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BOUNDARY INTEGRAL EQUATIONS ON UNBOUNDED ROUGH SURFACES: FREDHOLMNESS AND THE FINITE SECTION METHOD

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ABSTRACT. We consider a class of boundary integral equations that arise in the study of strongly elliptic BVPs in unbounded domains of the form $D = \{(x, z) \in \mathbf{R}^n \times \mathbf{R} : z > f(x)\}$ where $f : \mathbf{R}^n \to \mathbf{R}$ is a sufficiently smooth bounded and continuous function. A number of specific problems of this type, for example, acoustic scattering problems, problems involving elastic waves and problems in potential theory, have been reformulated as second kind integral equations $u + Ku = v$ in the space BC of bounded, continuous functions. Having recourse to the so-called limit operator method, we address two questions for the operator $A = I + K$ under consideration, with an emphasis on the function space setting BC. Firstly, under which conditions is A a Fredholm operator, and, secondly, when is the finite section method applicable to A?

1. Introduction. The boundary integral equation method is very well developed as a tool for the analysis and numerical solution of strongly elliptic boundary value problems in both bounded and unbounded domains, provided the boundary itself is bounded, e.g., [**6, 29, 38**].

In the case when both domain and boundary are unbounded, the theory of the boundary integral equation method is much less well developed. The reason for this is fairly clear, namely, that loss of compactness of the boundary leads to loss of compactness of boundary integral operators. To be more precise, classical applications of the

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boundary integral method, for example to potential theory in smooth bounded domains, lead to second kind boundary integral equations of the form $Au = v$ where the function v is known, u unknown, and the operator A is a compact perturbation of the identity, e.g., [**6**]. In more sophisticated applications, to more complex strongly elliptic systems or to piecewise smooth or general Lipschitz domains, compactness arguments continue to play an important role. For example, a standard method to establish that a boundary integral operator A is Fredholm of index zero is to show a Gårding inequality, i.e., to establish that A, as an operator on some Hilbert space, is a compact perturbation of an elliptic principal part, e.g., [**29**]. The case when the boundary is unbounded is difficult because this tool of compactness is no longer available.

To compensate for loss of compactness, only a few alternative tools are known. In the case of classical potential theory and some other strongly elliptic systems, invertibility and/or Fredholmness of boundary integral operators can be established via direct a priori bounds, using Rellich-type identities. In the context of boundary integral equation formulations, these arguments were first systematically exploited by Jerison and Kenig [**20**], Verchota [**41**] and Dahlberg and Kenig [**17**] (and see [**22, 30**]). The main objective in these papers is to overcome loss of compactness associated with *nonsmoothness* rather than unboundedness of the boundary, but the Rellich identity arguments used are applicable also when the boundary is infinite in extent, notably, and most straightforwardly, when the boundary is the graph of a Lipschitz function. For example, for classical potential theory, invertibility of the operator $A = I + K$, where I is the identity and K the classical double-layer potential operator, can be established when the boundary is the graph of a Lipschitz function, as discussed in [**17, 22, 30**]. The Rellich-identity estimates establish invertibility of A in the first instance in L^2, but, by combining these L^2 estimates with additional arguments, the invertibility of A also in L^p for $2 - \varepsilon < p < \infty$ can be established [**17, 22**]. Here ε is some positive constant which depends only on the space dimension and the Lipschitz constant of the boundary.

The same methods of argument can be extended to some other elliptic problems and elliptic systems, e.g., [**18, 31, 32**]. Recently L^2 solvability has also been established for a second kind integral equation formulation on the (unbounded) graph of a bounded Lipschitz function in a case (the Dirichlet problem for the Helmholtz equation with real wave number) when the associated weak formulation of the boundary value problem is noncoercive [**8, 39**]. (This lack of coercivity is relatively easily dealt with as a compact perturbation when the boundary is Lipschitz and compact, e.g., [**40**], but is much more problematic when the boundary is unbounded.)

