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PONTRYAGIN PRINCIPLE IN ABSTRACT SPACES

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ABSTRACT. Pontryagin's theory for an optimal control problem with dynamics described by an ODE has possible extensions to other systems, such as some PDEs. If the system with state x and control u is described abstractly by $Dx = M(x,u)$, then some other linear mappings D than differentiation can lead to a Pontryagin principle. Costate boundary conditions are obtained by calculating an adjoint mapping. If the domain is not compact, but the problem reaches a strict minimum, then under some continuity restrictions the control problem can be approximated closely by one for which Pontryagin's principle holds.

1. Introduction.

The optimal control problem:

$$\begin{gathered}\mathrm{MIN}_{x,u}\ F(x,u) := \int_0^T f(x(t),u(t),t)dt \text{ subject to} \\ x(0) = x_0,\ \dot{x}(t) = m(x(t),u(t),t)\ (0 \le t \le T), \\ u(t) \in \Gamma(t)\ (0 \le t \le T)\ \Leftrightarrow (\forall t) g(x(t),t) \le 0\end{gathered}$$

may be written as:

$$\mathrm{MIN}_{x,u}\ F(x,u) \text{ subject to } Dx = M(x,u),$$

where $Dx = w \Leftrightarrow x = x_0 + \int_0^t w(s)ds,\ M(x,u)(t) := m(x(t),u(t),t)$, and D is made continuous by giving a suitable graph norm to the space X of states. This formulation suggests a generalization in which the domain $[0,T]$ is replaced by a closed subset $E \subset \mathbf{R}^r$ $(r \ge 1)$,

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not necessarily compact, and a linear mapping D, made continuous by suitable choice of norms. The state x and the control u will be assumed to lie in specified normed spaces. This excludes some anomalous instances where the state is unbounded.

The standard necessary conditions for the stated optimal control problem (see Fleming & Rishel, 1975) assume $E = [0, T]$; u is piecewise continuous and x is piecewise smooth; f, m, and g are C^1 functions; a global minimum is reached at $(x, u) = (\bar{x}, \bar{u})$; then the Hamiltonian:

$$h(x(t), u(t), t, \lambda(t)) := f(x(t), u(t), t) + \lambda(t) m(x(t), u(t), t).$$

satisfies

$$h(\bar{x}(t), \cdot, t, \bar{\lambda}(t)) \to \text{MIN, subject to } g(u(t)) \leq 0, \quad \text{for almost all } t \in E,$$

where the *costate* $\bar{\lambda}(t)$ satisfies the adjoint differential equation:

$$-\dot{\bar{\lambda}}(t) = h_x(\bar{x}(t), \bar{u}(t), t, \bar{\lambda}(t)), \; \bar{\lambda}(T) = 0,$$

where subscript $_x$ denotes partial derivative. Two approaches will be followed. The first is based on the Karush-Kuhn-Tucker necessary conditions for a constrained optimum. With functions restricted to suitable normed spaces, boundary conditions follow from calculating an adjoint mapping, and some anomalous cases do not occur.The second approach uses the stability of a strict local minimum to small perturbations, including here perturbations of the domain of the functions. Several detailed proofs are given as lemmas in Section 6.

Some related results are given in Craven (1995), and Craven and Islam (1995), where a domain $[0, \infty)$ is mapped to a compact domain $[0, T]$, to which existing theory can be applied, assuming that $x(t)$ and $u(t)$ tend sufficiently fast (e.g. exponentially) to finite limits as $t \to \infty$. Craven (2000) gave an approach to Pontryagin for PDEs.

2. Approach via KKT conditions Assume for the abstract control problem (ACP) on a domain $E \subset \mathbf{R}^r$:

$$\text{MIN}_{x,u} \; F(x, u) \text{ subject to } Dx = M(x, u), \; B_1 x = 0 \; (x \in E_1), \; u \in \Gamma,$$

where $F(x,u) := \int_E f(x(z),u(z),z)dz$, $M(x,u)(z) := m(x(z),u(z),z)$, and $E_1 \subset \partial E$, that:

(a) the control u has finite norm $\|u\|_\infty$ on E, and the state x has finite *graph norm* $\|x\| := \|x\|_\infty + \|Dx\|_\infty$. (Thus, if $E \subset \mathbf{R}$, then u may be piecewise continuous, and x may be piecewise smooth. Sobolev norms may also be considered, with $\|\cdot\|_2$ replacing $\|\cdot\|_\infty$, but that is not pursued here.)

