JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 21, Number 4, Winter 2009

# A DIRECT SOLUTION OF THE ROBIN INVERSE PROBLEM

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Communicated by Yuesheng Xu

ABSTRACT. We present a direct, linear boundary integral equation method for the inverse problem of recovering the Robin coefficient from a single partial boundary measurement of the solution to the Laplace equation.

**1. Introduction.** Let $\Omega$ be a smooth bounded domain in $R^2$ with boundary $\partial\Omega = \Gamma$. Consider the Robin boundary value problem for the Laplace equation:

$$\begin{cases} \Delta U = 0 & \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu} + pU = g & \text{on } \Gamma. \end{cases} \tag{1.1}$$

Here $p = p(x)$ with support $\Gamma_1 \subset \Gamma$ is the Robin coefficient, and $g = g(x)$ is a prescribed input function, both of which are nonnegative functions on $\Gamma$ and have nonempty supports, usually disjoint. Then the Robin inverse problem is as follows. Given $U = u_0$ on $\Gamma_0 \subset \Gamma$ with $\Gamma_0 \cap \Gamma_1 = \varnothing$, find the Robin coefficient $p$ on $\Gamma_1$. This problem originates from the quantitative study of many nondestructive testing techniques, where certain material property modeled by $p$ on an inaccessible portion $\Gamma_1$ of the boundary is to be recovered from a measurement $u_0$ of the solution $U$ on an accessible part $\Gamma_0$ of the boundary. Applications of such include evaluation of metal-to-silicon contact quality in semiconductor transistors (e.g. [**10**]) and corrosion damage detection (e.g. [**5, 6**]).

There have been some theoretical and numerical studies for this inverse problem, most of which are based on the PDE model (e.g.

*Keywords and phrases.* Robin inverse problem, boundary integral equation, Tikhonov regularization
Received by the editors on May 14, 2007, and in revised form on Oct. 3, 2007.
DOI:10.1216/JIE-2009-21-4-545 

[**3, 5**]). Because the equation is Laplacian and both the unknown coefficient $p$ and data measurement $u_0$ are on the boundary, it is natural to formulate this boundary value problem (1.1) as an integral equation on $\Gamma$. This approach was adopted in [**4, 9**], and used to study numerically the inverse problem. While inverse problems are usually nonlinear and most solution methods are iterative, [**9**] proposed a linear integral equation approach to the Robin inverse problem, based on the introduction of a new variable. In this paper, we continue on this approach and present a more direct, much simpler method for recovering the Robin coefficient. Numerical examples will be presented to illustrate the effectiveness of this simple yet competitive method. Because of its simplicity, it can also be used to provide a quick, quality initial guess for more computationally-expensive iterative algorithms.

**2. Formulation by Boundary Integral Equations.** We assume that $p \in L^\infty(\Gamma)$ and $g \in L^2(\Gamma)$. A weak solution to (1.1) is defined as $U \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla U \cdot \nabla \phi \, dx + \int_\Gamma pU\phi \, ds = \int_\Gamma g\phi \, ds \quad \text{for all } \phi \in H^1(\Omega). \tag{2.1}$$

The unique existence of such weak solutions can be established by Lax-Milgram Theorem with the help of the trace theorem and a Poincaré-type inequality.

Let $\Phi = \Phi(x, y)$ stand for the fundamental solution for the Laplacian in $R^2$:

$$\Phi(x, y) = \frac{1}{2\pi} \ln \frac{1}{|x - y|} \quad \text{for} \quad x \neq y.$$

By the third Green identify and jump relations for single and double-layer potentials ([11]), we find that the trace of $U \in H^1(\Omega)$ on $\Gamma$, denoted by $u \in H^{1/2}(\Gamma)$, satisfies the boundary integral equation:

$$\frac{1}{2}u(x) + \int_\Gamma \left( \frac{\partial \Phi(x, y)}{\partial \nu_y} + p(y)\Phi(x, y) \right) u(y) \, ds_y = \int_\Gamma \Phi(x, y) \, g(y) \, ds_y, \quad x \in \Gamma. \tag{2.2}$$

In operator form, (2.2) can be written as

$$\left( \frac{1}{2}\mathcal{I} + \mathcal{D} \right) u + \mathcal{S}(pu) = \mathcal{S}g, \tag{2.3}$$

with the single and double-layer potential operators defined by

$$(\mathcal{S}u)(x) = \int_\Gamma \Phi(x,y)\ u(y)\ ds_y \qquad \text{and}$$
$$(\mathcal{D}u)(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_y} u(y)\ ds_y \qquad \text{for } x \in \Gamma.$$

Note that the operators have the following mapping properties (e.g. [1.1]): $\mathcal{S}$: $H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and $\mathcal{D}$: $H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$.

