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A PARAMETER CHOICE STRATEGY FOR A MULTI-LEVEL AUGMENTATION METHOD SOLVING ILL-POSED OPERATOR EQUATIONS

ZHONGYING CHEN, YING JIANG, LIHONG SONG AND HONGQI YANG

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ABSTRACT. We apply the multi-level augmentation method for solving operator equations of the first kind via the Tikhonov regularization method. We present a new a posteriori parameter choice strategy which leads to optimal convergence rates. Numerical experiments illustrate the efficiency of the method.

1. Introduction. In this paper we consider the problem of solving the first kind operator equation

$$\mathcal{K}x = y, \tag{1.1}$$

where $\mathcal{K}$ is a linear compact operator from a Hilbert space $\mathbf{X}$ to another Hilbert space $\mathbf{Y}$. Assume that $y \in D(\mathcal{K}^\dagger) := R(\mathcal{K}) + R(\mathcal{K})^\perp$, where $R(\mathcal{K})$ is the range of the operator $\mathcal{K}$, and $\mathcal{K}^\dagger$ the Moore-Penrose generalized inverse of $\mathcal{K}$. It is known that the minimum norm least squares solution $x_* = \mathcal{K}^\dagger y$ of the equation (1.1) exists. In the following, for simplicity, we assume $y \in R(\mathcal{K})$, cf. [**6**].

In practice, only approximate righthand side $y^\delta \in \mathbf{Y}$ with $\|y - y^\delta\| \le \delta$ is available, where $\delta > 0$ is an error level. Thus we need to solve the perturbed operator equation

$$\mathcal{K}x^\delta = y^\delta. \tag{1.2}$$

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It is well known that the above problem is ill-posed when the range $R(\mathcal{K})$ of the operator $\mathcal{K}$ is not closed in $\mathbf{Y}$. Regularization methods are normally used to obtain a stable approximate solution [**5, 6, 11, 15**]. The idea of the regularization methods is to determine an approximation $x^\delta \in \mathbf{X}$ to the solution x_* that depends continuously on the perturbation error δ of the righthand side. There are many regularization methods, such as the Tikhonov regularization, the Lavrentiev regularization and the Landweber iteration.

Regularized projection methods with *a priori* and *a posteriori* choices of regularization parameters were studied by many authors (see, for example, [**7, 10, 12, 13**]). Recently, multi-scale and wavelet methods were applied to solve ill-posed problems, see [**3, 8, 9**]. Multi-scale schemes are becoming efficient numerical methods for solving ill-posed problems. The multi-level augmentation method was developed in [**1**] for solving well-posed operator equations. The method is based on the multi-scale decomposition of the range space of the operator and the solution space of the equation and a matrix splitting strategy. The method was applied to solving ill-posed operator equations in [**2**]. It was combined with the Lavrentiev regularization method and led to fast solutions of discrete regularization methods for the equations. Theoretical analysis and numerical experiments show the accuracy and efficiency of the method. Tikhonov regularization method is a more general method and has been used widely. The choice of the regularization parameter is a key issue in Tikhonov regularization methods [**4, 6**]. So we apply the augmentation method to Tikhonov regularization method, and extend the idea of *a posteriori* choices of regularization parameters in [**2**]. The purpose of this paper is to develop a new *a posteriori* regularization parameter choice strategy for the multi-level augmentation method for solving ill-posed operator equations of the first kind, by using the Tikhonov regularization method. We will show that the method provides an optimal convergence order with great advantages in computation.

We organize this paper as follows. In Section 2, we describe the multi-level augmentation method for numerical solutions of ill-posed operator equations of the first kind. We present in Section 3 an *a posteriori* regularization parameter choice strategy. In Section 4, we establish an optimal order of convergence for the approximate solution obtained from the multi-level augmentation method using the *a* *posteriori* regularization parameter. In Section 5, we present numerical examples which illustrate the efficiency of the method and confirm the theoretical results obtained in Sections 3 and 4.

2. The multi-level augmentation method. The multi-level augmentation method had been applied to Lavrentive regularization method in [**2**]. In this section we describe the multi-level augmentation method for numerically solving ill-posed operator equations of the first kind stabilized by Tikhonov regularization. Similarly, cf. [**2**], we also show some error estimations due to the multi-level augmentation method.

The Tikhonov regularization method for solving (1.2) is: For some $\alpha > 0$, finding an approximation solution $x^\delta(\alpha)$ such that

$$(\alpha\mathcal{I} + \mathcal{A})x^\delta(\alpha) = \mathcal{K}^* y^\delta, \tag{2.1}$$

where $\mathcal{A} = \mathcal{K}^*\mathcal{K}$ and $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$. We denote $x(\alpha)$ the solution of the equation

$$(\alpha\mathcal{I} + \mathcal{A})x(\alpha) = \mathcal{K}^* y. \tag{2.2}$$

In order to solve (2.1) numerically, let $\{\mathbf{X}_n : n \in \mathbf{N}_0\}$, where $\mathbf{N}_0 := \{0, 1, 2, \dots\}$, be a sequence of finite-dimensional subspaces of $\mathbf{X}$ satisfying $\overline{\cup_{n\in\mathbf{N}_0}\mathbf{X}_n} = \mathbf{X}$, and let $\{\mathcal{P}_n : n \in \mathbf{N}_0\}$ be a sequence of orthogonal projections from $\mathbf{X}$ to $\mathbf{X}_n$. Denote $\mathcal{A}_n := \mathcal{P}_n\mathcal{A}\mathcal{P}_n$, $z_n^\delta := \mathcal{P}_n\mathcal{K}^* y^\delta$. The approximation problem for solving (2.1) is to find $x_n^\delta(\alpha)$ in $\mathbf{X}_n$ such that

$$(\alpha\mathcal{I} + \mathcal{A}_n)x_n^\delta(\alpha) = z_n^\delta. \tag{2.3}$$

It is clear that equations (2.1) and (2.3) have a unique solution respectively. Moveover, the following hold

$$\|(\mathcal{A} + \alpha\mathcal{I})^{-1}\| \le 1/\alpha, \quad \|(\mathcal{A}_n + \alpha\mathcal{I})^{-1}\| \le 1/\alpha, \tag{2.4}$$

$$\|(\mathcal{A} + \alpha\mathcal{I})^{-1}\mathcal{A}\| \le 1, \quad \|(\mathcal{A}_n + \alpha\mathcal{I})^{-1}\mathcal{A}_n\| \le 1, \tag{2.5}$$

and

$$\|(\mathcal{A}+\alpha\mathcal{I})^{-1}\mathcal{K}^*\| \le 1/(2\sqrt{\alpha}), \quad \|(\mathcal{A}_n+\alpha\mathcal{I})^{-1}\mathcal{P}_n\mathcal{K}^*\| \le 1/(2\sqrt{\alpha}). \tag{2.6}$$

In the following discussion, we impose two hypotheses, cf. [**2, 12**].

(H1) For some $\nu \in (0,1]$, $x_* = \mathcal{K}^\dagger y \in \mathrm{R}(\mathcal{A}^\nu)$, i.e., there is an $\omega \in \mathbf{X}$ such that $x_* = \mathcal{A}^\nu \omega$.

