^\infty(a,b)} h^{2-1/P} \\ & + C^2(Q,R)\|\partial_2\mathbf{k}\|_{L^\infty(a,b;L^R(a,b))}\|q'\|_{L^Q(a,b)} h^{2-1/R-1/Q} \end{aligned} \tag{37}$$

$$\begin{aligned} &\max_{k\in\{1,\dots,N\}} |r_k^2| \\ &\le C^1(P)(b-a)\|\partial_1\partial_2^2\mathbf{k}\|_{L^1(a,b;L^P(a,b))}\|q\|_{L^\infty(a,b)} h^{2-1/P} \\ &\quad + C^2(Q,R)(b-a)\|\partial_1\partial_2\mathbf{k}\|_{L^1(a,b;L^R(a,b))}\|q'\|_{L^Q(a,b)} h^{2-1/R-1/Q}, \end{aligned} \tag{38}$$

where

$$\begin{aligned} C^1(P) &= \frac{1}{2^{2-1/P}(3-1/P)(2-1/P)}, \\ C^2(R,Q) &= \frac{1}{2^{2-1/R}(3-1/R-1/Q)}. \end{aligned}$$

Proof. See the appendix. □

Now we show that the discrepancy principle (31) indeed yields convergence as $\delta \to 0$.

Theorem 2. *Let* $\mathbf{k} \in C([a,b]^2)$, $\partial_1 \mathbf{k} \in C([a,b]^2)$, $\partial_2 \mathbf{k} \in L^\infty(a,b; L^\infty(a,b))$, $(\partial_1\partial_2\mathbf{k})^* \in L^1(a,b;L^\infty(a,b))$, (4), $q \in L^\infty(a,b)$, *and* (2) *hold. Moreover, let* q_h^δ *be defined by* (6), (7), *and let* $N_* = N_*(\delta)$, $h_* = (b-a)/N_*$ *be chosen according to* (31) *with* $\tau > 1$.

Then

$$\|q_{h_*}^\delta - q\|_{L^\infty(a,b)} \longrightarrow 0 \quad \text{as } \delta \to 0. \tag{39}$$

Proof. The discrepancy principle (31) yields

$$\max_{k\in\{1,\dots,N_*\}} \left| \frac{1}{h_*}\int_{x_{k-1}}^{x_k} \mathbf{k}(x_k,\xi) q_{h_*}^\delta(\xi)\,d\xi + \int_a^{x_{k-1}} \frac{\mathbf{k}(x_k,\xi)-\mathbf{k}(x_{k-1},\xi)}{h_*} q_{h_*}^\delta(\xi)\,d\xi - \frac{f^\delta(x_k)-f^\delta(x_{k-1})}{h_*}\right| \le \frac{2\tau\delta}{h_*} \tag{40}$$

and

$$\max_{K\in\{1,\dots,N_*/\beta\}} \left| \frac{1}{\beta h_*}\int_{x_{K-1}}^{x_K} \mathbf{k}(x_K,\xi) q_{\beta h_*}^\delta(\xi)\,d\xi + \int_a^{x_{K-1}} \frac{\mathbf{k}(x_K,\xi)-\mathbf{k}(x_{K-1},\xi)}{\beta h_*} q_{\beta h_*}^\delta(\xi)\,d\xi - \frac{f^\delta(x_K)-f^\delta(x_{K-1})}{\beta h_*}\right| > \frac{2\tau\delta}{\beta h_*}. \tag{41}$$

Combining (1) with (40) and using the fact that

$$\left|\frac{(f-f^\delta)(x_k)-(f-f^\delta)(x_{k-1})}{h}\right| \le \frac{2\delta}{h}, \quad k=1,\dots,N \tag{42}$$

yields

$$
\begin{aligned}
(43) \quad \max_{k\in\{1,\dots,N_*\}} \Bigg| \frac{1}{h_*}\int_{x_{k-1}}^{x_k} \mathbf{k}\,(x_k,\xi)(q_{h_*}^\delta - q)(\xi)\,d\xi \\
+ \int_a^{x_{k-1}} \frac{\mathbf{k}\,(x_k,\xi)-\mathbf{k}\,(x_{k-1},\xi)}{h_*}(q_{h_*}^\delta - q)(\xi)\,d\xi \Bigg| \\
\le \frac{2(\tau+1)\delta}{h_*}, \quad k=1,\dots N_*,
\end{aligned}
$$