(b) F and M are differentiable, with partial derivatives F_x, etc. (See Lemma 1 in Section 4)

(c) D is a continuous linear mapping.

(d) a minimum is reached at $(x,u) = (\bar{x},\bar{u})$;

(e) the linearized mapping $M_x(\bar{x},\bar{u})$ is surjective. (A required constraint qualification)

Then necessary KKT conditions hold (see Craven 1995):

$$F_x(\bar{x},\bar{u}) + \Lambda(-D + M_u(\bar{x},\bar{u})) = 0,$$
$$[F_u(\bar{x},\bar{u}) + \Lambda M_u(\bar{x},\bar{u})](\Gamma - \bar{u}) \geq 0,$$

for some continuous linear mapping Λ, called a *Lagrange multiplier.* Define $H(x,u,\Lambda) := F(x,u) + \Lambda M(x,u)$. Λ may have a *density* $\lambda(\cdot)$, defined by $\langle \Lambda, w \rangle = \int_E \lambda(z)w(z)dz$.

Note that (c) is ensured by the choice of graph norm in (a). The choice of norms excludes some solutions of the differential equation where x is unbounded, or where Dx does not approach $D\bar{x}$ when x approaches $\bar{x}$. (There is a question of whether such excluded cases are *well-posed* problems in economics or engineering applications.)

The mapping D has an adjoint D^*, defined by; $< \Xi, Dx > = < D^*\Xi, x >$. If Λ has a *density* λ, then H has a density, called the *Hamiltonian*:

$$h(x(z),u(z),z,\lambda(z)) = f(x(z),u(z),z) + \lambda(z)m(x(z),u(z),z);$$

thus $H(.) = \int_E h(.)dz$. The first part of KKT, with:

$$< \Lambda, Dx > = \int_E \lambda(z)(Dx)(z)dz = \int_E (D^*\lambda)(z)x(z)dz,$$

gives the *adjoint equation* (ADE):

$$D^*\lambda(z) = (f + \lambda m)_x(\bar{x}, \bar{u}),$$

subject to boundary conditions on λ, obtained by some analog of integration by parts (see Section 3). If $\lambda(\cdot)$ exists then, given the assumptions (a) - (e), the boundary conditions on $\lambda(\cdot)$ depend on the construction of the adjoint D^*, and on the boundary conditions for the state $x(\cdot)$, but not on other details of the control problem (see Section 3). The existence of the density λ for a compact domain E is shown by solving the adjoint equation. But a density is not always available for a noncompact E. However, if the adjoint equation has a solution $\lambda(\cdot)$ satisfying the boundary condition obtained from the adjoint, and if $\int_E |\lambda(z)|dz$ is finite, tben this λ defines a continuous linear functional on bounded continuous functions, and the following deduction of Pontryagin's principle will apply.

The second part of KKT gives necessary conditions for a minimum of $K(.) := H(\bar{x}, ., \Lambda)$ over Γ, and these become also sufficient for this minimum if also $F(\bar{x}, \cdot$ is convex, $M(\bar{x}, \cdot)$ is linear, and Γ is convex. Suppose (if possible) that $h(\bar{x}(z), \cdot, z, \lambda(z))$ is not minimized at $\bar{u}(z)$ on a set of z of positive measure. Then a lemma of Pontryagin (see Lemma 2, based on the presentation in Craven, 1995, Theorem 7.2.6), shows that under the further assumption:

(f) $u \in \Gamma \Leftrightarrow G(u) \leq 0 \Leftrightarrow (\forall z \in E) g(u)(z) \leq 0$,

the minimum of $K(.)$ is contradicted. The following theorem has thus been proved:

Theorem 1. *For the control problem (ACP) with compact domain E, , assume hypotheses (a) - (f), and also that $F(\bar{x}, \cdot)$ is convex and $M(\bar{x}, \cdot)$ is linear. Then the adjoint differential equation (ADE) holds, with boundary conditions determined by the adjoint mapping D^*, and also Pontrygin's principle:*

$$h(\bar{x}(z), ., z, \lambda(z)) \to MIN \text{ on } \Gamma(z) := \{v : g(v, z) \leq 0\} \text{ for almost all } z.$$