(H2) There is a sequence of positive real numbers $\{\varepsilon_n : n \in \mathbf{N}_0\}$ satisfying

$$\frac{1}{\sigma}\varepsilon_n < \varepsilon_{n+1} < \varepsilon_n, \quad \text{and} \quad \varepsilon_n \longrightarrow 0, \quad (n \to +\infty),$$

for some $\sigma > 1$, such that

$$\|(\mathcal{I} - \mathcal{P}_n)\mathcal{A}^\nu\| \le a_\nu \varepsilon_n^\nu, \quad 0 < \nu \le 2. \tag{2.7}$$

We remark that, under the condition (H1), the following estimates hold (see, for example, [**2, 14**])

$$\|x(\alpha) - x_*\| \le c_\nu \|\omega\| \alpha^\nu, \quad \text{and} \tag{2.8}$$

$$\|x(\alpha) - x^\delta(\alpha)\| \le \frac{\delta}{2\sqrt{\alpha}}. \tag{2.9}$$

We now apply multi-level augmentation method to solve the equation (2.3). Assume that the subspaces $\mathbf{X}_n$, $n \in \mathbf{N}_0$, are nested, i.e., $\mathbf{X}_n \subset \mathbf{X}_{n+1}$, $n \in \mathbf{N}_0$. Let $\mathbf{W}_n$ be the orthogonal complement of $\mathbf{X}_{n-1}$ in $\mathbf{X}_n$, $n \in \mathbf{N}$, that is, $\mathbf{X}_n = \mathbf{X}_{n-1} \oplus^\perp \mathbf{W}_n$. Thus for any $k \in \mathbf{N}$ and $m \in \mathbf{N}_0$, we have the space decomposition

$$\mathbf{X}_{k+m} = \mathbf{X}_k \oplus^\perp \mathbf{W}_{k+1} \oplus^\perp \mathbf{W}_{k+2} \oplus^\perp \cdots \oplus^\perp \mathbf{W}_{k+m}. \tag{2.10}$$

We identify the vector $[x_0, x_1, \ldots, x_m]^T$ in $\mathbf{X}_k \times \mathbf{W}_{k+1} \times \cdots \times \mathbf{W}_{k+m}$ with vector $x_0 + x_1 + \cdots + x_m$ in $\mathbf{X}_k \oplus^\perp \mathbf{W}_{k+1} \oplus^\perp \cdots \oplus^\perp \mathbf{W}_{k+m}$. Let

$$\mathcal{Q}_{n+1} := \mathcal{P}_{n+1} - \mathcal{P}_n.$$

The operator $\mathcal{A}_{k+m}$ can be written as a matrix form

$$\mathcal{A}_{k,m} := \begin{bmatrix} \mathcal{P}_k\mathcal{A}\mathcal{P}_k & \mathcal{P}_k\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{P}_k\mathcal{A}\mathcal{Q}_{k+m} \\ \mathcal{Q}_{k+1}\mathcal{A}\mathcal{P}_k & \mathcal{Q}_{k+1}\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{Q}_{k+1}\mathcal{A}\mathcal{Q}_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{k+m}\mathcal{A}\mathcal{P}_k & \mathcal{Q}_{k+m}\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{Q}_{k+m}\mathcal{A}\mathcal{Q}_{k+m} \end{bmatrix}. \tag{2.11}$$

We split the operator $\mathcal{A}_{k,m}$ into the sum of two operators

$$\mathcal{A}_{k,m} = \mathcal{A}^L_{k,m} + \mathcal{A}^H_{k,m}, \tag{2.12}$$

where

$$\begin{aligned}\mathcal{A}^H_{k,m} &:= (\mathcal{P}_{k+m} - \mathcal{P}_k)\mathcal{A}\mathcal{P}_{k+m} \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{Q}_{k+1}\mathcal{A}\mathcal{P}_k & \mathcal{Q}_{k+1}\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{Q}_{k+1}\mathcal{A}\mathcal{Q}_{k+m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{k+m}\mathcal{A}\mathcal{P}_k & \mathcal{Q}_{k+m}\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{Q}_{k+m}\mathcal{A}\mathcal{Q}_{k+m} \end{bmatrix},\end{aligned}$$

and

$$\mathcal{A}^L_{k,m} := \mathcal{P}_k\mathcal{A}\mathcal{P}_{k+m} = \begin{bmatrix} \mathcal{P}_k\mathcal{A}\mathcal{P}_k & \mathcal{P}_k\mathcal{A}\mathcal{Q}_{k+1} & \cdots & \mathcal{P}_k\mathcal{A}\mathcal{Q}_{k+m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For a given positive parameter α, we denote $\mathcal{B}_{k,m}(\alpha) := \alpha\mathcal{I} + \mathcal{A}^L_{k,m}$, $\mathcal{C}_{k,m} := \mathcal{A}^H_{k,m}$. Then, the equation (2.3) with $n = k + m$ becomes

$$(\mathcal{B}_{k,m}(\alpha) + \mathcal{C}_{k,m})x^\delta_{k+m}(\alpha) = z^\delta_{k+m}. \tag{2.13}$$

We recall the multi-level augmentation scheme for solving (2.3) which can be described as follows.

Algorithm 2.1 (The multi-level augmentation scheme).

1. *For a fixed $k > 0$, solve (2.3) with $n = k$ to obtain $x^\delta_k(\alpha) \in \mathbf{X}_k$.*

2. *Set $x^\delta_{k,0}(\alpha) := x^\delta_k(\alpha)$ and compute matrices $\mathcal{B}_{k,0}(\alpha)$ and $\mathcal{C}_{k,0}$.*

3. *For $m \in \mathbf{N}$, assume that $x^\delta_{k,m-1}(\alpha)$ has been obtained and do the following.*

(a) *Augment the matrices $\mathcal{B}_{k,m-1}(\alpha)$ and $\mathcal{C}_{k,m-1}$ to form $\mathcal{B}_{k,m}(\alpha)$ and $\mathcal{C}_{k,m}$.*

(b) *Augment $x^\delta_{k,m-1}(\alpha)$ to form $\bar{x}^\delta_{k,m}(\alpha) := [x^\delta_{k,m-1}(\alpha)^T, 0]^T \in \mathbf{X}_{k+m}$.*

(c) *Solve* $x^\delta_{k,m}(\alpha)$ *from the equation*

$$\mathcal{B}_{k,m}(\alpha)x^\delta_{k,m}(\alpha) = z^\delta_{k+m} - \mathcal{C}_{k,m}\bar{x}^\delta_{k,m}(\alpha). \tag{2.14}$$

To estimate the error of the multi-level augmentation solution we first estimate the error between the projection solution $x^\delta_n(\alpha)$ and the solution $x(\alpha)$ of the equation (2.2) by modifying the proof of Proposition 3.1 in [**2**]. We denote

$$\gamma^\delta_{\alpha,n} := \frac{\delta}{2\sqrt{\alpha}} + \frac{(2a_{1+\nu} + a_1 a_\nu)\|\omega\|}{\alpha}\varepsilon_n^{1+\nu}.$$