Similar to the Neumann problem, the Robin problem has a necessary condition for its solution, as stated in the following lemma.

**Lemma 2.1.** If $u(x) \in L^2(\Gamma)$ satisfies the integral equation (2.2), then

$$\int_\Gamma p(x)u(x)\ ds_x = \int_\Gamma g(x)\ ds_x.$$

*Proof.* From (2.2) we have $(\frac{1}{2}\mathcal{I} + \mathcal{D})u = \mathcal{S}(g - pu)$. Let $\mathcal{D}'$ denote the dual of $\mathcal{D}$ (see (2.8) below). For $\psi \in N(\frac{1}{2}I + \mathcal{D}')$, we have

$$\left\langle \left(\frac{1}{2}\mathcal{I} + \mathcal{D}\right) u, \psi \right\rangle = \langle \mathcal{S}(g - pu), \psi \rangle .$$

Here $\langle \cdot, \cdot \rangle$ denotes the usual $L^2$ inner product on $\Gamma$. Since $\psi \in N(\frac{1}{2}\mathcal{I} + \mathcal{D}')$, the left side above is 0, thus we obtain

$$\langle \mathcal{S}(g - pu), \psi \rangle = 0, \quad \text{i.e.} \quad \langle g - pu, \mathcal{S}\psi \rangle = 0 \tag{2.4}$$

since $\Phi(x,y) = \Phi(y,x)$. Note that, for such $\psi$, the single-layer potential $v(x) = (\mathcal{S}\psi)(x)$ is a solution of the homogeneous interior Neumann problem in $\Omega$. By uniqueness, $v(x)$ is a constant. Hence from (2.4) we see that $\langle g - pu, 1 \rangle = \int_\Gamma \{g(x) - p(x)u(x)\} ds_x = 0$. □

Now suppose $u(x) \in H^{1/2}(\Gamma)$ solves the integral equation (2.2). With this $u$, we construct a solution to (1.1) as follows. Define

$$U(x) = \int_\Gamma \Phi(x,y)\ g(y)\ ds_y - \int_\Gamma \left( \frac{\partial \Phi(x,y)}{\partial \nu_y} + p(y)\Phi(x,y) \right) u(y)\ ds_y, \quad x \in R^2 \setminus \Gamma. \tag{2.5}$$

Then $U(x)$ is harmonic in $R^2 \setminus \overline{\Omega}$ and $\Omega$ respectively, and $U \in H^1(\Omega)$ and $U \in H^1(\Omega_r \setminus \overline{\Omega})$ for large enough $r$ ($\Omega_r$ denotes the disk centered at the origin with radius $r$) ([11, Theorem 6.11]). By taking the trace of $U$ on $\Gamma$ from outside $(+)$ and inside $(-)$ of $\Omega$ respectively, and from (2.2) for $u$, we find the traces $u^+$ and $u^-$ as:

$$u^+ = \mathcal{S}g - \left(\frac{1}{2}I + \mathcal{D}\right) u - \mathcal{S}(pu) = 0 \quad \text{and}$$

$$u^- = \mathcal{S}g - \left(-\frac{1}{2}I + \mathcal{D}\right) u - \mathcal{S}(pu) = u.$$

From Lemma 2.1, $\int_\Gamma \{g(y) - p(y)u(y)\}\, ds_y = 0$, hence, $U(x)$ by (2.5) can be also expressed as (for some $x_0 \in \Omega$)

$$U(x) = \int_\Gamma (\Phi(x,y) - \Phi(x,x_0)) \{g(y) - p(y)u(y)\}\, ds_y \\ - \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_y} u(y)\, ds_y, \quad x \in R^2 \setminus \overline{\Omega},$$