This result extends to a non-compact domain E if $\lambda(\cdot$ exists, satisfying the boundary conditions. But a discount factor is required in the objective (see Lemma 1) when E is non-compact. Note that the hypotheses (a) and (b) may seriously restrict the functions f and m; some solutions x of the differential equation are excluded by the choice of norms. The result also applies to dimension > 1, for example when $M(x,u) = A(x) + B(x)u$, and D is a linear partial differential operator such as the Laplacian, over a compact domain E. The differentiability requirement (a) usually requires that f includes a discount factor such as $e^{-\rho z}$ when $E = [0, \infty)$, or $\psi(z) \geq 0$ when $E \subset \mathbf{R}^2$, with $\int_E \psi(z) dz < \infty$.

The restriction to linear K and G may be weakened to invex, that is:

$$K(u) - K(\bar{u}) \geq K_u(\bar{u})\eta(u, \bar{u}); \quad G(u) - G(\bar{u}) \geq G_u(\bar{u})\eta(u, \bar{u})$$

for some scale function $\eta(\cdot, \cdot)$, or in particular to convex, since this implies the minimum of $K(\cdot)$. No such restriction is needed in one dimension (see e,g, Craven, 1995, and the theorem cited in the Introduction); but a required property, that $Dx = M(x,u)$ defines a solution with $\|x - \bar{x}\|_\infty \leq \text{const } \|u - \bar{u}\|_1$, seems not to hold for partial differential operators. An appropriate generalization for $r > 1$ is given in Section 6.

3. Boundary Conditions If $E = [0, T] \subset \mathbf{R}$, then integration by parts gives the boundary condition $\lambda(T)\bar{x}(T) = 0$. If instead $E = [0, \infty)$, then instead $\lambda(t)\bar{x}(t) \to 0$ as $t \to \infty$, hence in particular $\lambda(t) \to 0$ as $t \to \infty$ if $\bar{x}(t)$ is bounded away from 0 as $t \to \infty$.

If $E = [0.T]$ and $D = \frac{d^2}{dt^2}$, then integration by parts gives $D^* = \frac{d^2}{dt^2}$ with an integrated part $[\dot{\lambda}x - \lambda\dot{x}]_0^T$, which must vanish. If $x(0)$ and $\dot{x}(0)$ are specified by boundary conditions, then the boundary conditions $\lambda(T) = 0$ and $\dot{\lambda}(T) = 0$ are required.

If, for example, $E \subset \mathbf{R}^3$ is compact, and $D = \nabla^2$, the Laplacian, then the adjoint and boundary conditions may be constructed from Green's theorem:

$$\int_E (\lambda \nabla^2 x - x \nabla^2 \lambda) dv = \int_{\partial E} (\lambda \frac{\partial x}{\partial n} - x \frac{\partial \lambda}{\partial n}) ds,$$

in which dv and ds denote elements of volume and surface. Then $D^* = \nabla^2$, with boundary conditions $\lambda(z) = 0$ on that part of ∂E

where $\frac{\partial x}{\partial n}$ is not specified, and $\frac{\partial \lambda}{\partial n} = 0$ on that part of ∂E where x is not specified. If the domain E is unbounded, as for example:

$$E := \{(z_1, z_2) : 0 \leq z_1, 0 \leq z_2 \leq 1\} \subset \mathbf{R}^2,$$

and if Theorem 1 applies, then limiting boundary conditions hold. For the example::

$$\lambda(z_1, z_2)\frac{\partial \bar{x}}{\partial n}(z_1, z_2) \to 0 \text{ and } \bar{x}(z_1, z_2)\frac{\partial \lambda}{\partial n}(z_1, z_2) \to 0 \text{ as } z_1 \to \infty.$$

Suppose now that $\Gamma(z)$ is an interval in $\mathbf{R}$, or a polygonal (or polyhedral) region in higher dimension, the same for each z. Pontryagin's principle, optimizing a linear function of u, gives the optimal $\bar{u}(z)$ on the boundary of Γ. This leads to *generalized bang-bang control,* where E is partitioned into subsets E_i, and $\bar{u}(E_i)$ at a vertex of $\Gamma(z)$, though with faces or edges of $\Gamma(z)$, also possible (corresponding to *singular arcs* when $E \subset \mathbf{R}$.)