Lemma 2.2. *Assume that hypotheses* (H1) *and* (H2) *hold. Let* $x(\alpha)$ *and* $x^\delta_n(\alpha)$ *be the solutions of the equation* (2.2) *and the projection method* (2.3)*, respectively. Then there exists a positive integer* $N \in \mathbf{N}_0$ *such that when* $n \geq N$,

$$\|x^\delta_n(\alpha) - x(\alpha)\| \leq \gamma^\delta_{\alpha,n}. \tag{2.15}$$

Proof. We just need to do some modification of the proof of Proposition 3.1 in [**2**]. We replace $\|(\alpha\mathcal{I} + \mathcal{A}_n)^{-1}(f^\delta - f)\|$ by $\|(\alpha\mathcal{I} + \mathcal{A}_n)^{-1}\mathcal{P}_n\mathcal{K}^*(y^\delta - y)\|$, replace $\|(\alpha\mathcal{I} + \mathcal{A}_n)^{-1}(\mathcal{P}_n - \mathcal{I})f\|$ by $\|(\alpha\mathcal{I}+\mathcal{A}_n)^{-1}(\mathcal{P}_n-\mathcal{I})\mathcal{K}^*y\|$, and replace $\|[(\alpha\mathcal{I}+\mathcal{A}_n)^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}]f\|$ by $\|[(\alpha\mathcal{I} + \mathcal{A}_n)^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}]\mathcal{K}^*y\|$ in the proof of Proposition 3.1 of [**2**]. We also substitute

$$\|(\alpha\mathcal{I} + \mathcal{A}_n)^{-1}(f^\delta - f)\| \leq \frac{\delta}{\alpha}$$

by

$$\|(\alpha\mathcal{I} + \mathcal{A}_n)^{-1}\mathcal{P}_n\mathcal{K}^*(y^\delta - y)\| \leq \frac{\delta}{2\sqrt{\alpha}}. \tag{2.16}$$

Then, by going through the similar process, we get the conclusion. □

In the next theorem, we estimate the distance between $x^\delta_{k+m}(\alpha)$ and $x^\delta_{k,m}(\alpha)$ by modifying the proof of Proposition 3.3 in [**2**].

Theorem 2.3. *Assume that hypotheses* (H1) *and* (H2) *hold. Let $x^\delta_{k,m}(\alpha)$ and $x^\delta_{k+m}(\alpha)$ be the solutions of the multi-level augmentation method and the projection method* (2.3) *with $n = k+m$, respectively. Then, there exists a positive integer $N \in \mathbf{N}_0$ such that, when $k \geq N$, $m \in \mathbf{N}_0$ and α satisfies the conditions*

$$\alpha \geq (2\sigma+3)a_1\varepsilon_k \quad \text{and} \quad \varepsilon_k \leq 1; \tag{2.17}$$

the following estimations hold

$$\|x^\delta_{k+m}(\alpha) - x^\delta_{k,m}(\alpha)\| \leq \gamma^\delta_{\alpha,k+m}, \quad \text{and} \tag{2.18}$$

$$\|x(\alpha) - x^\delta_{k,m}(\alpha)\| \leq 2\gamma^\delta_{\alpha,k+m}. \tag{2.19}$$

Proof. Similar to equation (3.22) in [**2**], we have that

$$\|x^\delta_{k+m}(\alpha) - x^\delta_{k,m}(\alpha)\| \leq \frac{\|\mathcal{C}_{k,m}\|}{\alpha - \|\mathcal{C}_{k,m}\|}\|\bar{x}^\delta_{k,m}(\alpha) - x^\delta_{k+m}(\alpha)\|. \tag{2.20}$$

Since $x^\delta_{k,0}(\alpha) = x^\delta_k(\alpha)$, the estimation (2.18) holds when $m = 0$. Suppose that (2.18) holds for $m = r-1$. We come to prove that (2.18) holds for $m = r$. According to the definition of $\bar{x}^\delta_{k,r}(\alpha)$, we have

$$\begin{aligned}\|\bar{x}^\delta_{k,r}(\alpha) - x^\delta_{k+r}(\alpha)\| \leq{}& \|x^\delta_{k+r}(\alpha) - x(\alpha)\| \\ &+ \|x^\delta_{k+r-1}(\alpha) - x(\alpha)\| + \|x^\delta_{k,r-1}(\alpha) - x^\delta_{k+r-1}(\alpha)\|.\end{aligned} \tag{2.21}$$

By the hypotheses (H2), the condition (2.17) and Lemma 2.2, we obtain

$$\begin{aligned}\|\bar{x}^\delta_{k,r}(\alpha) - x^\delta_{k+r}(\alpha)\| \leq{}& \frac{1}{2\sigma+3}\gamma^\delta_{\alpha,k+r} + \frac{1}{2\sigma+3}\gamma^\delta_{\alpha,k+r-1} \\ &+\gamma^\delta_{\alpha,k+r-1}.\end{aligned} \tag{2.22}$$

Noting that $\varepsilon_{k+r-1} < \sigma\varepsilon_{k+r}$, $\sigma > 1$ and $\varepsilon_{k+r-1} < \varepsilon_k < 1$, we have

$$\gamma^\delta_{\alpha,k+r-1} \leq \sigma\gamma^\delta_{\alpha,k+r}. \tag{2.23}$$

Substituting (2.23) into (2.22), we get

$$\|\bar{x}^{\delta}_{k,r}(\alpha) - x^{\delta}_{k+r}(\alpha)\| \le \frac{2\sigma^2+4\sigma+1}{2\sigma+3}\gamma^{\delta}_{\alpha,k+r}. \tag{2.24}$$

Since α satisfies (2.17) and $\|\mathcal{C}_{k,m}\| \le 2a_1\varepsilon_k$, we have

$$\frac{\|\mathcal{C}_{k,m}\|}{\alpha - \|\mathcal{C}_{k,m}\|} \le \frac{2}{2\sigma+1}. \tag{2.25}$$

Substituting (2.24) and (2.25) into the righthand side of (2.20) with $m = r$ yields

$$\begin{aligned} \|x^{\delta}_{k+m}(\alpha) - x^{\delta}_{k,m}(\alpha)\| &\le \frac{2(2\sigma^2+4\sigma+1)}{(2\sigma+1)(2\sigma+3)}\gamma^{\delta}_{\alpha,k+m} \\ &\le \gamma^{\delta}_{\alpha,k+m}. \end{aligned} \tag{2.26}$$

From Lemma 2.2 and (2.26), we obtain the inequality (2.19). □

3. An a posteriori parameter choice strategy.

3. An a posteriori parameter choice strategy. In this section, we present an *a posteriori* parameter choice strategy which leads to the optimal convergence for the approximate solution obtained by the multi-level augmentation method with the parameter.

To do this, we consider the equation

$$(\alpha\mathcal{I} + \mathcal{A})v^{\delta}(\alpha) = x^{\delta}_{k,m}(\alpha) \tag{3.1}$$

and its approximate equations

$$(\alpha\mathcal{I} + \mathcal{A}_{k+m})v^{\delta}_{k+m}(\alpha) = x^{\delta}_{k,m}(\alpha), \tag{3.2}$$

where $x^{\delta}_{k,m}(\alpha) \in \mathbf{X}_{k+m}$ is the multi-level augmentation solution of the equation (2.3). In order to distinguish the multi-level augmentation solution of (3.2), we denote $v^{\delta}(\alpha)$ and $v^{\delta}_{k+m}(\alpha)$ the solutions of (3.1) and (3.2), respectively, which indeed depend on k and m. We also denote $v^{\delta}_{k,m}(\alpha)$ the multi-level augmentation solution of (3.2). The *a* *posteriori* regularization parameter α_* is determined by the following principle. To describe the strategy, we denote $\alpha_1 := \delta^2$.

Algorithm 3.1 (A posteriori parameter choice strategy). *Choose α_* as the regularization parameter as follows.*

(C1) *If $\|\alpha_1^{3/2} v_{k,m}^\delta(\alpha_1)\| \geq 3\delta$, set $\alpha_* := \alpha_1$.*

(C2) *Otherwise, let α_* be the minimum solution of the equation*

$$\|\alpha^{3/2} v_{k,m}^\delta(\alpha)\| = 3\delta \tag{3.3}$$

in $[\alpha_1, +\infty)$.