from which we see that $U(x)$ is bounded in $R^2 \setminus \overline{\Omega}$. Hence $U(x)$ is harmonic and bounded in $R^2 \setminus \overline{\Omega}$ with $u^+ = 0$ on $\Gamma$. Then by the uniqueness of the exterior Dirichlet problem, we conclude that $U(x) \equiv 0$ in $R^2 \setminus \overline{\Omega}$, thus $\partial U/\partial \nu^+ = 0$ on $\Gamma$. Therefore, from the jump relations of the normal derivatives of single and double-layer potentials, we find from (2.5) that

$$\frac{\partial U}{\partial \nu^-} = g - pu \quad \text{on} \quad \Gamma,$$

and therefore $\partial U/\partial \nu^- = g - pu^-$. That is, $U(x)$ given in (2.5) on $\Omega$ is indeed the $H^1(\Omega)$ solution to (1.1). Therefore, we have established the following equivalence result.

**Theorem 2.2.** *The boundary value problem* (1.1) *and the integral equation* (2.2) *are equivalent.*

The integral equation (2.2) is formulated directly for the trace of the solution $U$ to (1.1). We can also find the solution $U$ to (1.1) as a single-layer potential

$$U(x) = \int_\Gamma \Phi(x,y)\, \varphi(y)\, ds_y, \quad x \in \Omega, \tag{2.6}$$

when the density function $\varphi$ on $\Gamma$ solves the integral equation

$$\frac{1}{2}\varphi(x) + \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_x}\varphi(y)\, ds_y + p(x)\int_\Gamma \Phi(x,y)\,\varphi(y)\, ds_y = g(x), \quad x \in \Gamma. \tag{2.7}$$

In operator form, (2.7) becomes

$$\left(\frac{1}{2}\mathcal{I} + \mathcal{D}'\right)\varphi + p \cdot \mathcal{S}\varphi = g,$$

where the dual operator $\mathcal{D}'$ of $\mathcal{D}$ is given by

$$(\mathcal{D}'\varphi)(x) = \int_\Gamma \frac{\partial \Phi(x,y)}{\partial \nu_x}\varphi(y)\, ds_y, \quad x \in \Gamma. \tag{2.8}$$

We note that in [**2**] this formulation (2.6, 2.7) is used for the study of completion of Cauchy data for the Laplacian. In the following, we will use (2.2) for the analysis and direct solution of the inverse problem of finding $p$, while we will use (2.6, 2.7) to generate synthetic data $u_0$ with the addition of random noise for the numerical examples.

**3. A Direct Linear Method for the Inverse Problem.** In this section we present a direct solution method for the inverse problem of recovering the coefficient function $p(x)$ on $\Gamma_1$ from a single boundary measurement $u_0$ of $u$ on $\Gamma_0$.

Similar to [**9**], we introduce a new variable:

$$v(x) = p(x)u(x). \tag{3.1}$$

The support of $v$ is contained in $\Gamma_1$. Then equation (2.3) becomes linear in both $u$ and $v$:

$$\left(\frac{1}{2}\mathcal{I} + \mathcal{D}\right)u + \mathcal{S}_1 v = \mathcal{S}g, \tag{3.2}$$

where $(\mathcal{S}_1 v)(x) = \int_{\Gamma_1} \Phi(x,y)v(y)ds_y$ for $x \in \Gamma$. Denote the restriction operator from $\Gamma$ to $\Gamma_0$ by $\mathcal{R}_0 : L^2(\Gamma) \to L^2(\Gamma_0)$. That is, $(\mathcal{R}_0 u)(x) = u(x)$ for $x \in \Gamma_0$. Then the measurement of $u$ on $\Gamma_0$ can be expressed as

$$\mathcal{R}_0 u = u_0. \tag{3.3}$$

We cast the inverse problem as a direct problem of finding $p$ from (3.1)–(3.3). Since $u$ on the other part of the boundary is unknown, we will view (3.2)–(3.3) as a system to find both $u$ on $\Gamma$ and $v$ on $\Gamma_1$. We write them as a system of operator equations:

$$\begin{bmatrix} \frac{1}{2}\mathcal{I}+\mathcal{D} & \mathcal{S}_1 \\ \mathcal{R}_0 & \mathcal{O} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \mathcal{S}g \\ u_0 \end{bmatrix} \quad \text{or} \quad \mathcal{A}w = f. \tag{3.4}$$

Here $\mathcal{O}$ denotes the zero operator from $L^2(\Gamma_1)$ to $L^2(\Gamma_0)$. Once $u$ on $\Gamma$ and $v$ on $\Gamma_1$ are found from (3.4), we can use the simple relation (3.1) to find the Robin coefficient $p$ on $\Gamma_1$.