For optimal control on $[0, \infty)$, there are various examples in the economics literature of problems where the boundary condition $\lambda(t)\bar{x}(t) \to \infty$ does not hold. One such, from Aseev and Kryazhiminskiy (2006), is the following:

$$\text{MAX } \int_0^\infty e^{-t}[1 + \gamma(x(t))u(t)dt \text{ subject to } u(t) \in [\frac{1}{2}, 1],$$
$$x(0) = 1,\ \dot{x}(t) = u(t).$$

The optimum is $u(t) = 1, x(t) = t$ $(x \geq 0)$. This does not contradict the present result, since this $x(\cdot)$ is unbounded. A change of variable from $x(t)$ to a new state variable $\theta(t) := (1 + t)^{-1}x(t)$ gives a new adjoint equation:

$$\dot{\lambda}(t) = (1 + t)^{-1}\lambda(t) + e^{-t}\gamma'(t).$$

If $\lambda(\infty) = 0$ is assumed, then:

$$\lambda(t) = -(1 + t)^{-1}\int_t^\infty (1 + s)^{-1}e^{-s}\gamma'(s)ds.$$

This integral will be finite at $t = 0$ provided that $\gamma(\cdot)$ does not increase too rapidly, and then the Pontryagin condition will hold. But this does not happen for all $\gamma(\cdot)$, in agreement with the cited reference. If $\gamma(\cdot)$ increases too fast, then the differentiability condition (b) may not hold (see Section 6.)

Various other examples have been given, e.g. Halkin 1974, where the boundary condition for $\lambda(\cdot)$ does not hold. This and other examples do not include a discount factor in the objective. In such cases, F would not be differentiable, and Theorem 1 is not contradicted.

While the Pontryagin necessary conditions are not always sufficient for a minimum, they may characterize a point as a *quasimin* (see Craven, 1995.) This assumes that all the boundary conditions are included.

4. Truncated domain Some unbounded (non-compact) domains E are also of interest. When can a control problem on an unbounded domain be approximated closely by a problem on a compact domain? Consider in particular the approximation of $[0,\infty)$ by $[0,T]$ (for large T), and the approximation of

$$E = [0,\infty) \subset \mathbf{R};\ E_q = [0,T] \text{ with } q = 1/T;$$
$$f \text{ and } m \text{ multiplied by the indicator } \chi_{E_q}(.);$$
$$E = \{(t_1,t_2) : 0 \le t_1, 0 \le t_2 \le 1\} \subset \mathbf{R}^2;$$
$$E_q = \{(t_1,t_2) : 0 \le t_1 \le 1/q, 0 \le t_2 \le 1\}$$
$$\text{with } q = 1/T; f \text{ and } m \text{ multiplied by the indicator } \chi_{E_q}(.).$$

In both unbounded cases, part of the boundary ∂E has receded 'to infinity', and boundary conditions are required also for the 'recession directions'. For the two-dimensional example, boundary conditions are required for $\lim_{t_1\to\infty}\lambda(t_1,t_2)$.

Later theorems require the following lemmas. They are proved in Section 6.

Lemma 1: Differentiability *For $E = [0,\infty)$, (or $[0,T]$,) assume that :*

$$|m(\bar{x}(t)+v(t),\bar{u}(t),t) - m(\bar{x}(t),\bar{u}(t),t)$$
$$-m_x(\bar{x}(t),\bar{u}(t),t)v(t)| < \epsilon|v(t)|$$
$$\textit{when } |v(t)| < \delta(\epsilon),\ \textit{independent of } t,$$

(thus m is differentiable with respect to x, uniformly in $t \in [0, \infty)$), and also $f(x(t), u(t), t) = e^{-\rho t} \tilde{f}(x(t), u(t), t)$, with $\tilde{f}$ differentiable with respect to x, uniformly in $t \in [0, \infty)$. Then:

$$|M(\bar{x} + v, \bar{u}) - M(\bar{x}, \bar{u}) - M_x(\bar{x}, \bar{u})v| < \epsilon \|v\|_\infty \quad \text{when } \|v\|_\infty < \delta(\epsilon),$$

with a similar statement for F.