We will prove the existence of the solution of (3.3) by using the intermediate value theorem. To this end, we first define the function $g_{k,m}^\delta(\alpha)$ on $(0, +\infty)$ by

$$g_{k,m}^\delta(\alpha) := \left\| \alpha^{3/2} v_{k,m}^\delta(\alpha) \right\|,$$

and show that it is a continuous function on $(0, +\infty)$.

Lemma 3.2. *For a given k, $m \in \mathbf{N}_0$ and $\delta > 0$, $g_{k,m}^\delta(\alpha)$ is a continuous function on $(0, +\infty)$.*

Proof. We first prove that $x_{k,m}^\delta(\alpha)$ depends continuously on α. It is clear that when $m = 0$, $x_{k,m}^\delta(\alpha)$ depends continuously on α. Assume that for $r \in \mathbf{N}$, $x_{k,r-1}^\delta(\alpha)$ depends continuously on α. Then for $\alpha_1, \alpha_2 > 0$, according to (2.14), we have that

$$\begin{aligned} \|x_{k,r}^\delta(\alpha_1) - x_{k,r}^\delta(\alpha_2)\| &\leq \|(\mathcal{B}_{k,r}^{-1}(\alpha_1) - \mathcal{B}_{k,r}^{-1}(\alpha_2)) z_{k+r}^\delta\| \\ &\quad + \|\mathcal{B}_{k,r}^{-1}(\alpha_1)\mathcal{C}_{k,r}\bar{x}_{k,r}^\delta(\alpha_1) - \mathcal{B}_{k,r}^{-1}(\alpha_2)\mathcal{C}_{k,r}\bar{x}_{k,r}^\delta(\alpha_2)\| \\ &\leq \|(\mathcal{B}_{k,r}^{-1}(\alpha_1) - \mathcal{B}_{k,r}^{-1}(\alpha_2)) z_{k+r}^\delta\| \\ &\quad + \|(\mathcal{B}_{k,r}^{-1}(\alpha_1) - \mathcal{B}_{k,r}^{-1}(\alpha_2))\mathcal{C}_{k,r}\bar{x}_{k,r}^\delta(\alpha_2)\| \\ &\quad + \|\mathcal{B}_{k,r}^{-1}(\alpha_1)\mathcal{C}_{k,r}(\bar{x}_{k,r}^\delta(\alpha_1) - \bar{x}_{k,r}^\delta(\alpha_2))\|. \end{aligned} \tag{3.4}$$

Since

$$\|\mathcal{B}_{k,r}^{-1}(\alpha_1) - \mathcal{B}_{k,r}^{-1}(\alpha_2)\| = \|(\alpha_1 - \alpha_2)\mathcal{B}_{k,r}^{-1}(\alpha_1)\mathcal{B}_{k,r}^{-1}(\alpha_2)\|,$$

and $x_{k,r-1}^{\delta}(\alpha)$ depends continuously on α, we conclude that the right-hand side of (3.4) is going to zero as $\alpha_1 \to \alpha_2$. Thus, $x_{k,r}^{\delta}(\alpha)$ depends continuously on α. Similar arguments can be used to obtain that $v_{k,m}^{\delta}(\alpha)$ depends continuously on α. This means that $g_{k,m}^{\delta}(\alpha)$ is a continuous function on $(0, +\infty)$. $\square$

Then, we will prove that there are two points in $(0, +\infty)$ at which the values of $g_{k,m}^{\delta}(\alpha)$ are nonpositive and nonnegative, respectively. To do this, we denote

$$\begin{aligned}\phi_{k,m}^{\delta}(\alpha) &:= \alpha^2 v_{k,m}^{\delta}(\alpha),\\ \psi(\alpha) &:= \alpha^2(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{K}^* y,\\ \psi^{\delta}(\alpha) &:= \alpha^2(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{K}^* y^{\delta},\end{aligned}$$

and establish some estimates. As a preparation, we first provide upper bounds of

$$\widetilde{E}_{k+m}^{\delta}(\alpha) := \inf\left\{\left\|v^{\delta}(\alpha) - v\right\| : v \in \mathbf{X}_{k+m}\right\} \quad \text{and} \quad \|v_{k,m}^{\delta}(\alpha) - v_{k+m}^{\delta}(\alpha)\|.$$

Lemma 3.3. *Assume that hypotheses* (H1) *and* (H2) *hold. Then there exists a positive integer $N \in \mathbf{N}_0$ such that when $k \geq N$, $m \in \mathbf{N}_0$, α and ε_k satisfy condition* (2.17)*, the following estimate holds*:

$$\widetilde{E}_{k+m}^{\delta}(\alpha) \leq \frac{\delta}{\alpha^{3/2}} + \frac{(5a_{1+\nu} + a_1 a_\nu)\|\omega\|}{\alpha^2}\varepsilon_{k+m}^{1+\nu}, \tag{3.5}$$

and

$$\|v_{k,m}^{\delta}(\alpha) - v_{k+m}^{\delta}(\alpha)\| \leq \frac{\delta}{\alpha^{3/2}} + \frac{(5a_{1+\nu} + 2a_1 a_\nu)\|\omega\|}{\alpha^2}\varepsilon_{k+m}^{1+\nu}. \tag{3.6}$$

Proof. The proofs of (3.5) and (3.6) are similar to the proof of Proposition 4.1 in [**2**] and the proof of Theorem 2.3, respectively. $\square$

Lemma 3.4. *Assume that hypotheses* (H1) *and* (H2) *hold. Then there exists a positive integer* $N \in \mathbf{N}_0$ *such that when* $k \geq N$, $m \in \mathbf{N}_0$, α *and* ε_k *satisfy condition* (2.17); *then the following hold*:

(i) $$\|\psi(\alpha)\| \leq \|\omega\|\alpha^{1+\nu},$$

(ii) $$\|\psi^\delta(\alpha) - \psi(\alpha)\| \leq \sqrt{\alpha}\delta/2.$$

(iii) $$\|\phi^\delta_{k,m}(\alpha) - \psi(\alpha)\| \leq 2\sqrt{\alpha}\delta + (10a_{1+\nu} + 5a_1 a_\nu)\|\omega\|\varepsilon^{1+\nu}_{k+m}.$$

Proof. (i) Since $y = \mathcal{K}x_* = \mathcal{K}(\mathcal{K}^*\mathcal{K})^\nu\omega$, we have that

$$\|\psi(\alpha)\| = \|\alpha^{1+\nu}[\alpha(\alpha\mathcal{I} + \mathcal{A})^{-1}]^{1-\nu}[(\alpha\mathcal{I} + \mathcal{A})^{-1}\mathcal{A}]^{1+\nu}\omega\|. \tag{3.7}$$

By (2.4) and (2.5),

$$\big\|[\alpha(\alpha\mathcal{I} + \mathcal{A})^{-1}]^{1-\nu}\big\| \leq 1, \quad \big\|[(\alpha\mathcal{I} + \mathcal{A})^{-1}\mathcal{A}]^{1+\nu}\big\| \leq 1;$$

thus, we obtain

$$\|\psi(\alpha)\| \leq \|\omega\|\alpha^{1+\nu}. \tag{3.8}$$

(ii) Using (2.4) and (2.6), we have

$$\|\psi^\delta(\alpha) - \psi(\alpha)\| = \|\alpha^2(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{K}^*(y^\delta - y)\| \leq \frac{\sqrt{\alpha}\delta}{2}.$$