The system (3.4) is a linear system for $w = (u, v)^T$, but is ill-posed. We will apply the classical Tikhonov regularization method to address the ill-posedness. First we establish the injectivity of the operator $\mathcal{A}$ and the denseness of its range.

**Theorem 3.1.** *The operator $\mathcal{A}$: $L^2(\Gamma) \times L^2(\Gamma_1) \to L^2(\Gamma) \times L^2(\Gamma_0)$ is injective. Furthermore, if the operator $\mathcal{S}$ is injective, then $\mathcal{A}$ has dense range.*

*Proof.* If $\mathcal{A}w = 0$ for some $w = (u, v)^T \in L^2(\Gamma) \times L^2(\Gamma_1)$, then

$$\frac{1}{2}u + \mathcal{D}u + \mathcal{S}_1 v = 0 \quad \text{and} \quad \mathcal{R}_0 u = 0.$$

From the first equation, we see that $u$ is the boundary value of the harmonic function in $\Omega$ (also denoted by $u$ for simplicity) with Neumann boundary condition $\partial u/\partial \nu = -\tilde{v}$, where $\tilde{v}$ denotes the zero extension of $v$ on $\Gamma_1$ to the entire $\Gamma$. In particular, $\partial u/\partial \nu = 0$ on $\Gamma_0$ since $\Gamma_1 \cap \Gamma_0 = \varnothing$. But the second equation above also gives $u = 0$ on $\Gamma_0$. Hence, by Holmgren's uniqueness theorem, $u = 0$, and consequently $v = 0$. Therefore $w = 0$ and $\mathcal{A}$ is injective.

To show that $\mathcal{A}$ has dense range, we prove that $\mathcal{A}'$ is injective as follows. Note that

$$\mathcal{A}' = \begin{bmatrix} \frac{1}{2}\mathcal{I}+\mathcal{D}' & \mathcal{R}_0' \\ \mathcal{S}_1' & \mathcal{O}' \end{bmatrix}$$

where $\mathcal{R}_0' : L^2(\Gamma_0) \to L^2(\Gamma)$ is the zero extension operator from $\Gamma_0$ to $\Gamma$, $\mathcal{S}_1' : L^2(\Gamma) \to L^2(\Gamma_1)$ is $\mathcal{S}$ restricted to $\Gamma_1$, and $\mathcal{O}' : L^2(\Gamma_0) \to L^2(\Gamma_1)$

is the zero operator. If $\mathcal{A}'z = 0$ for some $z = (\xi, \eta)^T \in L^2(\Gamma) \times L^2(\Gamma_0)$, then

$$\frac{1}{2}\xi + \mathcal{D}'\xi + \mathcal{R}_0'\eta = 0 \quad \text{and} \quad \mathcal{S}_1'\xi = 0.$$

From the first equation, the single-layer potential $u = \mathcal{S}\xi$ on $\Omega$ is the solution to the Nuemann boundary value problem with $\partial u/\partial \nu = -\mathcal{R}_0'\eta$. In particular, $\partial u/\partial \nu = 0$ on $\Gamma_1$. The second equation above also gives $u = 0$ on $\Gamma_1$. Hence, by Holmgren's theorem again, we find that $u = 0$ and thus $\xi = 0$ since $\mathcal{S}$ is injective; consequently $\eta = 0$. Therefore $z = (\xi, \eta)^T = 0$ and $\mathcal{A}'$ is injective. Thus $\mathcal{A}$ has dense range. □

*Remark* 3.2. The injectivity of $\mathcal{S}$ depends on the domain $\Omega$. One sufficient condition is based on the "transfinite diameter" of $\Omega$ [**12**], while another requires that there be $x_0 \in \Omega$ such that $|x - x_0| \neq 1$ for all $x \in \Omega$ [**7**, Theorem 3.16]. The unit disk is an example where $\mathcal{S}$ has a nontrivial nullspace containing all constant functions.