Lemma 2: Minimum of Hamiltonian *Define $k(w, z) := h(\bar{x}(z), w, z, \lambda(z))$. Assume that $u \in \Gamma \Leftrightarrow (\forall z) g(u(z), z) \leq 0$. Suppose that $k(., z)$ is not minimized over $\Gamma(z) := \{w : g(w, z) \leq 0\}$ at $\bar{u}(z)$ on some set $A^{\#} \subset E$ of positive measure. Then $H(\bar{x}, \cdot, \Lambda)$ cannot reach a quasimin at $\bar{u}$.*

Lemma 3: Quasimin of abstract Hamiltonian *Define $Q(x, u, \Lambda) := H(x, u, \Lambda) - \Lambda Dx$, Let $x = \Phi(u)$ and $\bar{x} = \Phi(\bar{u})$ with $u \in \Gamma$. Assume that:*

(k) $$\|x - \bar{x}\|_\infty \leq] \ \textit{const} \|u - \bar{u}\|_\delta$$

Assume the Lipschitz condition, for some $\delta \in (1, 2)$:

(1) $$\|H_x(\tilde{x}, u, \Lambda) - H_x(\bar{x}, \bar{u}, \Lambda)\| \leq A\|x - \bar{x}\|_\infty + B\|u - \bar{u}\|_\delta.$$

Then $H(\bar{x}, ., \Lambda)$ reaches a quasimin at $\bar{u}$, subject to $u \in \Gamma(z) := \{z \in E : g(u(z) \leq 0\}$ and $u - \bar{u} \in V := \|u - \bar{u}\|_\infty < b$, for some $b > 0$.

Lemma 4: Stability of adjoint equation *For the adjoint equation when the domain $E = [0, \infty)$ and $x(t) \in \mathbf{R}^r$:*

$$\dot{x}(t) = a(t) + \lambda(t) b(t); \quad a(t) := f_x(\bar{x}(t), \bar{u}(t), t), \ b(t) := m_x(\bar{x}(t), \bar{u}(t), t),$$

assume that

(g) *$a(t) = e^{-\rho t} \tilde{a}(t)$ with $\rho > 0$ and $\tilde{a}(\cdot)$ bounded;*

(h) $\int_t^{t+\tau} b(s)ds = \bar{b}\tau + k(t,\tau)\tau$ *where* $\bar{b}$ *is a constant matrix and* $\|k(t,\tau)\| < \zeta$ *for sufficiently large* t *and* τ;

(i) $-\rho + \mu + \zeta < 0$. *where* μ *is the largest real part of any eigenvalue of* $\bar{b}$.

Then the adjoint equation on $[0, T]$ *with terminal condition* $\lambda(T) = 0$ *has a solution* $\lambda^{[T]}(\cdot)$, *with* $\|\lambda^{[T]}(t)\| \leq \psi(t)$ *for all* T *and* t, *where* $\int_0^\infty \psi(t)dt$ *is finite.*

Perturbations. There is a general result for constrained minimization subject to a small perturbation. It assumes that the unperturbed problem reaches a *strict* local minimum. Consider the minimization of $\phi(x, q)$ with respect to x, subject to $\zeta(x, q) \in K$, where q is a small perturbation parameter, and K is a convex cone. Assume that the feasible set Γ_q of this perturbed problem P_q is nonempty.

A function $\phi(.)$ reaches a *strict minimum* over a set S at $\bar{x}$ if:

$$\phi(x) - \phi(\bar{x}) \geq \rho(r) > 0 \text{ whenever } x \in X \text{ and } \|x - \bar{x}\| = r > 0$$

for some function $\rho(\cdot)$ of positive r. sufficiently small.

Craven (1995, Section 7.4) showed that if $\phi(x, q)$ and $\zeta(x, q)$ are uniformly continuous in (x, q), when x is in a bounded neighbourhood of Γ_q and $|q| < \delta$; if the unperturbed problem P_0 reaches a strict minimum at $\bar{x}$; and $\phi(., q)$ reaches a minimum on $x \in \Gamma_q : \|x - \bar{x}\| < r$ for r sufficiently small when $0 \neq |q| < \delta$, then the perturbed problem P_q reaches a minimum at a point $\bar{x}(q)$, where $\bar{x}(q) \to \bar{x}$ as $q \to 0$.