In this paper we consider the application of another tool which is available for the study of integral equations on unbounded domains, namely the *limit operator method* [**23, 35, 36**]. The results we obtain are applicable to the boundary integral equation formulation of many strongly elliptic boundary value problems in unbounded domains of the form

$$D = \{(x, z) \in \mathbf{R}^n \times \mathbf{R} : z > f(x)\} \tag{1}$$

where $n \geq 1$ and $f : \mathbf{R}^n \to \mathbf{R}$ is a given bounded and continuous function, in short, $f \in \mathrm{BC}$, so that the unbounded boundary is the graph of some bounded function. The results we prove are relevant to the case where the boundary is fairly smooth (Lyapunov), that is, f is differentiable with a bounded and α-Hölder continuous gradient for some $\alpha \in (0, 1]$; i.e., for some constant $C > 0$, $|\nabla f(x) - \nabla f(y)| \leq C|x - y|^\alpha$ holds for all $x, y \in \mathbf{R}^n$. This restriction to relatively smooth boundaries has the implication, for many boundary integral operators on ∂D, for example, the classical double-layer potential operator, see Section 2 below, that loss of compactness arises from the unboundedness of ∂D rather than its lack of smoothness. To be precise, the boundary integral operators we consider, while not compact are nevertheless *locally compact* (in the sense of subsection 3.1), and this local compactness will play a key role in the results we obtain. Throughout, we let

$$f_+ = \sup_{x \in \mathbf{R}^n} f(x) \text{ and } f_- = \inf_{x \in \mathbf{R}^n} f(x)$$

denote the highest and the lowest elevation of the infinite boundary ∂D. It is convenient to assume, without loss of generality, that $f_- > 0$, so that D is entirely contained in the half space $H = \{(x, z) \in \mathbf{R}^n \times \mathbf{R} : z > 0\}$.

Let us introduce the particular class of second kind integral equations on $\mathbf{R}^n$ that we consider in this paper. As we will make clear through detailed examples in Section 2, equations of this type arise naturally when many strongly elliptic boundary value problems in the domain D are reformulated as boundary integral equations on ∂D. To be specific, boundary value problems arising in acoustic scattering problems [**9, 12, 13, 14**], in the scattering of elastic waves [**2, 3**], and in the study of unsteady water waves [**33**], have all been reformulated as second kind boundary integral equations which, after the obvious parametrization, can be written as

$$u + Ku = v,$$

where K is the integral operator

$$(Ku)(x) = \int_{\mathbf{R}^n} k(x,y)\, u(y)\, dy, \quad x \in \mathbf{R}^n \tag{2}$$

with kernel k. Further, in all the above examples, the kernel k has the following particular structure which will be the focus of our study, that

$$k(x,y) = \sum_{i=1}^{j} b_i(x)\, k_i\big(x-y, f(x), f(y)\big)\, c_i(y), \tag{3}$$

where

$$b_i \in \mathrm{BC}, \quad k_i \in C\big((\mathbf{R}^n \setminus \{0\}) \times [f_-, f_+]^2\big) \text{ and } c_i \in L^\infty \tag{4}$$

for $i = 1, \ldots, j$, and

$$|k(x,y)| \le \kappa(x-y), \quad x, y \in \mathbf{R}^n, \tag{5}$$

for some $\kappa \in L^1$. Here and throughout L^p is our abbreviation for $L^p(\mathbf{R}^n)$, for $1 \le p \le \infty$, and we denote the norm on L^p by $\|\cdot\|_p$. By $L(L^p)$ and $L(\mathrm{BC})$ we will denote the Banach space of bounded linear operators on L^p and on BC, respectively. We note that (2)–(5) imply that $K \in L(L^p)$ for $1 \le p \le \infty$ with $\|K\|_{L(L^p)} \le \|\kappa\|_1$. In particular,

$$\|K\|_{L(L^\infty)} = \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} |k(x,y)|\, dy \le \|\kappa\|_1,$$

and we note that $Ku \in \mathrm{BC}$ for $u \in L^\infty$.

In the cases cited above, see Section 2 below, the structure (3)–(4) is a simple consequence of the invariance with respect to translations in the plane $\mathbf{R}^n$ of the fundamental solutions used in the integral equation formulations. This property follows in turn from invariance in the $\mathbf{R}^n$ plane of the coefficients in the differential operator. In each case the bound (5) follows from the Hölder continuity of f, which ensures that $k(x, y)$ is only weakly singular at $x = y$, and from the particular choice of fundamental solution used in the integral equation formulation (a Green's function for the half-space H in each case), which ensures that $k(x, y)$ decreases sufficiently rapidly as $|x - y| \to \infty$. Throughout, we will denote the set of all operators K satisfying (2)–(5) for a particular function $f \in \mathrm{BC}$ (but any choices of j and of the functions b_i, k_i, c_i and κ) by $\mathcal{K}_f$.

There are two main aims of this paper. The major aim is to apply results from the so-called limit operator method [**23, 26, 35, 36**] to operators satisfying (2)–(5), to address, at least partially, the following two questions for the operator $A = I + K$.