Now we apply the classical Tikhonov method to find a stable approximate solution $w_\alpha$ to $\mathcal{A}w = f$, i.e. $w_\alpha$ solves the regularized system

$$-\alpha\mathcal{H}w + \mathcal{A}'\mathcal{A}w = \mathcal{A}'f, \tag{3.5}$$

where we choose the regularization operator $\mathcal{H}$ as $\mathcal{H}w = (D_p^2 u, D_0^2 v)^T$ ($D_p^2$ is the second derivative operator with periodic boundary condition, while $D_0^2$ is the second derivative operator with zero boundary conditions). The positive constant $\alpha$ is the regularization parameter. This regularized solution can also be viewed as the minimizer of the Tikhonov functional (quadratic) that consists of a data fidelity term and a regularization term [**7, 8**].

From (2.1) it can be easily shown that the solution $U \in H^1(\Omega)$ is non-negative on $\Omega$ (by setting $\phi = \min\{U, 0\}$ and using the coercivity of the associated bilinear form), and, if the solution is more regular, classical maximum principles can be applied to yield positivity of $u$ on any compact subset of $\Gamma_1$ [**3**]. Hence, in such situations, it may be valid to solve $p$ from the relation (3.1) by simple division: $p(x) = v(x)/u(x)$. However, when solving $w_\alpha$ from (3.5), the component $u_\alpha$ is not guaranteed to be positive, hence we must exercise care when computing $p$ from this relation. Based on a Tikhonov regularization consideration for the possibly ill-posed problem of solving $p$ from

$p(x)u(x) = v(x)$, we find an approximate solution $p_{\alpha,\beta}(x)$ for the Robin coefficient $p(x)$ as

$$p_{\alpha,\beta}(x) = \frac{v_\alpha^+(x)u_\alpha^+(x)}{\beta + (u_\alpha^+(x))^2}, \quad x \in \Gamma_1, \tag{3.6}$$

for some small $\beta > 0$, where $v^+ = \max\{v, 0\}$ denotes the non-negative part of a function $v$. In nearly all of our numerical examples, $u_\alpha(x)$ is indeed positive and we are able to set $\beta$ to 0. There are other regularization methods for this problem, a common one being to express $p$ in terms of appropriate basis functions such as $B$-splines ([**2, 4**]).

We remark that this direct method is much simpler than the least squares method presented in [**9**]. The least squares formulation there was applied only to equation (3.3) for $v$ after solving $u$ in terms of $v$ from (3.2) with extra care to deal with the fact that $\frac{1}{2}\mathcal{I} + \mathcal{D}$ has a one-dimensional nullspace. Moreover, although it was $v$ that was to be solved, regularization in the form of $u$ was needed. Thus it was necessary there to compute the inverse operator/matrix to represent $u$ in terms of $v$. On the contrary, here our method (3.5) is for a larger system for $(u, v)$ in a much more direct fashion, and the regularization is applied to the system to address the combined ill-posedness in the equations.

**4. Numerical Examples.** On the boundary $\Gamma$, we use a regular 1-periodic parametrization with counterclockwise orientation

$$x(t) = (x_1(t), x_2(t)), \quad 0 \le t \le 1,$$

where $x_1(t),\ x_2(t) \in C_p^2[0,1]$ and $|x'(t)| > 0$ for $0 \le t \le 1$. For $x = (x_1, x_2)$, we denote $x^\perp = (x_2, -x_1)$. We also set $u(t) = u(x(t))$ for simplicity. Then the integral operators in (2.2) and (2.7) can be expressed explicitly in terms of their kernels as

$$(\mathcal{S}u)(t) = \int_0^1 K_s(t,s)\ u(s)\ ds \quad \text{with} \quad K_s(t,s) = \frac{|x'(s)|}{2\pi} \ln \frac{1}{|x(t) - x(s)|},$$

$$(\mathcal{D}u)(t) = \int_0^1 K_c(t,s)\; u(s)\; ds \quad \text{with} \;\; K_c(t,s)$$
$$= \begin{cases} \dfrac{1}{2\pi} \dfrac{x'(s)^\perp \cdot (x(t) - x(s))}{|x(t) - x(s)|^2}, & t \neq s, \\ \dfrac{1}{4\pi} \dfrac{x'(t)^\perp \cdot x''(t)}{|x'(t)|^2}, & t = s, \end{cases}$$
$$(\mathcal{D}'u)(t) = \int_0^1 K_c'(t,s)\; u(s)\; ds \quad \text{with} \;\; K_c'(t,s) = K_c(s,t) \frac{|x'(s)|}{|x'(t)|}$$