This may be applied to a control problem CP on a noncompact domain E, with a truncated problem CP_q to a compact domain E_q as the perturbed problem. Two examples of truncated domains E_q are given above.

If F and M satisfy Lipschitz conditions, which hold if f and g are differentiable uniformly in t and f includes a discount factor (see Lemma 1), then they satisfy the uniform continuity conditions for the perturbation result. Then the perturbation result of Craven (1995) has the following consequence for the control problem:

Theorem 2 *Assume that CP reaches a strict minimum at* $(\bar{x}, \bar{u})$; $F(.,.)$ *and* $M(.,.)$ *satisfy Lipschitz conditions; and* CP_q *(with* $q \neq 0$ *and u in bounded neighbourhoods) reaches a minimum. Then* CP_q *is minimized at* $(\bar{x}_q, \bar{u}_q)$, *where* $(\bar{x}_q, \bar{u}_q) \to (\bar{x}, \bar{q})$ *as* $q \to 0$.

The boundary conditions make $\lambda_q(.) = 0$ on the (hyper)plane truncating E to E_q. If Pontryagin theory applies to the truncated problem CP_q, and if $\lim_{q \to 0} \lambda_q(.)$ produces an integrable function (which does not always happen), then Pontgryagin necessary conditions have been constructed for CP, including the boundary condition "at infinity".

For the case of $E = [0, \infty)$, sufficient conditions for an integrable $\lambda(\cdot)$ are obtained in Lemma 4. Consider $x(t) \in \mathbf{R}$, and $u(\cdot)$ bounded. In the equation for $\dot{x}(t)$, unless $m_x(\bar{x}(t), \bar{u}(t), t)$ is negative in some average sense (to be made precise) for large t, then the state $x(t)$ will typically (except for special initial conditions) grow exponentially, so is not bounded. If $x(t) \in \mathbf{R}^r$ with $r > 1$, then $m_x(\bar{x}(t), \bar{u}(t), t)$ is a matrix, and the negative requirement becomes an eigenvalue restriction. In either case, the requirement on m_x to ensure stability, when the differential equation is solved with t increasing from 0 is similar to that required for stability of the adjoint equation, when solved with t decreasing from ∞, and a boundary condition on $\lambda(\infty)$.

Theorem 3 *For* $E = [0, \infty)$ *and* $D = \frac{d}{dt}$, *assume the hypotheses (a) -(f) of Theorem 1, the hypotheses of Theorem 2, and also the hypotheses (g), (h), (i) of Lemma 4. Then there exists a costate* $\lambda(\cdot)$ *satisfying* $\lambda(t) \to 0$ *as* $t \to \infty$

Proof. Since $(x_q, u_q) \to (\bar{x}, \bar{u})$ from Theorem 2, and $\|\lambda_q(t)\| \leq \psi(t)$ with $\int_0^\infty \psi(t)dt < \infty$ from Lemma 4 (with $q = 1/T$), the integral expression (see Lemma 4) for $\lambda_q(t)$ shows that $\lambda_q(t)$ tends to a limit $\lambda(t)$ with $\int_0^\infty b\lambda(t)dt < \infty$. As remarked in Section 2, this λ is then a density for Λ. Then Theorem 1 gives Pontryagin's principle. □

5. Compact domain in higher dimension The proof of Pontryagin's principle in Craven (1995) assumed that the mapping Φ from a control u to the corresponding state x is Lipschitz from $\|\cdot\|_\infty$ to $\|\cdot\|_1$. While this holds when $E = [0, T]$, as a consequence of Gronwall's inequality, this proof does not extend to higher dimensions. Consider, in particular, the equation $Dx = u$ in $\mathbf{R}^2$ or $\mathbf{R}^3$, where D is the Laplacian. The Green's function for D is then unbounded, so the Lipschitz result does not follow for the $\|\cdot\|_1$ norm, Assume now that:

(j) : $$x(z) = \Phi[u](z) = \int_E k(z, u(s))dz,$$

where $|k(z, u(s)) - k(z, \bar{u}(s)| \leq \psi(s)$ with $\psi \in L^{\delta/(\delta-1)}$ and $\delta > 2$.
Then $|x(z)| \leq$ const $\|u - \bar{u}\|_\delta$, for some $\delta \in (1, 2)$.