whereas from (34), (41) we get

$$
\begin{aligned}
(44) \quad \max_{K\in\{1,\dots,N_*/\beta\}} \Bigg| \frac{1}{\beta h_*}\int_{x_{K-1}}^{x_K} ([I-P_{\beta h_*}]\mathbf{k}\,(x_K,\cdot))(\xi) q_{\beta h_*}^\delta(\xi)\,d\xi \\
+ \int_a^{x_{K-1}} [I-P_{\beta h_*}]\frac{\mathbf{k}\,(x_K,\cdot)-\mathbf{k}\,(x_{K-1},\cdot)}{\beta h_*}(\xi) q_{\beta h_*}^\delta(\xi)\,d\xi \Bigg| \\
> \frac{2\tau\delta}{\beta h_*},
\end{aligned}
$$

if $h_* < b-a$.

If $h_* \to 0$ as $\delta \to 0$, then according to (13), (14), (32), (35), (36), the left hand side and therewith also the right hand side in (44) can be estimated from above by

$$
\begin{aligned}
\frac{2\tau\delta}{\beta h_*} \le \overline{C}(\mathbf{k})(\beta h_*)\|q_{\beta h_*}^\delta\|_{L^\infty(a,b)} \le C(\mathbf{k})\overline{C}(\mathbf{k})\left(\sup_{|x-\xi|\le h} |f(x)-f(\xi)| + \delta\right) \\
\longrightarrow 0 \quad \text{as } \delta \to 0,
\end{aligned}
$$

with

$$
\overline{C}(\mathbf{k}) = \frac{1}{2}\left(\|\partial_2\mathbf{k}\|_{L^\infty(a,b;L^\infty(a,b))}\|(\partial_1\partial_2\mathbf{k})^*\|_{L^1(a,b;L^\infty(a,b))}\right),
$$

which by (43) and (4), (17), (29) yields (39).

On the other hand, if h_* stays bounded away from zero as $\delta \to 0$, then we directly get $\delta/h_* \to 0$ as $\delta \to 0$ and can apply the same argument as above with (43) to conclude (39).

A subsequence-subsequence argument can be used to cover all possible cases: namely for an arbitrary sequence $(\delta_n)_{n\in\mathbf{N}}$ with $\delta_n \to 0$ and an arbitrary subsequence $(\delta_{n_j})_{j\in\mathbf{N}}$, we either have $h_*(\delta_{n_j})$ as $j \to \infty$ or existence of some $\underline{h} > 0$ and another subsequence indicated by n_{j_l}, $l \in \mathbf{N}$, such that $h_*(\delta_{n_{j_l}}) \geq \underline{h}$ for all $l \in \mathbf{N}$. □

Remark 2. Note that, with the a priori choice of Theorem 1, for L^1 convergence (Case(a) in Theorem 1) we only need $q \in L^1$. This relaxation as compared to the L^∞ situation (Case (b) in Theorem 1) does not seem to be possible for the a posteriori results. Also in the remainder of this paper we confine ourselves to L^∞ estimates, to avoid lengthy but partly straightforward computations.

Convergence rates can be obtained under additional smoothness assumptions. We first state a rates result with a priori chosen N_* for the discrete error norms.

Theorem 3. *Let q_h^δ be defined by* (6), (7).

(a) *Under the assumptions of Lemma* 1 (a) *and with the a priori choice*

$$N_* \sim \delta^{-1/(2-1/P)} \tag{45}$$

we get

$$\max_{k\in\{1,\dots,N\}} \frac{1}{h_*}\left| \int_{x_{k-1}}^{x_k} [q - q_{h_*}^\delta](\xi)\, d\xi \right| = O(\delta^{(1-1/P)/(2-1/P)}).$$

(b) *Under the assumptions of Lemma* 1 (b) *and with the a priori choice*

$$N_* \sim \delta^{-1/(3-\kappa)} \tag{46}$$

we get

$$\max_{k\in\{1,\dots,N\}} \frac{1}{h_*}\left| \int_{x_{k-1}}^{x_k} [q - q_{h_*}^\delta](\xi)\, d\xi \right| = O(\delta^{(2-\kappa)/(3-\kappa)}).$$

where $\kappa = \max\{1/P, 1/R + 1/Q\}$.