Fredholmness and invertibility. Under what conditions is the operator A invertible? More generally, under what conditions is the operator A Fredholm; that is, $Au = 0$ has a finite-dimensional solution space only, and the range of A is closed and has finite co-dimension? So, if A is Fredholm, then the equation $Au = v$ is solvable for all v in a closed subspace of finite co-dimension, and the solution u is unique up to perturbations in a finite-dimensional space.

Applicability of the finite section method. If A is invertible, under which conditions is it possible to replace the equation $Au = v$, i.e.,

$$u(x) + \int_{\mathbf{R}^n} k(x, y)\, u(y)\, dy = v(x), \quad x \in \mathbf{R}^n, \tag{6}$$

by the finite truncations

$$u_\tau(x) + \int_{|y| \le \tau} k(x, y)\, u_\tau(y)\, dy = v(x), \quad x \in \mathbf{R}^n, \tag{7}$$

with a large $\tau > 0$? In the case when we study equations (6) and (7) in the function space $X = L^\infty$ or $X = \mathrm{BC}$, we say that the method of replacing (6) by (7) is *applicable* if the latter equations are uniquely solvable for all sufficiently large τ and their solutions u_τ converge *strictly* (which means uniformly on every compact set) to the solution u of the original problem (6), for every righthand side $v \in X$. If this is the case, then we can approximately solve a boundary integral equation on the unbounded surface ∂D by instead solving a boundary integral equation on a large finite truncation of ∂D.

The second aim of the paper, of interest in its own right and helpful to the aim of applying known limit operator results, is to relate operators in the class $\mathcal{K}_f$, for some $f \in \mathrm{BC}$, to classes of integral operators that have been studied previously in the literature.

Throughout the paper, although many of our results can be extended to other function spaces, especially to L^p for $1 \leq p \leq \infty$, we will concentrate on the case where we view A as on operator on BC. In part we make this restriction just for brevity. Our other reasons for this focus are that, while the function space BC has been the main setting of many of the application-related papers already cited above [**2, 3, 9, 12, 13, 14, 33**], little has been said in the limit operator literature about Fredholmness and the finite section method in the space BC; indeed, only recently has the L^p setting for the limiting cases $p = 1, \infty$ been addressed [**23–26, 28**]. For $1 < p < \infty$, the Fredholmness of operators $I + K$ on L^p with K locally compact was meanwhile studied using limit operator techniques [**36**]. We will prove some new results in subsection 3.2 relating, for very general classes of operators, the applicability of the finite section method on BC to its applicability on L^∞.

The structure and main results of the paper are as follows. In Section 2 we consider three examples of strongly elliptic boundary value problems in the domain D and exhibit the structure (3)–(5). In Section 3 we introduce limit operators and related concepts, then in subsection 3.1 we recall recently established sufficient criteria for Fredholmness and necessary conditions for invertibility of an operator A on L^∞ and BC. These criteria, expressed in terms of invertibility of limit operators of the operator A, apply to large classes of operators, but in particular to operators of the class $\mathcal{K}_f$. In subsection 3.2 we make a preliminary study of the finite section method in the space BC, showing that it is applicable if and only if it is stable and that it is stable on BC if and only if it is stable on L^∞.

To apply the results of subsection 3.1 to operators of the class $\mathcal{K}_f$, it is necessary to show that the class of operators considered in subsection 3.1 includes $\mathcal{K}_f$, and to consider the limit operators of operators in $\mathcal{K}_f$. As a step in this direction and of interest in its own right, we show in subsection 4.1 that the closure of $\mathcal{K}_f$ in $L(\mathrm{BC})$ is a Banach algebra, in fact the Banach algebra generated by particular combinations of multiplication and convolution operators which we identify. This simplifies the study of the limit operators of $K \in \mathcal{K}_f$, since limit operators of multiplication and convolution operators are well understood, see Section 2. Note that the observation, which is part of our result, that $K \in \mathcal{K}_f$ can be approximated by finite sums of products of convolution and multiplication operators, has been utilized in particular cases as a computational tool for matrix compression and fast matrix-vector multiplication, see [**42** and the references therein]. In subsection 4.2, using the results of subsection 4.1, we identify explicitly the limit operators of operators of the class $\mathcal{K}_f$, in the case when the functions f and the functions b_i and c_i in (3) are sufficiently well-behaved (f, b_i and c_i all uniformly continuous will do), in particular showing that each limit operator of $K \in \mathcal{K}_f$ is in $\mathcal{K}_{\tilde{f}}$ for some $\tilde{f}$ related to the original function f.