(iii) It follows from (2.2) and (3.2) that

$$\begin{aligned}
&\|\phi^\delta_{k,m}(\alpha) - \psi(\alpha)\| \\
&\quad \leq \alpha^2\|v^\delta_{k,m}(\alpha) - v^\delta_{k+m}(\alpha)\| + \alpha^2\|v^\delta_{k+m}(\alpha) - (\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1}x(\alpha)\| \\
&\qquad + \alpha^2\|(\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1}x(\alpha) - (\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{K}^*y\| \\
&\quad \leq \alpha^2\|v^\delta_{k,m}(\alpha) - v^\delta_{k+m}(\alpha)\| + \alpha^2\|(\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1}[x^\delta_{k,m}(\alpha) - x(\alpha)]\| \\
&\qquad + \alpha^2\|[(\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}](\alpha\mathcal{I} + \mathcal{A})^{-1}\mathcal{K}^*y\|.
\end{aligned} \tag{3.9}$$

Noting that

$$\begin{aligned}
&[(\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}](\alpha\mathcal{I} + \mathcal{A})^{-1}\mathcal{K}^*y \\
&\qquad = (\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1}\mathcal{P}_{k+m}\mathcal{A}(\mathcal{I} - \mathcal{P}_{k+m})(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{A}^{1+\nu}\omega \\
&\qquad\quad + (\alpha\mathcal{I} + \mathcal{A}_{k,m})^{-1}(\mathcal{I} - \mathcal{P}_{k+m})\mathcal{A}(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{A}^{1+\nu}\omega,
\end{aligned}$$

using (2.4) and (H2), we conclude that

$$\begin{aligned}(3.10)\quad &\|[(\alpha\mathcal{I}+\mathcal{A}_{k,m})^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1})](\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{K}^*y\|\\ &\qquad\leq \frac{a_1a_\nu\|\omega\|}{\alpha^2}\varepsilon_{k+m}^{1+\nu}+\frac{a_{1+\nu}\|\omega\|}{\alpha^2}\varepsilon_{k+m}^{1+\nu}.\end{aligned}$$

Combining (3.9) with (3.6), (2.19) and (3.10) yields the third estimate of this lemma. ◻

In the next lemma we show that there is an $\alpha \in (0,+\infty)$ such that $\alpha^{-1/2}\|\psi^\delta(\alpha)\| \geq 6\delta$. To do this, we denote the spectrum radius of $\mathcal{K}$ by ρ and require the following additional condition.

(H3) $\|\mathcal{K}^*y^\delta\| \geq 24\|\mathcal{K}^*\|\delta$.

Lemma 3.5. *Assume that hypothesis* (H3) *holds. If* $\alpha=\rho^2$, *then*

$$\alpha^{-1/2}\|\psi^\delta(\alpha)\| \geq 6\delta.$$

Proof. Let $\{\beta_i; u_i, v_i : i \in \mathbf{N}\}$ be the singular system of $\mathcal{K}$. Then the singular systems of operators $(\alpha\mathcal{I}+\mathcal{A})$ and $(\alpha\mathcal{I}+\mathcal{A})^{-2}$ are $\{\alpha+\beta_i^2; u_i, u_i : i\in\mathbf{N}\}$ and $\{1/(\alpha+\beta_i^2)^2; u_i, u_i : i \in \mathbf{N}\}$, respectively.

By the singular value decomposition, we have that

$$\alpha^{-1/2}\psi^\delta(\alpha)=\alpha^{3/2}(\alpha\mathcal{I}+\mathcal{A})^{-2}\mathcal{K}^*y^\delta=\alpha^{3/2}\sum_{i\in\mathbf{N}}\frac{1}{(\alpha+\beta_i^2)^2}(\mathcal{K}^*y^\delta,u_i)u_i.$$

Thus,

$$\begin{aligned}(3.11)\qquad \|\alpha^{-1/2}\psi^\delta(\alpha)\| &= \left(\sum_{i\in\mathbf{N}}\frac{\alpha^3}{(\alpha+\beta_i^2)^4}|(\mathcal{K}^*y^\delta,u_i)|^2\right)^{1/2}\\ &\geq \frac{\alpha^{3/2}}{(\alpha+\rho^2)^2}\|\mathcal{K}^*y^\delta\|.\end{aligned}$$

By the assumption (H3), when $\alpha=\rho^2$,

$$\|\alpha^{-1/2}\psi^\delta(\alpha)\| \geq \frac{6}{\rho}\|\mathcal{K}^*\|\delta \geq 6\delta,$$

which completes the proof of this lemma. ◻

The following theorem shows the existence of the solution of (3.3), which ensures that the *a posteriori* parameter can be determined by the Algorithm 3.1. To do this, we set $\alpha_0 := \min\{(\delta^2/4\|\omega\|^2), \delta^2\}$.

Theorem 3.6. *Assume that hypotheses* (H1)–(H3) *hold. Then there exists a positive integer N such that when $k \geq N$ and $m \in \mathbf{N}_0$, the equation* (3.3) *has a solution $\alpha \in [\alpha_1, \rho^2]$ except for $g^\delta_{k,m}(\alpha_1) \geq 3\delta$, in which case* (3.3) *has a solution $\alpha \in [\alpha_0, \alpha_1]$.*

Proof. It follows from (H2) that there exists a positive integer N such that when $k \geq N$ and $m \in \mathbf{N}_0$,

$$\varepsilon_k < 1, \tag{3.12}$$

$$(3\sigma + 1)a_1\varepsilon_k \leq \alpha_0, \tag{3.13}$$

and

$$\varepsilon_{k+m} \leq \eta^\delta_\nu := \min\left\{\frac{\delta\sqrt{\alpha_0}}{2(10a_{1+\nu} + 5a_1a_\nu)\|\omega\|}, 1\right\}. \tag{3.14}$$

By the definition of $\phi^\delta_{k,m}(\alpha)$,

$$\begin{aligned} g^\delta_{k,m}(\alpha) &= \alpha^{-1/2}\|\phi^\delta_{k,m}(\alpha)\| \\ &\geq \alpha^{-1/2}\|\psi^\delta(\alpha)\| - \alpha^{-1/2}\|\psi^\delta(\alpha) - \phi^\delta_{k,m}(\alpha)\| \\ &\geq \alpha^{-1/2}\|\psi^\delta(\alpha)\| - \alpha^{-1/2}(\|\psi^\delta(\alpha) - \psi(\alpha)\| + \|\psi(\alpha) - \phi^\delta_{k,m}(\alpha)\|). \end{aligned}$$

From (2.4) and (2.6) we have that

$$\|\psi^\delta(\alpha) - \psi(\alpha)\| = \|\alpha^2(\alpha\mathcal{I} + \mathcal{A})^{-2}\mathcal{K}^*(y^\delta - y)\| \leq \frac{\delta}{2}.$$