Proof. In Case (a), the assertion follows from (20), (27), (35), (36), in Case (b) from (20), (27), (37), (38). □

To obtain results in the L^∞ norm, it remains to estimate the discretization error $\inf_{\tilde{q}_h \in Q_h} \|q - \tilde{q}_h\|_{L^\infty}$, which is of course somewhat standard but nevertheless will be shortly done here for the sake of completeness:

Lemma 2. *Let $q \in L^\infty(a,b)$ and $q' \in L^Q(a,b)$ for some $Q \in (1,\infty]$. Then,*

$$\max_{k\in\{1,\dots,N\}} \max_{\sigma\in[x_{k-1},x_k]} \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} q(\xi)\, d\xi - q(\sigma) \right| \leq \frac{1}{2-1/Q} h^{1-1/Q} \|q'\|_{L^Q(a,b)}, \tag{47}$$

Proof. For any $k \in \{1,\dots,N\}$, $\sigma \in [x_{k-1},x_k]$, we have

$$\begin{aligned}
\left| \frac{1}{h} \int_{x_{k-1}}^{x_k} q(\xi)\, d\xi - q(\sigma) \right|
&= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} (q(\xi) - q(\sigma))\, d\xi \right| \\
&= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} \int_\sigma^\xi q'(\rho)\, d\rho\, d\xi \right| \\
&\leq \frac{1}{h} \int_{x_{k-1}}^{x_k} |\xi - \sigma|^{1-1/Q}\, d\xi \|q'\|_{L^Q(a,b)} \\
&\leq \frac{1}{2-1/Q} h^{1-1/Q} \|q'\|_{L^Q(a,b)} \qquad \square
\end{aligned}$$

Inserting these estimates in the proofs of Theorems 1 and 2 we get the following two results on convergence rates

Theorem 4. *Let q_h^δ be defined by* (6), (7). *Under the assumptions of Lemma* 1 (a) *with $q' \in L^P(a,b)$ and with the a priori choice* (45), *we get*

$$\|q - q_{h_*}^\delta\|_{L^\infty(a,b)} = O(\delta^{(1-1/P)/(2-1/P)}).$$

Proof. Instead of (29), we use

$$
\begin{aligned}
(48)\quad & \max_{k\in\{1,\dots,N\}} \frac{1}{h}\left|\int_{x_{k-1}}^{x_k} [q-q_h^\delta](\xi)\,d\xi\right| \\
&= \max_{k\in\{1,\dots,N\}} \left|\frac{1}{h}\int_{x_{k-1}}^{x_k} q(\xi)\,d\xi - q(\rho_h) + [q-q_h^\delta](\rho_h)\right| \\
&\geq \max_{k\in\{1,\dots,N\}} \max_{\rho\in[x_{k-1},x_k]} |[q-q_h^\delta](\rho)| \\
&\quad - \max_{k\in\{1,\dots,N\}} \max_{\sigma\in[x_{k-1},x_k]} \left|\frac{1}{h}\int_{x_{k-1}}^{x_k} q(\xi)\,d\xi - q(\sigma)\right| \\
&\geq \|q-q_h^\delta\|_{L^\infty(a,b)} - \frac{1}{2-1/P} h^{1-1/P}\|q'\|_{L^P(a,b)}
\end{aligned}
$$

where $\rho_h \in [x_{k-1}, x_k]$ is such that $[q-q_h^\delta](\rho_h) = \max_{\rho\in[x_{k-1},x_k]} |[q - q_h^\delta](\rho)|$ and (note that by $P > 1$, q is continuous) and we have used the fact that q_h^δ is constant on $(x_{k-1}, x_k]$, as well as (47). □

Theorem 5. *Under the assumptions of Theorem* 2 *and Lemma* 1 (a) *with* $f' \in L^\infty(a,b)$ *and the a posteriori choice* (31) *with*

$$
\begin{aligned}
(49)\quad 2\tau > \widetilde{C}(\mathbf{k}) &:= \frac{2}{\gamma}\exp\left((2(b-a)/\gamma)\|\partial_1\mathbf{k}\|_{L^\infty((a,b)^2)}\right) \\
&\quad \times \left(\|\partial_2\mathbf{k}\|_{L^\infty(a,b;L^P(a,b))} + \|(\partial_1\partial_2\mathbf{k})^*\|_{L^1(a,b;L^P(a,b))}\right)
\end{aligned}
$$

we get

$$\|q-q_{h_*}^\delta\|_{L^\infty(a,b)} = O(\delta^{(1-1/P)/(2-1/P)}),$$

provided $h_* < b-a$.