Finally, in Section 5, we put the results of Sections 3 and 4 together with recent results on the finite section method in L^∞. Our first main result relates invertibility and Fredholmness of $A = I+K$ as an operator on BC to invertibility of the (explicitly identified) limit operators of A, for $K \in K_f$. Our second result, specific to the case $n = 1$, is a necessary and sufficient criterion for applicability of the finite section method in terms of invertibility of the restrictions to half-lines of the limit operators of A. As a specific example, we consider the case when f and the coefficients b_i and c_i are slowly oscillating at infinity when these criteria become very explicit. We also apply our results to the first example of Section 2 (a boundary integral equation for the Dirichlet problem for the Laplace equation in a nonlocally perturbed half-plane).

We finish this introduction by noting that there exist tools which are related to the limit operator method which have been developed by the first author and his collaborators for studying invertibility and the stability and convergence of approximation methods for integral equations on unbounded domains, see [**4, 15, 16**] and the references therein. These methods can be and have been applied to boundary integral equations of the class that we consider in this paper [**2, 3, 9, 12, 13, 14, 33**]. We note, however, that no systematic study of operators of the class $\mathcal{K}_f$ has been made in these papers. Moreover, the results in these papers are complementary to those we exhibit here; in particular, they lead to sufficient but not necessary conditions for invertibility and applicability of the finite section method and do not provide criteria for Fredholmness.

2. Examples. We start with some concrete physical problems that have been modeled as elliptic boundary value problems and reformulated as second kind boundary integral equations, the integral operator in each case exhibiting the structure (2)–(5).

Example 2.1. *Potential theory.* In [**33**] Preston, Chamberlain and Chandler-Wilde consider the two-dimensional Dirichlet boundary value problem: Given $\varphi_0 \in \mathrm{BC}(\partial D)$, find $\varphi \in C^2(D) \cap \mathrm{BC}(\overline{D})$ such that

$$\begin{aligned} \triangle\varphi &= 0 \quad \text{in } D, \\ \varphi &= \varphi_0 \quad \text{on } \partial D, \end{aligned}$$

which arises in the theory of classical free surface water wave problems. In this case $n = 1$ and the authors suppose that f is differentiable with bounded and $\alpha-$Hölder continuous first derivative for some $\alpha \in (0, 1]$, i.e., for some constant $C > 0$, $|f'(x) - f'(y)| \le C|x - y|^\alpha$ for $x, y \in \mathbf{R}$.

Now let

$$G(\mathbf{X}, \mathbf{Y}) = \Phi(\mathbf{X}, \mathbf{Y}) - \Phi(\mathbf{X}^r, \mathbf{Y}), \quad \mathbf{X}, \mathbf{Y} \in \mathbf{R}^2,$$

denote the Green's function for the half plane H where

$$\Phi(\mathbf{X}, \mathbf{Y}) = -\frac{1}{2\pi} \ln |\mathbf{X} - \mathbf{Y}|, \quad \mathbf{X}, \mathbf{Y} \in \mathbf{R}^2,$$

with $|\,.\,|$ denoting the Euclidean norm in $\mathbf{R}^2$, is the standard fundamental solution for Laplace's equation in two dimensions, and $\mathbf{X}^r = (x_1, -x_2)$ is the reflection of $\mathbf{X} = (x_1, x_2)$ with respect to ∂H. For the solution of the above boundary value problem the following double layer potential ansatz is made in [**33**]:

$$\varphi(\mathbf{X}) = \int_{\partial D} \frac{\partial G(\mathbf{X}, \mathbf{Y})}{\partial \mathbf{n}(\mathbf{Y})} \, \tilde{u}(\mathbf{Y}) \; ds(\mathbf{Y}), \quad \mathbf{X} \in D,$$

where $\mathbf{n}(\mathbf{Y}) = (f'(y), -1)$ is a vector normal to ∂D at $\mathbf{Y} = (y, f(y))$, and the density function $\tilde{u} \in \mathrm{BC}(\partial D)$ is to be determined. In [**33**] it is shown that φ satisfies the above Dirichlet boundary value problem if and only if

$$(I - K)\tilde{u} = -2\varphi_0, \tag{8}$$

where

$$(K\tilde{u})(\mathbf{X}) = 2 \int_{\partial D} \frac{\partial G(\mathbf{X},\mathbf{Y})}{\partial \mathbf{n}(\mathbf{Y})} \tilde{u}(\mathbf{Y})\, ds(\mathbf{Y}), \quad \mathbf{X} \in \partial D. \tag{9}$$