Therefore, by using Lemma 3.4 (iii) and Lemma 3.5, we conclude that when $\alpha = \rho^2$,

$$g^\delta_{k,m}(\alpha) \geq 6\delta - \frac{\delta}{2} - 2\delta - \frac{(10a_{1+\nu} + 5a_1a_\nu)\|\omega\|\varepsilon^{1+\nu}_{k+m}}{\rho},$$

which with (3.14) yields

$$g_{k,m}^{\delta}(\rho^2) \geq 3\delta. \tag{3.15}$$

On the other hand, according to Lemma 3.4 we have that

$$\begin{aligned} g_{k,m}^{\delta}(\alpha) &= \alpha^{-1/2}\|\phi_{k,m}^{\delta}(\alpha)\| \\ &\leq \alpha^{-1/2}\|\psi(\alpha)\| + \alpha^{-1/2}\|\phi_{k,m}^{\delta}(\alpha) - \psi(\alpha)\| \\ &\leq \|w\|\alpha^{1/2+\nu} + 2\delta + \frac{(10a_{1+\nu} + 5a_1a_\nu)\|\omega\|\epsilon_{k+m}^{1+\nu}}{\sqrt{\alpha}}. \end{aligned} \tag{3.16}$$

Noting that

$$\|w\|\alpha_0^{1/2+\nu} \leq \frac{\delta}{2}, \qquad \text{and} \qquad \frac{(10a_{1+\nu} + 5a_1a_\nu)\|\omega\|\varepsilon_{k+m}^{1+\nu}}{\sqrt{\alpha_0}} \leq \frac{\delta}{2},$$

we conclude

$$g_{k,m}^{\delta}(\alpha_0) \leq 3\delta. \tag{3.17}$$

Since $g_{k,m}^{\delta}(\alpha)$ is a continuous function on the interval $[\alpha_0, \rho^2]$, the equation (3.3) has a solution $\alpha \in [\alpha_1, \rho^2]$ when $g_{k,m}^{\delta}(\alpha_1) \leq 3\delta$. Otherwise, it has a solution $\alpha \in [\alpha_0, \alpha_1]$. $\square$

4. Convergence rate analysis. In this section, we establish the optimal convergence rate for the approximate solution stabilized by Tikhonov regularization method and obtained by the multi-level augmentation method with the *a posteriori* parameter choice strategy given in the last section.

As preparation, we show that there is a positive constant c independent of α and δ such that $c\delta^{2/(1+2\nu)} \leq \alpha$, where $\alpha \in [\alpha_0, \rho^2]$ is a solution of (3.3). For convenience, we denote $\alpha_2 := \delta^{2/(1+2\nu)}$.

Lemma 4.1. *Assume that hypotheses* (H1)–(H3) *hold, and N is chosen such that* (3.12), (3.13) *and* (3.14) *hold. Let $\alpha \in [\alpha_0, \rho^2]$ be a solution of* (3.3). *Then there exists a positive constant c independent of α and δ such that*

$$c\alpha_2 \leq \alpha.$$

Proof. Since (3.13) implies that α satisfies (2.17), it follows from Lemma 3.4 that

$$\begin{aligned}(4.1)\qquad \left\|\phi^{\delta}_{k,m}(\alpha)\right\| &\le \|\psi(\alpha)\| + \left\|\phi^{\delta}_{k,m}(\alpha) - \psi(\alpha)\right\| \\ &\le \|\omega\|\,\alpha^{1+\nu} + 2\sqrt{\alpha}\delta + (10a_{1+\nu} + 5a_1 a_\nu)\|\omega\|\varepsilon^{1+\nu}_{k+m}.\end{aligned}$$

Noting that ε_{k+m} satisfies (3.14) and α is a solution of (3.3), we conclude from (4.1) that

$$3\sqrt{\alpha}\delta = \left\|\phi^{\delta}_{k,m}(\alpha)\right\| \le \|\omega\|\,\alpha^{1+\nu} + 2\sqrt{\alpha}\delta + \frac{1}{2}\sqrt{\alpha_0}\delta.$$

This leads to

$$\frac{\sqrt{\alpha}}{2}\delta \le \|\omega\|\,\alpha^{1+\nu},$$

which yields the result of this lemma with $c := [1/(2\|w\|)]^{2/(1+2\nu)}$. □

Let α_* be chosen according to Algorithm 3.1. In the case (C1) and the case (C2) with $\alpha_* \le \alpha_2$, it is easy to obtain the optimal convergence rate directly, see Theorem 4.5. To deal with case (C2) with $\alpha_2 \le \alpha_*$, we need the following lemmas. To describe them, we define $F_\beta(\alpha) := \left\|x^\delta(\alpha) - x^\delta(\beta)\right\|$.

Lemma 4.2. *For any given $\beta > 0$, $F_\beta(\cdot)$ is differentiable on $(\beta, +\infty)$.*

Proof. A simple computation leads, for any $\alpha + h, \alpha \in (\beta, +\infty)$, to

$$F_\beta(\alpha+h) - F_\beta(\alpha) = \frac{\left\langle x^\delta(\alpha+h) - x^\delta(\alpha), x^\delta(\alpha+h) + x^\delta(\alpha) - 2x^\delta(\beta)\right\rangle}{\|x^\delta(\alpha+h) - x^\delta(\beta)\| + \|x^\delta(\alpha) - x^\delta(\beta)\|}.$$

Note that

$$x^\delta(\alpha') - x^\delta(\alpha'') = (\alpha'' - \alpha')(\alpha'\mathcal{I} + \mathcal{A})^{-1}(\alpha''\mathcal{I} + \mathcal{A})^{-1}\mathcal{K}^* y^\delta$$

and

$$\begin{aligned}\lim_{h\to 0}\left\|x^\delta(\alpha+h) - x^\delta(\beta)\right\| &= \left\|x^\delta(\alpha) - x^\delta(\beta)\right\| \\ &= (\beta - \alpha)\|(\alpha\mathcal{I} + \mathcal{A})^{-1}(\beta\mathcal{I} + \mathcal{A})^{-1}\mathcal{K}^* y^\delta\| \ne 0,\end{aligned}$$

we obtain

$$\lim_{h\to 0}\frac{F_\beta(\alpha+h)-F_\beta(\alpha)}{h}=-\frac{\left\langle(\alpha\mathcal{I}+\mathcal{A})^{-2}\mathcal{K}^*f^\delta, x^\delta(\alpha)-x^\delta(\beta)\right\rangle}{\|x^\delta(\alpha)-x^\delta(\beta)\|}, \tag{4.2}$$

which means that $F(\alpha)$ is differentiable on $(\beta,+\infty)$. □

Next we estimate the value of $F'(\alpha)$ on $(\alpha_2,\alpha_*]$.

Lemma 4.3. *Assume that hypotheses* (H1)–(H3) *hold, and N is chosen such that* (3.13) *and* (3.14) *hold. Let α_* be chosen according to* (C2) *of Algorithm* 3.1. *Then for any $\alpha,\beta\in(\alpha_2,\alpha_*]$ with $\beta<\alpha$,*

$$\alpha^2\left|F'_\beta(\alpha)\right|\le 6\sqrt{\alpha}\delta. \tag{4.3}$$

Proof. It follows from (4.2) and the definition of $\psi^\delta(\alpha)$ that

$$|F'_\beta(\alpha)|\le\left\|(\alpha\mathcal{I}+\mathcal{A})^{-2}\mathcal{K}^*y^\delta\right\|=\alpha^{-2}\|\psi^\delta(\alpha)\|.$$