Proof. By (35), (36) with q replaced by q_h^δ, and (32), we can estimate the left hand side and therewith the right hand side in (44) from above as follows:

$$\frac{2\tau\delta}{\beta h_*} \leq \widetilde{C}(\mathbf{k})(\beta h_*)^{1-1/P}\left(\|f'\|_{L^\infty(a,b)} + \frac{\delta}{\beta h_*}\right).$$

Therewith, by (49),

$$(\beta h_*)^{2-1/P} \geq \frac{2\tau - \widetilde{C}(\mathbf{k})}{\widetilde{C}(\mathbf{k})\|f'\|_{L^\infty(a,b)}}\delta =: \overline{C}\delta,$$

which by (43) implies that

$$\max_{k\in\{1,\dots,N_*\}} \left| \frac{1}{h_*}\int_{x_{k-1}}^{x_k} \mathbf{k}\,(x_k,\xi)(q^\delta_{h_*}-q)(\xi)\,d\xi \right.$$
$$\left. + \int_a^{x_{k-1}} \frac{\mathbf{k}\,(x_k,\xi)-\mathbf{k}\,(x_{k-1},\xi)}{h_*}(q^\delta_{h_*}-q)(\xi)\,d\xi \right|$$
$$\le 2(\tau+1)\delta\beta(\overline{C}\delta)^{-1/(2-1/P)}$$
$$= 2(\tau+1)\beta\overline{C}^{-1/(2-1/P)}\,\delta^{(1-1/P)/(2-1/P)}.$$

By (4), (17), (48), this yields the assertion. □

Remark 3. Note that in Theorems 3, 4, the regularity assumptions are made directly in terms of q, while in Theorem 5, they are hidden in the assumption $f' \in L^\infty(a,b)$.

3. Application to a thermoacoustic inverse problem. In combustion technology, unsteady heat fluctuations can influence pollutant emission, reliability, and especially noise production: The physical background for the latter phenomenon is the fact that oscillatory heat release acts as a source of sound in compressible flows. Therewith, it is of high interest to reconstruct the oscillatory heat release distribution from measurements of the sound pressure at combustor walls, cf., e.g., [**1, 14, 15**]. This represents an ill-posed problem and therefore has to be regularized, see e.g., [**4, 6, 8, 12, 13, 16, 17**].

Following [**1**], we work in frequency domain, assuming time harmonic behavior at frequency ω, and formulate the problem as a one-dimensional differential equation

$$p_{xx} + Z_1 p_x + Z_2 p = Z_3(ikq + Mq_x), \quad x \in (0,L), \tag{50}$$

which is justified in an appropriate experimental setup, cf. [**1**]. Here p denotes the acoustic pressure, q the heat release, and the constants Z_1, Z_2, Z_3 are given by

$$Z_1 = -\frac{2kM}{1-M^2}\; i$$
$$Z_2 = \frac{k^2}{1-M^2}$$
$$Z_3 = -\frac{\tilde{\gamma}-1}{\overline{c}(1-M^2)}$$

where $\tilde{\gamma}$ is the ratio of specific heats, $k = \omega/\overline{c}$ the wave number, $M = \overline{u}/\overline{c}$ the mean Mach number, $\overline{u}$ the mean axial velocity, $\overline{c}$ the mean speed of sound, and L the length of the combustor. Here, to keep notation similar to existing literature on this application, we write a subscript x for the derivative with respect to space, although (50) is obviously an ordinary differential equation. Considering (50) as an ODE for q, we see that q is only uniquely determined if we specify in addition to (50) an initial value for q. We will simply set

$$q(0) = 0, \tag{51}$$

which is physically justified by the fact that $q(0)$ can be regarded as a selectable offset value.

Unique identifiability of q from measurements of p follows from the Picard-Lindelöf theorem provided p is sufficiently smooth. One can even derive the following explicit formula

$$\begin{aligned} q(x) = {} & \exp(-i(k/M)x)\Big(q(0) + \tfrac{1}{MZ_3}(-p_x(0) + (i(k/M) - Z_1)p(0))\Big) \\ & + \tfrac{1}{MZ_3}\left(p_x(x) - (i(k/M) - Z_1)p(x)\right) \\ & + \tfrac{1}{MZ_3}\left(-(k/M)^2 - Z_1 i(k/M) + Z_2\right) \\ & \times \int_0^x \exp(i(k/M)(\xi - x))p(\xi)\,d\xi, \end{aligned} \tag{52}$$

cf. [**7**].

The problem of identifying the heat release from pressure measurements according to (50) is ill-posed in the sense that small perturbations in the data can lead to large deviations in the solution. As a matter of fact, in place of the exact pressures p only measured values p^δ are available, that are contaminated with noise (which here and below is indicated by a superscript δ).