In accordance with the parametrization $\mathbf{X} = (x, f(x))$ of ∂D, we define

$$u(x) := \tilde{u}(\mathbf{X}) \quad \text{and} \quad b(x) := -2\varphi_0(\mathbf{X}), \quad x \in \mathbf{R},$$

and rewrite equation (8) as the equation

$$u(x) - \int_{-\infty}^{+\infty} k(x,y)\, u(y)\, dy = b(x), \quad x \in \mathbf{R}, \tag{10}$$

on the real axis for the unknown function $u \in \mathrm{BC}(\mathbf{R})$, where

$$\begin{aligned}
&k(x,y)\\
&= 2\frac{\partial G(\mathbf{X},\mathbf{Y})}{\partial \mathbf{n}(\mathbf{Y})} \sqrt{1+f'(y)^2} = -\frac{1}{\pi}\left(\frac{(\mathbf{X}-\mathbf{Y})\cdot \mathbf{n}(\mathbf{Y})}{|\mathbf{X}-\mathbf{Y}|^2} - \frac{(\mathbf{X}^r-\mathbf{Y})\cdot \mathbf{n}(\mathbf{Y})}{|\mathbf{X}^r-\mathbf{Y}|^2}\right)\\
&= -\frac{1}{\pi}\left(\frac{(x-y)f'(y) - f(x) + f(y)}{(x-y)^2 + (f(x)-f(y))^2} - \frac{(x-y)f'(y) + f(x) + f(y)}{(x-y)^2 + (-f(x)-f(y))^2}\right)\\
&= -\frac{1}{\pi}\left(\frac{x-y}{(x-y)^2 + (f(x)-f(y))^2} - \frac{x-y}{(x-y)^2 + (f(x)+f(y))^2}\right) f'(y)\\
&\quad + \frac{1}{\pi}\left(\frac{f(x)-f(y)}{(x-y)^2 + (f(x)-f(y))^2} + \frac{f(x)+f(y)}{(x-y)^2 + (f(x)+f(y))^2}\right).
\end{aligned}$$

Clearly $k(x,y)$ is of the form (3) with $j = 2$ and property (4) satisfied. From Lemma 2.1 and inequality (5) in [**33**] we moreover get that the inequality (5) holds with

$$\kappa(x) = \begin{cases} c\,|x|^{\alpha-1} & \text{if } 0 < |x| \le 1, \\ c\,|x|^{-2} & \text{if } |x| > 1, \end{cases}$$

where $\alpha \in (0,1]$ is the Hölder exponent of f', and c is some positive constant. $\square$

Example 2.2. *Wave scattering by an unbounded rough surface.* In [**10**] Chandler-Wilde, Ross and Zhang consider the corresponding problem for the Helmholtz equation in two dimensions. Given $\varphi_0 \in \mathrm{BC}(\partial D)$, they seek $\varphi \in C^2(D) \cap \mathrm{BC}(\overline{D})$ such that

$$\begin{aligned} \triangle\varphi + k^2\varphi = 0 \quad & \text{in } \ D, \\ \varphi = \varphi_0 \quad & \text{on } \ \partial D, \end{aligned}$$

and such that φ satisfies an appropriate radiation condition and constraints on growth at infinity. Again, $n = 1$ and the surface function f is assumed to be differentiable with a bounded and α-Hölder continuous first derivative for some $\alpha \in (0,1]$. This problem models the scattering of acoustic waves by a sound-soft rough surface; the same problem arises in time-harmonic electromagnetic scattering by a perfectly conducting rough surface.

The authors reformulate this problem as a boundary integral equation which has exactly the form (8)–(9), except that $G(\mathbf{X}, \mathbf{Y})$ is now defined to be the Green's function for the Helmholtz equation in the half-plane H which satisfies the impedance condition $\partial G/\partial x_2 + ikG = 0$ on ∂H. As in Example 2.1, this boundary integral equation can be written in the form (10) with $k(x, y)$ of the form (3) with $j = 2$ and property (4) satisfied, and also here inequality (5) holds with

$$\kappa(x) = \begin{cases} c\,|x|^{\alpha-1} & \text{if } 0 < |x| \le 1, \\ c\,|x|^{-3/2} & \text{if } |x| > 1, \end{cases}$$

where $\alpha \in (0,1]$ is the Hölder exponent of f', and c is some positive constant. $\square$