Thus, by using Lemma 3.4 we have

$$\begin{aligned}\alpha^2|F'_\beta(\alpha)| &\le \left\|\phi^\delta_{k,m}(\alpha)-\psi(\alpha)\right\|+\left\|\psi^\delta(\alpha)-\psi(\alpha)\right\|+\left\|\phi^\delta_{k,m}(\alpha)\right\|\\ &\le 2\sqrt{\alpha}\delta+(10a_{1+\nu}+5a_1a_\nu)\|\omega\|\varepsilon^{1+\nu}_{k+m}+\frac{\sqrt{\alpha}\delta}{2}\\ &\quad+\left\|\phi^\delta_{k,m}(\alpha)\right\|.\end{aligned} \tag{4.4}$$

Since α_* is chosen according to (C2) and $\alpha\le\alpha_*$,

$$\left\|\phi^\delta_{k,m}(\alpha)\right\|=\|\alpha^2 v^\delta_{k,m}(\alpha)\|\le 3\sqrt{\alpha}\delta. \tag{4.5}$$

Combining (3.14), (4.4) and (4.5) yields the result of this lemma. □

Lemma 4.4. *Suppose that the assumptions of Lemma* 4.3 *hold. Then*

$$\|x^\delta(\alpha_2)-x^\delta(\alpha_*)\|=\mathcal{O}(\delta^{(2\nu)/(2\nu+1)}),\quad \textit{as}\ \ \delta\to 0. \tag{4.6}$$

Proof. Let $p > 1$ and $\beta_i = p^{i-1}\alpha_2$, $i \in \mathbf{N}$. Assume that $\beta_N \le \alpha_* \le \beta_{N+1}$. Then there exist $\xi_i \in (\beta_i, \beta_{i+1})$, $i = 1, 2, \ldots, N$, such that

$$\begin{aligned}
&\|x^\delta(\alpha_2) - x^\delta(\alpha_*)\| \\
&\qquad \le \sum_{i=1}^{N-1} \|x^\delta(\beta_i) - x^\delta(\beta_{i+1})\| + \|x^\delta(\beta^N) - x^\delta(\alpha_*)\| \\
&\qquad = \sum_{i=1}^{N-1} [F_{\beta_i}(\beta_{i+1}) - F_{\beta_i}(\beta_i)] + F_{\beta_N}(\alpha_*) - F_{\beta_N}(\beta_N) \\
&\qquad \le \sum_{i=1}^{N-1} (\beta_{i+1} - \beta_i)|F'_{\beta_i}(\xi_i)| + (\alpha_* - \beta_N)|F'_{\beta_N}(\xi_N)|.
\end{aligned}$$

From Lemma 4.3, we have that, for any $i = 1, 2, \ldots, N$,

$$|F'_{\beta_i}(\xi_i)| \le \frac{6\delta}{\xi_i\sqrt{\xi_i}} \le \frac{6\delta}{p^{i-1}\alpha_2\sqrt{p^{i-1}\alpha_2}}.$$

Therefore,

$$\|x^\delta(\alpha_2) - x^\delta(\alpha_*)\| \le (p-1)\sum_{i=1}^{N} \frac{6\delta}{\sqrt{p^{i-1}\alpha_2}} \le c\frac{\delta}{\sqrt{\alpha_2}}, \tag{4.7}$$

with $c := 6\sqrt{p}(\sqrt{p}+1)$. $\square$

We finally estimate the error of the multi-level augmentation solution $x^\delta_{k,m}(\alpha_*)$, which shows that the *a posteriori* parameter strategy embodied in Algorithm 3.1 can lead to an optimal convergence rate.

Theorem 4.5. *Assume that hypotheses* (H1)–(H3) *hold, and N is chosen such that* (3.12), (3.13) *and* (3.14) *hold. Let α_* be chosen according to Algorithm* 3.1. *Then for $k \ge N$ and $m \in \mathbf{N}_0$,*

$$\|x_* - x^\delta_{k,m}(\alpha_*)\| = \mathcal{O}(\delta^{(2\nu)/(2\nu+1)}), \quad \delta \to 0.$$

Proof. In the case (C1), from Theorem 3.6 we know that there is a solution α of (3.3) in $[\alpha_0, \alpha_1]$. According to Lemma 4.1, we have

$c\alpha_2 \le \alpha \le \alpha_* = \alpha_1 \le \alpha_2$. In the case (C2) with $\alpha_* \le \alpha_2$, Lemma 4.1 also leads to $c\alpha_2 \le \alpha_* \le \alpha_2$. Noting that $\alpha_2 = \delta^{2/(1+2\nu)}$, in either case we have

$$\alpha_*^\nu \le \delta^{(2\nu)/(1+2\nu)}, \quad \frac{\delta}{\sqrt{\alpha_*}} \le \frac{\delta}{\sqrt{c\alpha_2}} = \frac{1}{\sqrt{c}}\delta^{(2\nu)/(1+2\nu)}, \tag{4.8}$$

and

$$\begin{aligned}\frac{\varepsilon_{k+m}^{1+\nu}}{\alpha_*} &\le \frac{\delta\sqrt{\alpha_0}}{2c(10a_{1+\nu}+5a_1a_\nu)\|\omega\|\alpha_2} \\ &\le \frac{1}{2c(10a_{1+\nu}+5a_1a_\nu)\|\omega\|}\delta^{(2\nu)/(1+2\nu)},\end{aligned}\tag{4.9}$$

in the last inequality we used (3.14). It follows from (2.8) and Theorem 2.3 that

$$\begin{aligned}\|x_* - x_{k,m}^\delta(\alpha_*)\| &\le \|x(\alpha_*) - x_*\| + \|x_{k,m}^\delta(\alpha_*) - x(\alpha_*)\| \\ &\le c_\nu\|w\|\alpha_*^\nu + \frac{\delta}{\sqrt{\alpha_*}} + \frac{(4a_{1+\nu}+2a_1a_\nu)\|\omega\|}{\alpha_*}\varepsilon_{k+m}^{1+\nu},\end{aligned}$$

which with (4.8) and (4.9) yields the estimate of this theorem. In the case (C2) with $\alpha_2 \le \alpha_*$, we consider

$$\begin{aligned}\|x_* - x_{k,m}^\delta(\alpha_*)\| &\le \|x_{k,m}^\delta(\alpha_*) - x^\delta(\alpha_*)\| \\ &\quad + \|x^\delta(\alpha_*) - x^\delta(\alpha_2)\| + \|x^\delta(\alpha_2) - x_*\|.\end{aligned}\tag{4.10}$$

Using (2.9), Theorem 2.3 and the fact that $\alpha_2 \le \alpha_*$, we have that

$$\|x_{k,m}^\delta(\alpha_*) - x^\delta(\alpha_*)\| = \mathcal{O}(\delta^{(2\nu)/(2\nu+1)}), \quad \delta \to 0. \tag{4.11}$$

By Lemma 4.4, the following holds

$$\|x^\delta(\alpha_2) - x^\delta(\alpha_*)\| = \mathcal{O}(\delta^{(2\nu)/(2\nu+1)}), \quad \delta \to 0. \tag{4.12}$$

From (2.8) and (2.9) we obtain

$$\|x^\delta(\alpha_2) - x_*\| = \mathcal{O}(\delta^{(2\nu)/(2\nu+1)}), \quad \delta \to 0. \tag{4.13}$$

Combining (4.10)–(4.13) proves the estimate of this theorem. ◻

5. Numerical examples. In this section, we present some numerical examples to illustrate the effectiveness of the algorithm described in preceding sections. In our two examples, we compare our parameter choice strategy with a priori parameter choice strategy to show the efficiency of our algorithm.