Integrating twice with respect to space, we can reformulate (50) as a Volterra integral equation of the first kind (50) where

$$\begin{aligned} f(x) = {} & -p(x) - \int_0^x (Z_1 + Z_2(x-\xi))p(\xi)\,d\xi \\ & + (1 + Z_1 x)p(0) + x p_x(0) + Z_3 M x q(0) \end{aligned} \tag{53}$$

and the kernel takes the simple (convolution type) form

$$\mathbf{k}\,(x,\xi) = -Z_3(M + ik(x-\xi)) \tag{54}$$

cf. [**1**]. Since here $\mathbf{k}\,(x,x) = -Z_3 M > 0$, we deal with a *one-smoothing* integral operator here.

3.1. Numerical experiments. In this section we test the performance of method (6), (31) and compare it to a classical method for Volterra integral equations of the first kind, namely Lavrent'ev's method, which, given a small regularization parameter $\alpha > 0$ defines the solution q_α to the second kind (hence well-posed) Volterra integral equation

$$\alpha q(x) + \int_0^x \mathbf{k}\,(x,\xi) q(\xi)\, d\xi = f^\delta(x) \tag{55}$$

as a regularized approximation to the solution of (1). For a convergence analysis of this method we refer to [**2**] and further references in [**10**]. According to Theorem 1 in [**10**] (quoted from [**2**]) $\alpha = \alpha(\delta)$ should be chosen such that

$$\alpha \to 0 \quad \text{and} \quad \delta/\alpha \longrightarrow 0 \quad \text{as } \delta \to 0.$$

Here we computed solutions using Lavrent'ev's method with several regularization parameters for each noise level and display the result that yielded the smallest error in order to provide a really fair comparison to the proposed method.

In our computations, we used the values $M = 0.1$, $k = 0.5$, $\tilde{\gamma} = 1.2$ taken from [**1**]. Moreover, we set $\beta = 2$ and $\tau = 1.1$.

As a first test example, we considered

$$p(x) = \exp(i(\Omega/L)x),$$

with the exact solution according to (52) given by

$$\begin{aligned} q(x) &= \exp(-i(k/M)x)q(0) \\ &\quad + \frac{Z_2 - (\Omega/L)^2 + Z_1 i\Omega/L}{i((k/M) + (\Omega/L))MZ_3} \left(\exp(i(\Omega/L)x) - \exp(-i(k/M)x)\right). \end{aligned} \tag{56}$$