for $0 \leq t, s \leq 1$. The kernel $K_s$ is weakly singular while $K_c$ and $K_c'$ are continuous. The two boundary integral equations (2.2) and (2.7) then become

$$\frac{1}{2}u(t) + \int_0^1 \{K_c(t,s) + p(s)K_s(t,s)\}\, u(s)\, ds = \int_0^1 K_s(t,s)\; g(s)\, ds \tag{4.1}$$

and

$$\frac{1}{2}\varphi(t) + \int_0^1 \{K_c'(t,s) + p(t)K_s(t,s)\}\, \varphi(s)\, ds = g(t), \tag{4.2}$$

respectively, for $0 \leq t \leq 1$. When using formulation (4.2), we obtain $u$ from $\varphi$ by

$$u(t) = \int_0^1 K_s(t,s)\; \varphi(s)\; ds. \tag{4.3}$$

We employ the Nyström's method with trigonometric interpolation. The singularity in $K_s(t,s)$ can be rearranged as

$$\ln|x(t) - x(s)| = \ln\left(2|\sin(\pi(t-s))|\right) + K_0(t,s)$$

with continuous kernel

$$K_0(t,s) = \begin{cases} \ln \dfrac{|x(t) - x(s)|}{2|\sin(\pi(t-s))|}, & t \neq s \\ \ln \dfrac{|x'(t)|}{2\pi}, & t = s \end{cases}$$

so that integrals involving this singularity can be dealt with by exact integration. For detailed description and anlysis of this numerical method, we refer to [**8**, Chapter 12]. We note that this is one of many existing well-established numerical methods for integral equations of second kind (e.g. [**1, 8**]); it serves our purpose for the initial investigation of solution method for the inverse problem, and if it becomes necessary in future study or for specific problems in application, we can easily adopt other numerical methods suitable for more specific need.

For the sake of simplicity, in our examples below, we take $\Omega$ as the elliptic region bounded by $x_1^2/a^2 + x_2^2/b^2 = 1$ with $(a, b) = (1, 0.2)$. The ellipse has the standard parametrization

$$x = x(t) = (a\cos(2\pi t), b\sin(2\pi t)), \quad 0 \le t \le 1.$$

We set the function $g(t)$ as

$$g(t) = 1 \quad \text{for } t \in [0.4, 0.6] \quad \text{and} \quad g(t) = 0 \quad \text{elsewhere.}$$

The two segments $\Gamma_1$ and $\Gamma_0$ are chosen as

$$\Gamma_1 = \{x(t) : t \in [0.1, 0.4]\} \quad \text{and} \quad \Gamma_0 = \{x(t) : t \in [0.6, 0.9]\}$$

Discretization mesh size is set to $h = 1/200$ for solving the system (3.4). To generate the synthetic data $u_0$ on $\Gamma_0$, we use the other formulation (2.6)–(2.7) with mesh size $h/2$, and add to it uniformly distributed random noise of noise level $\delta$ (relative to the $L^2$-norm of data $u_0$). The recovered Robin coefficient $p$ from (3.5)–(3.6) depends on the specific realization of the added random noise in the data. In order to provide a better illustration as to how different noise levels in the measurement affect the overall quality of the recovered $p$, we present results of several recovered Robin coefficient $p$ together in each plot in Figure 4.1 from several sets of synthetic data within the same noise level but with different realizations of the random noise. The regularization parameter $\alpha$ is chosen by experiments. As can be seen, for smaller noise level, the method is capable of recovering the Robin coefficient very well. In Figure 4.2 we present several recovered $p$ for different profiles of true $p$, again from several sets of data with same noise level in each graph.

We further remark on results of our direct method in comparison with the results from methods presented in [**9**] by similar integral

equation formulations. The results by our direct method here are slightly better in general than the direct least-squares method by [**9**]; the main difference in implementation between the two methods is the size of linear systems involved: The system (3.5) is twice the size of the equation for a single $u$ or $v$ as in [**9**]. The iterative quadratic programming method by [**9**] is more robust and produces better results in general. However, it is worth noting that our direct method here is far more economical computationally, yet it is capable of producing results of comparable quality, noticeably in cases with simpler profiles (e.g. Figure 4.1 here versus [**9**, Figure 4]). Because of its simple and economical nature, our direct method here in general can provide a quick quality initial guess for iterative methods that are computationally more intensive, such as the quadratic programming method or methods from the PDE approach.