Example 2.3. *Wave propagation over a flat inhomogeneous surface.* The propagation of mono-frequency acoustic or electromagnetic waves over flat inhomogeneous terrain has been modeled in two dimensions by the Helmholtz equation

$$\triangle\varphi + k^2\varphi = 0$$

in the upper half plane $D = H$ (so $f \equiv 0$ in (1)) with a Robin, or impedance, condition

$$\frac{\partial \varphi}{\partial x_2} + ik\beta\varphi = \varphi_0$$

on the boundary line ∂D. Here k, the wavenumber, is constant, $\beta \in L^\infty(\partial D)$ is the surface admittance describing the local properties of the ground surface ∂D, and the inhomogeneous term φ_0 is in $L^\infty(\partial D)$ as well.

Similarly to Example 2.2, in fact using the same Green's function $G(\mathbf{X}, \mathbf{Y})$ for the Helmholtz equation, Chandler-Wilde, Rahman and Ross [**12**] reformulate this problem as a boundary integral equation on the real line,

$$u(x) - \int_{-\infty}^{+\infty} \tilde{\kappa}(x-y)z(y)u(y)\,dy = \psi(x), \quad x \in \mathbf{R}, \tag{11}$$

where $\psi \in \mathrm{BC}$ is given and $u \in \mathrm{BC}$ is to be determined. The function $\tilde{\kappa}$ is in $L^1 \cap C(\mathbf{R} \setminus \{0\})$, and $z \in L^\infty$ is closely connected with the surface admittance β by $z = i(1-\beta)$.

Note that the kernel function of the integral operator in (11) is of the form (3) with $j = 1$. The validity of (4) and (5) is trivial in this case. $\square$

3. Limit operators and finite sections. The key to both the Fredholm property and the applicability of the finite section method is the behavior of our operator A at infinity. The tool we shall use to study this behavior is the so-called limit operator method.

3.1. *Limit operators.* Roughly speaking, a limit operator of A is a local representative of A at infinity–a possibly simpler operator that reflects how A acts out there. For its definition we need the following preliminaries.

For every $h \in \mathbf{R}^n$, let V_h denote the *shift operator* acting on $L^p(\mathbf{R}^n)$ by $(V_h u)(x) = u(x-h)$ for all $x \in \mathbf{R}^n$. For every measurable set $U \subset \mathbf{R}^n$, let P_U refer to the operator of multiplication by the characteristic function of U, and write P_τ for P_U if U is the ball around the origin with radius $\tau > 0$. We say that a sequence $(f_k) \subset L^\infty$

converges strictly to $f \in L^\infty$ if $\sup \|f_k\|_\infty < \infty$ and $\|P_m(f_k - f)\|_\infty \to 0$ for all $m \in \mathbf{N}$ as $k \to \infty$. Finally, a sequence of bounded linear operators (A_k) on L^p is said to $\mathcal{P}$*-converge* to A if $\sup \|A_k\| < \infty$ and $\|P_m(A_k - A)\|$ and $\|(A_k - A)P_m\|$ tend to zero for all $m \in \mathbf{N}$ as $k \to \infty$. The $\mathcal{P}$-limit of a sequence is unique if it exists.

Definition 3.1. If $p \in [1, \infty]$, A is a bounded linear operator on L^p and $h = (h_1, h_2, \dots) \subset \mathbf{Z}^n$ is a sequence tending to infinity, i.e., $|h_k| \to \infty$, then the $\mathcal{P}$-limit of the sequence $V_{-h_k} A V_{h_k}$, if it exists, is called the *limit operator of* A *with respect to* h, and it is denoted by A_h.

Example 3.2. As a simple example, if $A = M_b$ is the operator of multiplication by the function $b \in L^\infty$, considered as acting on L^p, and if $h = (h_1, h_2, \dots)$ tends to infinity, then the limit operator A_h exists if and only if the sequence $V_{-h_k} b = b(. + h_k)$ strictly converges to a function, say $b^{(h)}$, as $k \to \infty$, in which case $A_h = M_{b^{(h)}}$ is the operator of multiplication by $b^{(h)}$. □

Example 3.3. The limit operators of M_b, as operators on L^p, are particularly simple if b is what we call a slowly oscillating function. A function $b \in L^\infty$ is *slowly oscillating* if

$$\operatorname{ess\,sup}_{|y| \le 1} |b(x + y) - b(x)| \longrightarrow 0 \ \text{ as } \ x \to \infty.$$

In this case, using the notation of Example 3.2, the strict limit $b^{(h)}$, whenever it exists, is just a constant function with value in the local essential range of b at infinity, and conversely, every function of this type is a strict limit $b^{(h)}$ with a suitable sequence $h = (h_1, h_2, \dots) \subset \mathbf{Z}^n$ tending to infinity [**27**].