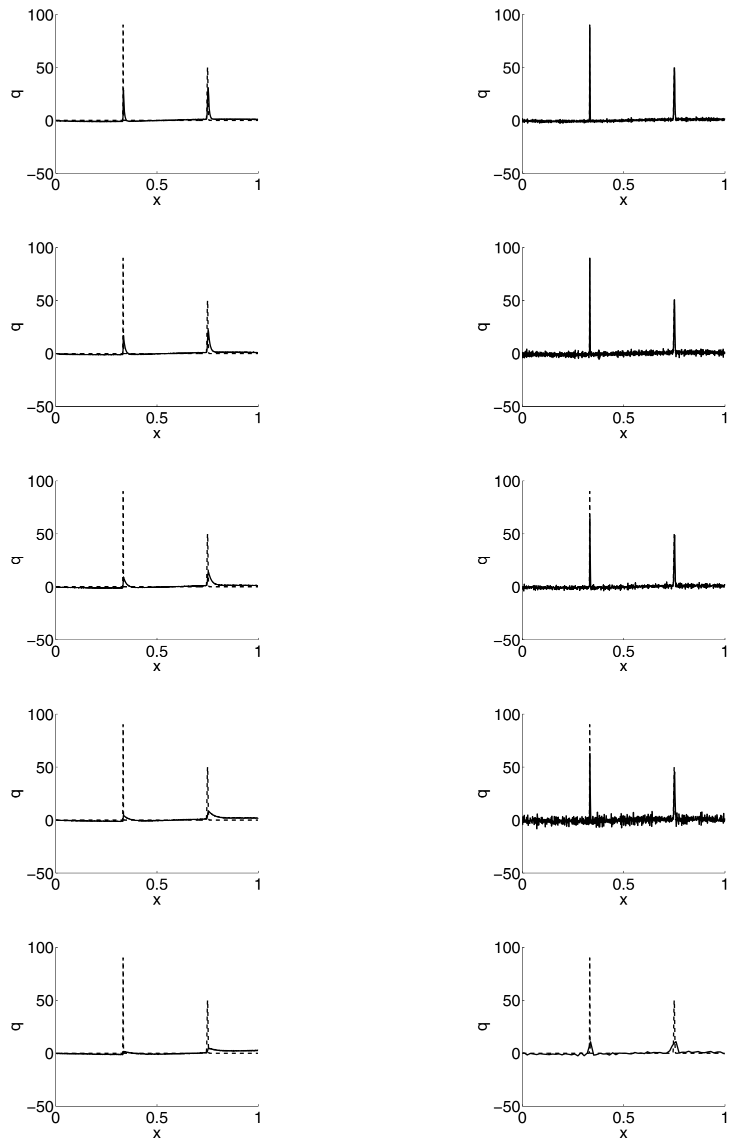

FIGURE 1. Method (55) (left) and (6) (right) for test example (56) applied to data with $1/4, 1/2, 1, 2$, and 4% noise (solid line) versus exact solution (dashed line).

Note that availability of an analytic formula for the solution helps us to avoid an inverse crime, (which would mean to produce non-representative numerical results by restriction of the whole problem to a finite dimensional subspace). Synthetic noise of relative level $\|p^\delta - p\|/\|p\| = \delta_p * 0.01$ with $\delta_p = (1/4), (1/2), 1, 2,$ and 4 percent in the data is generated by adding rescaled standard normally distributed random numbers to the exact values of p.

Figure 1 shows the respective results for Lavrent'ev's method and the midpoint rule for test example (56) with $L = 1$, $\Omega = 2\pi$.

As a second test example, we consider a heat release distribution with two peaks

$$q(x) = A_1 \exp\left(\frac{(x-x_1)^2}{\sigma_1^2}\right) + A_2 \exp\left(\frac{(x-x_2)^2}{\sigma_2^2}\right) \tag{57}$$

with corresponding pressure distribution according to (50) computed by finite differences on a fine grid in order to avoid an inverse crime.

Figure 2 shows the respective results for Lavrent'ev's method and the midpoint rule for test example (57) with $x_1 = L/3$, $x_2 = 3L/4$, $\sigma_1 = 1.e-3$, $\sigma_2 = 3.e-3$, $A_1 = 100$, $A_2 = 50$.

Method (6) appears to be more robust against noise as compared to (55) for the smoother test example (56). For the less regular test example (57), performance was worse for both methods, as expected from the well known fact that convergence of regularization methods depends on smoothness of the solution. Method (6) very well locates the peaks and to some extent even their heights, but yields somewhat oscillatory solutions. Method (55) succeeds in avoiding oscillations but gives poor approximations to the peak heights.

4. Conclusions and remarks. In this paper, we have carried out a convergence analysis of the midpoint rule for Volterra integral equations of the first kind, that so far has been studied for exact data only. We have established a regularizing property as well as convergence rates under additional regularity assumptions. Enhanced rates might be obtainable for higher order quadrature rules, however, limitations might arise in view of the converse results from [**11**]. We have shown numerical tests for an application in combustion technology.