To this end, we consider the problem of solving the first kind integral equation

$$(\mathcal{K}x)(t) = y(t), \quad t \in [0,1], \tag{5.1}$$

where $\mathcal{K} : L^2(0,1) \to L^2(0,1)$ is a linear compact operator defined by

$$(\mathcal{K}x)(t) := \int_0^1 k(t,s)x(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{5.2}$$

with a smooth kernel $k(t,s)$ defined on $[0,1] \times [0,1]$.

Let $\mathbf{X}_n$ be the finite space of piecewise linear polynomials on $[0,1]$ with knots at $j/2^n, j = 1,2,\dots,2^n-1$. We decompose the $\mathbf{X}_n$ into the direct sum of subspaces

$$\mathbf{X}_n = \mathbf{X}_0 \oplus^{\perp} \mathbf{W}_1 \oplus^{\perp} \cdots \oplus^{\perp} \mathbf{W}_n,$$

where $\mathbf{X}_0$ is the linear polynomial space on $[0,1]$, and for $n \in \mathbf{N}$, $\mathbf{W}_n$ is the orthogonal complement of $\mathbf{X}_{n-1}$ in $\mathbf{X}_n$. We choose a basis for $\mathbf{X}_0$

$$\omega_{00}(t) = 1 \text{ and } \omega_{01}(t) = 2\sqrt{3}\left(t - \frac{1}{2}\right), \quad t \in [0,1],$$

and a basis for $\mathbf{W}_1$

$$\omega_{10}(t) = 6\left(t - \frac{1}{2}\right) - 2\,\mathrm{sgn}\left(t - \frac{1}{2}\right), \quad t \in [0,1],$$

and

$$\omega_{11}(t) = 4\sqrt{3}\left|t - \frac{1}{2}\right|, \quad t \in [0,1].$$

Then the subspace $\mathbf{W}_i = \mathrm{span}\,\{\omega_{ij} : j = 1,2,\dots,2^i\}$ with the basis recursively generated by

$$\omega_{ij}(t) = \sqrt{2}\omega_{i-1,j}(2t), \quad j = 1,2,\dots,2^{i-1},$$

and

$$\omega_{i,2^{i-1}+j}(t) = \sqrt{2}\omega_{i-1,j}(2t-1), \quad j = 1, 2, \dots, 2^{i-1}.$$

In the following examples, we denote the errors $e^{\delta}_{k,m}(\alpha_*) := \|x_* - x^{\delta}_{k,m}(\alpha_*)\|_{L^2}$, and $e^{\delta}_{k+m}(\alpha_2) := \|x_* - x^{\delta}_{k+m}(\alpha_2)\|_{L^2}$, where $\delta := \|f - f^{\delta}\|_{L^2}$, $\alpha_2 = \delta^{2/(2\nu+1)}$ is the *a priori* parameter, α_* is the regularization parameters chosen by Algorithm 3.1.

Example 1. Consider the problem (5.1)–(5.2) with

$$k(t,s) = \sin(s+t),$$

and

$$y(t) = \frac{1}{8}\sin\left(\frac{2t+9}{2}\right) + \frac{5}{8}\sin\left(\frac{2t+1}{2}\right) + \frac{1}{2}\cos\left(\frac{2t-1}{2}\right) - \frac{1}{4}\cos\left(\frac{2t+3}{2}\right) - \frac{1}{4}\cos\left(\frac{2t-5}{2}\right) - \frac{1}{4}\sin\left(\frac{2t+5}{2}\right).$$

In the case

$$x_*(t) = \mathcal{K}^{\dagger}f = \cos\left(\frac{2t-1}{2}\right) - \frac{1}{2}\sin\left(\frac{2t+5}{2}\right) + \frac{1}{2}\sin\left(\frac{2t+1}{2}\right).$$

Moreover, $x_*(t) = \mathcal{K}^*\mathcal{K}\omega$, $\omega = 1$, which means $\nu = 1$. In the numerical implementation using the multi-level augmentation scheme, we choose $k = 5$ and $m = 5$. Numerical results listed in Table 1 confirm that the solutions from our methods get the optimal convergence rate $\delta^{2/3}$.

δ	$\delta^{2/3}$	α_*	α_2	$e^{\delta}_{k,m}(\alpha_*)$	$e^{\delta}_{k+m}(\alpha_2)$
5.637×10^{-3}	3.167×10^{-2}	3.220×10^{-2}	3.167×10^{-2}	5.893×10^{-2}	5.878×10^{-2}
4.549×10^{-4}	5.914×10^{-3}	5.681×10^{-3}	5.914×10^{-3}	1.206×10^{-2}	1.143×10^{-2}
1.824×10^{-4}	3.216×10^{-3}	2.692×10^{-3}	3.216×10^{-3}	7.354×10^{-3}	5.438×10^{-3}
6.065×10^{-5}	1.544×10^{-3}	1.373×10^{-3}	1.544×10^{-3}	3.263×10^{-3}	2.728×10^{-3}
8.455×10^{-5}	1.926×10^{-3}	1.176×10^{-3}	1.926×10^{-3}	4.025×10^{-3}	3.866×10^{-3}

TABLE 1. Numerical Results of Example 1.

δ	$\delta^{1/2}$	α_*	α_2	$e^\delta_{k,m}(\alpha_*)$	$e^\delta_{k+m}(\alpha_2)$
3.040×10^{-2}	1.744×10^{-1}	2.725×10^{-1}	3.040×10^{-2}	1.237×10^{-1}	8.031×10^{-2}
2.213×10^{-3}	4.660×10^{-2}	3.290×10^{-2}	2.213×10^{-3}	2.560×10^{-2}	2.141×10^{-2}
2.390×10^{-4}	1.549×10^{-2}	9.099×10^{-4}	2.390×10^{-4}	1.693×10^{-2}	2.869×10^{-3}
1.232×10^{-5}	3.511×10^{-3}	7.530×10^{-4}	1.232×10^{-5}	1.499×10^{-3}	4.7162×10^{-4}
3.029×10^{-6}	1.544×10^{-3}	1.373×10^{-3}	3.029×10^{-6}	7.571×10^{-4}	7.3140×10^{-5}

TABLE 2. Numerical Results of Example 2.

Example 2. Consider the problem (5.1)–(5.2) with

$$k(t,s) = (1+ts)e^{ts},$$

and

$$y(t) = \frac{e^{t+1}(t^2+t+1)-1}{(t+1)^2}.$$

The unique solution of this problem is $x_* = e^t$. At this time

$$\int_0^1 (1+ts)e^{ts}\,\mathrm{d}s = e^t,$$

and the operator $\mathcal{K} : L^2[0,1] \to L^2[0,1]$ is obviously self-adjoint. Thus we can conclude that $x_* \in R((\mathcal{K}^*\mathcal{K})^{1/2})$, $\nu = 1/2$. Table 2 shows the experiment results. In this case, the optimal convergence rate is $\delta^{1/2}$.

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Department of Scientific Computing and Computer Applications, Sun Yat-sen University, Guangzhou 510275, China
Email address: lnsczy@mail.sysu.edu.cn

Department of Scientific Computing and Computer Applications, Sun Yat-sen University, Guangzhou 510275, China
Email address: yjiang02@syr.edu

Centre for Signal Processing, School of Electrical & Electronic Engineering, Nanyang Technological University, Singapore 639798
Email address: lhsong@ntu.edu.sg

Department of Scientific Computing and Computer Applications, Sun Yat-sen University, Guangzhou 510275, China
Email address: mcsyhq@mail.sysu.edu.cn