If, for example, $n = 1$ and $b(x) = \sin \sqrt{|x|}$ and h_k is the integer part of $k^2\pi^2$ for $k = 1, 2, \dots$, then $b^{(h)}$ exists and is the zero function. It is easily seen that all limit operators of M_b are of the form cI with a constant $c \in [-1, 1]$, and vice versa. □

A bounded linear operator A on L^p is *band-dominated* if

$$\sup \|P_U A P_V\| \longrightarrow 0 \ \text{ as } \ d \to \infty,$$

where the supremum is taken over all measurable $U, V \subset \mathbf{R}^n$ with $\operatorname{dist}(U,V) := \inf_{u\in U, v\in V} |u-v| \geq d$. An operator A on L^p is called *rich* if, from every sequence $h \subset \mathbf{Z}^n$ tending to infinity, we can choose a subsequence g such that the limit operator A_g exists. We note that the set of all band-dominated operators and the set of all rich operators are both Banach subalgebras of $L(L^p)$, see e.g., [**26**]. For $A, A_1, A_2, \ldots \in L(L^p)$, we will write $A_m \rightrightarrows A$ if $\|A_m - A\| \to 0$ as $m \to \infty$. Finally, A is called *locally compact* (on L^p) if $P_\tau A$ and AP_τ are compact for all $\tau > 0$.

Lemma 3.4. *Let $A, B, A^{(1)}, A^{(2)}, \ldots$ be band-dominated operators on L^p, and let $h \subset \mathbf{Z}^n$ be a sequence that tends to infinity. Then the following hold.*

a) *If A_h and B_h exist, then $(A+B)_h$ exists and is equal to $A_h + B_h$.*

b) *If A_h and B_h exist, then $(AB)_h$ exists and is equal to $A_h B_h$.*

c) *If $A^{(m)} \rightrightarrows A$ as $m \to \infty$ and the limit operators $(A^{(m)})_h$ exist for all sufficiently large m, then A_h exists and $(A^{(m)})_h \rightrightarrows A_h$ holds as $m \to \infty$.*

Proof. These are basic results that can be found in any text on limit operators, e.g., [**34**, Proposition 1]. ◻

Note that operators in the class $\mathcal{K}_f$, for some $f \in \mathrm{BC}$, are band-dominated and locally compact as operators on L^p for $1 \leq p \leq \infty$. In the case $p = \infty$ this will be shown in Section 4. In subsection 4.2 we will study the limit operators of operators $K \in \mathcal{K}_f$ and will show that such operators are rich (on L^∞) if f is uniformly continuous and if the operators of multiplication by b_i and c_i (b_i and c_i as in (3)) are rich. The latter, for example, is the case if each of b_i and c_i is uniformly continuous. We note also that if, for each $K \in \mathcal{K}_f$, K^* is defined to be the integral operator given by (2), with $k(x,y)$ replaced by $k(y,x)$, it follows easily from Fubini's theorem that

$$\int_{\mathbf{R}^n} \phi \; K^*\psi \, dx = \int_{\mathbf{R}^n} \psi \; K\phi \, dx,$$

for $\phi \in L^p$, $\psi \in L^q$, $1 \leq p \leq \infty$, where $(1/p) + (1/q) = 1$. Since L^q can be identified with the dual of L^p for $1 \leq p < \infty$, the operator

$K^* \in L(L^q)$ is the adjoint of $K \in L(L^p)$ for $1 \le p < \infty$. The case $p = \infty$ is an anomaly here, but we can say that $K^* \in L(L^1)$ is the *pre-adjoint* of $K \in L(L^\infty)$ (which just means that K is the adjoint of K^*). This observation is relevant to the next theorem. Note that, in the case that the functions c_i in the definition (3) of $K \in \mathcal{K}_f$ are continuous, K^* is also in $\mathcal{K}_f$.

The following theorems are the known results on Fredholmness and invertibility from the theory of limit operators that we will apply in Section 5 to operators $K \in \mathcal{K}_f$, after studying the limit operators of $K \in \mathcal{K}_f$ in Section 4. The first theorem, a sufficient condition for Fredholmness, is a rather deep result. The second result, which is much more straightforward, is a necessary condition for invertibility.