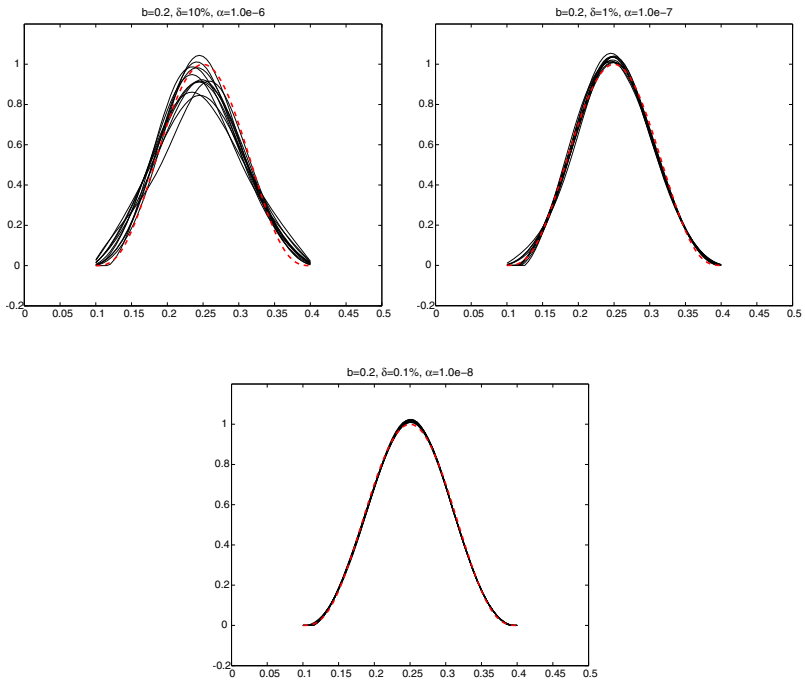


FIGURE 4.1. Results of 10 recovered $p$ (solid) from 10 measurements with same noise level in each plot, for the same true profile $p$ (dashed).

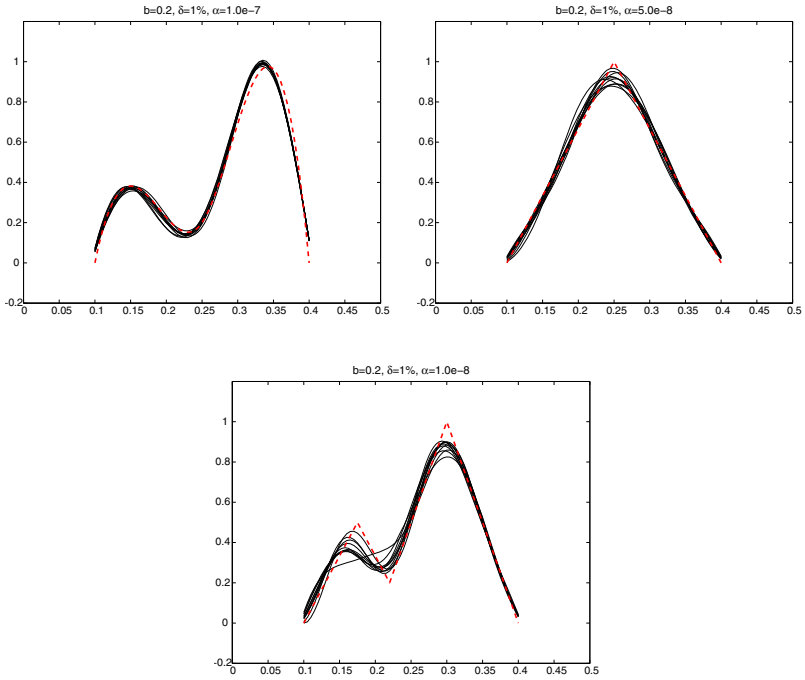


FIGURE 4.2. Results of 10 recovered $p$ (solid) from 10 measurements with same noise level $\delta = 1\%$, for different true profile $p$ (dashed).

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