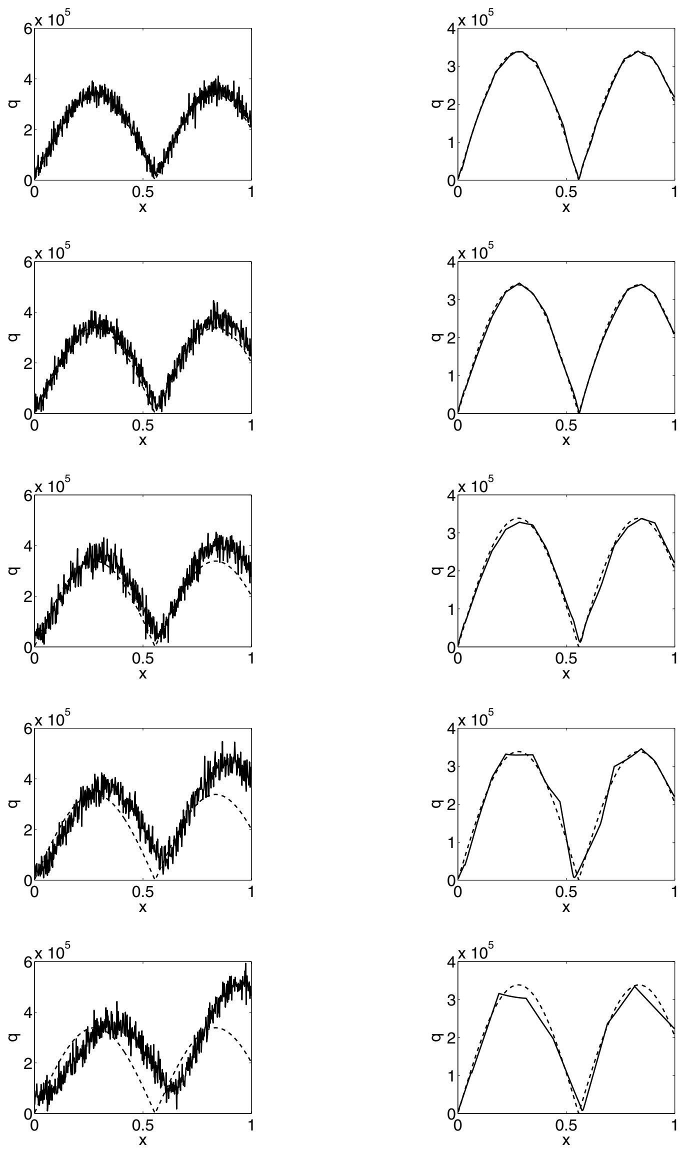

FIGURE 2. Method (55) (left) and (6) (right) for test example (57) applied to data with $1/4, 1/2, 1, 2$, and 4 per cent noise (solid line) versus exact solution (dashed line).

APPENDIX

Proof. (Lemma 1). Assertion (a) can be seen by applying Hölder's inequality to the estimates

$$
\begin{aligned}
|r_k^1| &= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \partial_2 \mathbf{k}\,(x_k,\tau)\, d\tau q(\xi)\, d\xi \right| \\
&= \frac{1}{h} \Bigg| - \int_{x_{k-1}}^{x_{k-1/2}} \int_{x_{k-1}}^{\tau} q(\xi)\, d\xi \partial_2 \mathbf{k}\,(x_k,\tau)\, d\tau \\
&\qquad + \int_{x_{k-1/2}}^{x_k} \int_{\tau}^{x_k} q(\xi)\, d\xi \partial_2 \mathbf{k}\,(x_k,\tau)\, d\tau \Bigg| \\
&\le \frac{1}{2} \|q\|_{L^\infty(a,b)} \int_{x_{k-1}}^{x_k} |\partial_2 \mathbf{k}\,(x_k,\tau)|\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
|r_k^2| &= \left| \frac{1}{h} \sum_{j=1}^{k-1} \int_{x_{j-1}}^{x_j} \int_{x_{j-1/2}}^{\xi} \int_{x_{k-1}}^{x_k} \partial_1 \partial_2 \mathbf{k}\,(x,\tau)\, dx\, d\tau q(\xi)\, d\xi \right| \\
&= \frac{1}{h} \Bigg| \int_{x_{k-1}}^{x_k} \sum_{j=1}^{k-1} \Bigg(- \int_{x_{j-1}}^{x_{j-1/2}} \int_{x_{j-1}}^{\tau} q(\xi)\, d\xi \partial_1 \partial_2 \mathbf{k}\,(x,\tau)\, d\tau \\
&\qquad + \int_{x_{j-1/2}}^{x_j} \int_{\tau}^{x_j} q(\xi)\, d\xi \partial_1 \partial_2 \mathbf{k}\,(x,\tau)\, d\tau \Bigg) dx \Bigg| \\
&\le \frac{1}{2} \|q\|_{L^\infty(a,b)} \int_{x_{k-1}}^{x_k} \sum_{j=1}^{k-1} \int_{x_{j-1}}^{x_j} |\partial_1 \partial_2 \mathbf{k}\,(x,\tau)|\, d\tau\, dx.
\end{aligned}
$$

To prove assertion (b), we denote by q^{sy} the part of q that is symmetric with respect to $x_{j(\xi)-1/2}$

$$
q^{sy}(\xi) = \frac{1}{2} \left(q(\xi) + q(2x_{j(\xi)-1/2} - \xi) \right),
$$

which implies

$$
\int_{x_{k-1}}^{x_k} (\xi - x_{k-1/2}) q^{sy}(\xi)\, d\xi = 0,
$$

$$
q(\xi) - q^{sy}(\xi) = \frac{1}{2} \left(q(\xi) - q(2x_{j(\xi)-1/2} - \xi) \right)
$$

and therewith get

(58)