Theorem 4 *Assume that $E \subset \mathbf{R}^r$ is compact, hypotheses (a) - (f) of Theorem 1 hold, and also (j), and (l) of Lemma 3. with $(\bar{x}, \bar{u})$ a global minimum. Then the Pontryagin principle holds for (ACP).*

Proof. Lemma 3 on the abstract Hamiltonian H shows, using hypotheses (j) (which implies (k)), that $H(\bar{x}, \cdot, \Lambda)$ has a quasimin at $\bar{u}$ (in the $\|\cdot\|_1$ norm), when restricted to the domain $\Gamma \cap V$, where V is a neighbourhood of $\bar{u}$ in the $\|\cdot\|_\infty$ norm. From this, Lemma 2 (in Section 6) proves Pontryagin's principle.

Thus, in particular Pontryagin's principle holds for some elliptic PDEs. (See Section 3 for some discussion of boundary conditions,) □

Remark The proof of Pontryagin's principle in Craven (1995) assumed that Φ was Lipschitz from $\|.\|_1$ to $\|.\|_\infty$. This holds for compact $E \subset \mathbf{R}$, using Gronwall's inequality (see Martin, 1976) when m_x and m_u are bounded. But it does not hold in higher dimensions; hypothesis (j) replaces it.

6. Proofs of Lemmas.

Proof of Lemma 1.

$$\begin{aligned} &|(M(\bar{x} + v, \bar{u}) - M(\bar{x}, \bar{u}) - M_x(\bar{x}, \bar{u})v)(t)| \\ &\leq |m(\bar{x}(t) + v(t), \bar{u})(t), t) - m(\bar{x}(t), \bar{u})(t), t) - m_x(\bar{x}(t), \bar{u}(t), t)v(t)| \\ &\leq \epsilon|v(t)| \leq \epsilon\|v\|_\infty \text{ if } \|v\|_\infty < \delta(\epsilon), \end{aligned}$$

Similarly

$$\begin{aligned} &|F(x,\bar{u}) - F(\bar{x},\bar{u}) - F_x(\bar{x},\bar{u})(x-\bar{x}) \\ \leq &\int_E |f(x(t),\bar{u}(t),t) - f(\bar{x}(t),\bar{u}(t),t) - f_x(\bar{x}(t),\bar{u}(t),t) \\ &\leq \int_E e^{\rho t}\epsilon dt = \epsilon/\rho \text{ when } (\forall t)|x(t)-\bar{x}(t)| < \delta(\epsilon). \end{aligned}$$

□

Proof of Lemma 2. The set $A^{\#}$ can be assumed to lie in some compact subset of E. Thus

$$(\forall z \in A^{\#})(\exists v(z) \in \Gamma(z))\ k(v(z),z) < k(\bar{u}(z),z).$$

Here $\|\bar{u}(\cdot)\|_\infty < \infty$, and $\|v(\cdot) - \bar{u}(\cdot)\| < \infty$ may be assumed. Define $\phi : E \to \mathbf{R}_+$ as zero when $k(v(z),z) = k(\bar{u}(z),z)$, and elsewhere by $k(\bar{u}(z),z) - k(v(z),z) = \phi(z)|v(z) - \bar{u}(z)|$. Obtain $A_0^{\#}$ from $A^{\#}$ be excluding the set of zero measure of those z not points of density, or where ϕ is not approximately continuous. Fix $z_0 \in A_0^{\#}$. Then there is a set $A \subset A_0^{\#}$ of positive measure, such that $\lim_{z\to z_0, z\in A}\phi(z) = \phi(z_0)$. Then $\phi(z) \geq \sigma := \frac{1}{2}\phi(z_0)$ for all $z \in A$ and $|z - z_0|$ sufficiently small.