$$
\begin{aligned}
|r_k^1| &= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} \left[\mathbf{k}\,(x_k, \xi) - \mathbf{k}\,(x_k, x_{j(\xi)-1/2}) \right] q(\xi)\, d\xi \right| \\
&= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} (\xi - x_{k-1/2}) q^{sy}(\xi)\, d\xi \partial_2 \mathbf{k}\,(x_k, x_{k-1/2}) \right. \\
&\quad + \frac{1}{h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \left[\partial_2 \mathbf{k}\,(x_k, \tau) - \partial_2 \mathbf{k}\,(x_k, x_{k-1/2}) \right]\, d\tau q^{sy}(\xi)\, d\xi \\
&\quad \left. + \frac{1}{h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \partial_2 \mathbf{k}\,(x_k, \tau)\, d\tau [q(\xi) - q^{sy}(\xi)]\, d\xi \right| \\
&= \left| \frac{1}{h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \int_{x_{k-1/2}}^{\tau} \partial_2^2 \mathbf{k}\,(x_k, \rho)\, d\rho\, d\tau q^{sy}(\xi)\, d\xi \right. \\
&\quad \left. + \frac{1}{2h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \partial_2 \mathbf{k}\,(x_k, \tau)\, d\tau \int_{2x_{k-1/2}-\xi}^{\xi} q'(\rho)\, d\rho\, d\xi \right| \\
&\leq \int_{x_{k-1/2}}^{x_k} \int_{x_{k-1/2}}^{\tau} |\partial_2^2 \mathbf{k}\,(x_k, \rho)|\, d\rho\, d\tau \|q^{sy}\|_{L^\infty(a,b)} \\
&\quad + \frac{1}{2h} \int_{x_{k-1}}^{x_k} \int_{x_{k-1/2}}^{\xi} \partial_2 \mathbf{k}\,(x_k, \tau)\, d\tau (2|\xi - x_{k-1/2}|)^{1-1/Q}\, d\xi \\
&\qquad \times \|q'\|_{L^Q(a,b)} \\
&\leq \int_{x_{k-1/2}}^{x_k} |\tau - x_{k-1/2}|^{1-1/P}\, d\tau \|\partial_2^2 \mathbf{k}\,(x_k, \cdot)\|_{L^P(a,b)} \|q\|_{L^\infty(a,b)} \\
&\quad + \frac{1}{2h} \int_{x_{k-1}}^{x_k} |\xi - x_{k-1/2}|^{1-1/R} (2|\xi - x_{k-1/2}|)^{1-1/Q}\, d\xi \\
&\qquad \times \|\partial_2 \mathbf{k}\,(x_k, \cdot)\|_{L^R(a,b)} \|q'\|_{L^Q(a,b)} \\
&= C^1(P) \|\partial_2^2 \mathbf{k}\,(x_k, \cdot)\|_{L^P(a,b)} \|q\|_{L^\infty(a,b)} h^{2-1/P}
\end{aligned}
$$

(59)

$$
\quad + C^2(Q, R) \|\partial_2 \mathbf{k}\,(x_k, \cdot)\|_{L^R(a,b)} \|q'\|_{L^Q(a,b)} h^{2-1/R-1/Q}
$$

as well as

$$
\|r_k^2| = \left| \int_{x_{k-1}}^{x_k} \sum_{j=1}^{k-1} \int_{x_{j-1}}^{x_j} \left[\partial_1 \mathbf{k}\,(\sigma, \xi) - \partial_1 \mathbf{k}\,(\sigma, x_{j(\xi)-1/2}) \right] q(\xi)\, d\xi\, d\sigma \right|
$$

$$\le (b-a)|\max_{j\in\{1,\dots,N\}} \underbrace{\left|\frac{1}{h}\int_{x_{j-1}}^{x_j}\left[\partial_1\mathbf{k}(\sigma,\xi)-\partial_1\mathbf{k}(\sigma,x_{j(\xi)-1/2})\right]q(\xi)\,d\xi\right|}_{\text{see (58), (59)}}$$

$$\le (b-a)\Big(C^1(P)\|\partial_1\partial_2^2\mathbf{k}\,(\sigma,\cdot)\|_{L^P(a,b)}\|q\|_{L^\infty(a,b)}h^{2-1/P}$$
$$+\,C^2(Q,R)\|\partial_1\partial_2\mathbf{k}\,(\sigma,\cdot)\|_{L^R(a,b)}\|q'\|_{L^Q(a,b)}h^{2-1/R-1/Q}\Big). \qquad \square$$

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