Define a curve $\{u_\beta : \beta \geq 0\} \subset \Gamma$ by:

$$u_\beta(z) = v(z)$$
$$\text{when } z \in A_\beta := \{z \in A : |z - z_0| \leq \psi(\beta)\}, \text{ and } \bar{u}(z) \text{ otherwise,}$$

with $\psi(\cdot) := \omega^{-1}(\beta)$, where $\omega(\theta) := \int_{\{z\in A:|z-z_0|\leq\theta\}} |v(\cdot) - \bar{u}(\cdot)$; the integral is finite since A lies in a compact subset. Now $u_\beta \in \Gamma$ by the hypothesis on $\Gamma(z)$. Then $K(\cdot) := H(\bar{x},\cdot,\Lambda)$ satisfies:

$$\begin{aligned} -K(\bar{u}_\beta) + K(\bar{u}) &\geq \int_{A_\beta} \phi(z)|u_\beta(z) - \bar{u}(z)|dz \\ &\geq \sigma \int_{A_\beta} |u_\beta(z) - \bar{u}(z)|dz = \sigma\|u_\beta - \bar{u}\|_1 = \sigma\beta. \end{aligned}$$

Thus

$$K(u_\beta) - K(\bar{u}) \leq -\sigma \|u_\beta - \bar{u}\|_1$$

This contradicts a *quasimin*, which requires :

$$K(u_\beta) - K(\bar{u}) \geq \epsilon \|u_\beta - \bar{u}\|_1 \text{ for } \|u_\beta - \bar{u}\|_1 \text{ sufficiently small.}$$

□

Proof of Lemma 3.

$$\begin{aligned} H(\bar{x}, u, \Lambda) - H(\bar{x}, \bar{u}, \Lambda) &= F(x, u) - F(\bar{x}, \bar{u}) + Q(\bar{x}, u, \Lambda) - Q(x, u, \Lambda), \\ &= F(x, u) - F(\bar{x}, \bar{u}) + Q_x(\tilde{x}, u, \Lambda)(\bar{x} - x) \\ &\qquad \text{for some } \tilde{x} \in [x, \bar{x}] \\ &\geq 0 + [Q_x(\tilde{x}, u, \Lambda) - Q_x(\bar{x}, \bar{u}, \Lambda)](\bar{x} - x) \\ &\qquad \text{since } Q_x(\bar{x}, \bar{u}, \Lambda) = 0. \\ &= [H_x(\tilde{x}, u, \Lambda) - H_x(\bar{x}, \bar{u}, \Lambda)](\bar{x} - x). \end{aligned}$$

For $u - \bar{u} \in V$,

$$\|u - \bar{u}\|_\delta \leq (\int_E b^\delta |u(z) - \bar{u}(z)| dz)^{1/\delta} = (b|E|)^{1/\delta} \|u - \bar{u}\|_1^{1/\delta}.$$

From (k) and (l),

$$\|[Q_x(\tilde{x}, u, \Lambda) - Q_x(\bar{x}, \bar{u}, \Lambda)](\bar{x} - x)\| \leq \text{ const } \|u - \bar{u}\|_1^{2/\delta} < \epsilon \|u - \bar{u}\||_1$$

when $\|u - \bar{u}\|_1$ is small enough. Hence:

$$\Delta := H(\bar{x}, u, \Lambda) - H(\bar{x}, \bar{u}, \Lambda) \geq \mathbf{o}(\|u - \bar{u}\|_1$$

as $\|u - \bar{u}\|_1 \to 0$ with $u \in \Gamma$. Thus $H(\bar{x}, ., \Lambda)$ reaches a *quasimin* at $\bar{u}$, subject to $u \in \Gamma$. □

Proof of Lemma 4. The equation has solution:

$$\lambda^{[T]}(t) = e^{-\rho t} \int_0^{T-t} e^{-\rho\tau} \tilde{a}(t+\tau) \exp[\int_t^{t+\tau} b(s)ds] d\tau.$$

Since $\| \exp[\int_t^{t+\tau} b(s)ds]\| \leq$ const $e^{-\mu\tau}$ and $\| \exp(k(t,\tau)\| \leq e^{-\zeta\tau}$ for sufficiently large t and τ, there holds for some constant c :

$$\|\lambda^{[T]}(t)\| \leq e^{-\rho t} \int_0^{\infty} e^{-\rho\tau} \|\tilde{a}(\cdot)\|_{\infty} e^{\mu\tau} c e^{\zeta\tau} d\tau \ = \ \mathbf{O}(e^{-\rho\tau})$$

since the integral converges, given $-\rho + \mu + \zeta < 0$. The result follows with $\psi(t) = \text{const}\times e^{-\rho t}$. □

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