Theorem 3.5. *If $A = I + K$ and, as an operator on L^∞, K is band-dominated, rich and locally compact, then the following holds. If all limit operators of A are invertible on L^∞, then A is Fredholm as an operator on L^∞. If also $K(L^\infty) \subset$ BC, then A is also Fredholm if restricted to* BC.

Proof. Let all limit operators of A be invertible on L^∞. From [**23**, Theorem 2] and the *if* part of Theorem 1.1 in [**28**], alias [**23**, Theorem 1], it follows that A is invertible at infinity, as defined in [**23, 28**]. Note that the *if* portion of Theorem 1.1 does require a rich operator but not the existence of a pre-adjoint, also see [**28**, Remark 3.5].

This, together with K being locally compact, implies that $A = I + K$ is Fredholm on L^∞, by [**10**, subsection 3.3]. Further, if also $K(L^\infty) \subset$ BC, then, by Lemma 3.9 c) below, A is Fredholm if restricted to BC. □

Note that, for $1 < p < \infty$, the invertibility of all limit operators of A (and the uniform boundedness of their inverses) is even necessary and sufficient for the Fredholmness of $A = I + K$ on L^p, see [**34**].

Theorem 3.6. *If, as an operator on L^∞, A is band-dominated, possesses a pre-adjoint in $L(L^1)$ and is invertible, then all limit operators of A are invertible on L^∞.*

Proof. This follows from the *only if* portion of Theorem 1.1 in [**28**]. The operator need not be rich for this implication, as pointed out in Remark 3.5 of [**28**]. ◻

We introduce two types of linear operators which will serve as basic building blocks for the operators we study in the rest of the paper. Firstly, for a function $b \in L^\infty$, let M_b denote the *multiplication operator* $u \mapsto bu$. Secondly, for $\kappa \in L^1$, with Fourier transform a given by

$$a(\xi) = F\kappa(\xi) = \int_{\mathbf{R}^n} e^{i\xi\cdot y}\kappa(y)\,dy, \quad \xi \in \mathbf{R}^n,$$

where $\cdot$ is the Euclidean inner product on $\mathbf{R}^n$, let C_a denote the *operator of convolution with* κ, defined by

$$(C_a u)(x) = \int_{\mathbf{R}^n} \kappa(x-y)\,u(y)\,dy, \quad x \in \mathbf{R}^n.$$

Moreover, let $FL^1 = \{F\kappa : \kappa \in L^1\}$. It is well known, e.g., Jörgens [**21**], that, for $1 \le p \le \infty$, the spectrum of C_a as an element of $L(L^p)$ is $\{a(\xi) : \xi \in \mathbf{R}^n\} \cup \{0\}$.

We will denote the set of all $b \in L^\infty$ for which M_b is a rich operator by $L^\infty_\$$. So, by Example 3.2, $b \in L^\infty_\$$ if and only if every sequence in $\mathbf{Z}^n$ tending to infinity has an infinite subsequence $h = (h_m)$ such that there exists a function $c \in L^\infty$ with

$$\left\| \, b|_{h_m+U} \, - \, c|_U \right\|_\infty \longrightarrow 0 \;\text{ as }\; m \to \infty \tag{12}$$

for every compact set $U \subset \mathbf{R}^n$. A straightforward computation shows that the operator $C_a M_b$ with $a \in FL^1$ is rich as an operator on L^∞ if the above holds with (12) replaced by the much weaker condition

$$\left\| \, b|_{h_m+U} - c|_U \right\|_1 \longrightarrow 0 \;\text{ as }\; m \to \infty. \tag{13}$$

We denote the set of all $b \in L^\infty$ with this property by $L^\infty_{\text{SC\$}}$ and write $\tilde{b}^{(h)}$ for the function c with property (13) for all compact sets U. Recall from Example 3.2 that we write $b^{(h)}$ for the function c in (12).

3.2 *The finite section method for* BC. In this section we will briefly introduce an approximation method for operators on the space of bounded and continuous functions $\mathrm{BC} \subset L^\infty$. The operators of interest to us will be of the form

$$A = I + K,$$

where K shall be bounded and linear on L^∞ with the condition $Ku \in \mathrm{BC}$ for all $u \in L^\infty$. Typically, K will be some integral operator. One of the simplest examples is a convolution operator $K = C_a$ with $a \in FL^1$. The following lemma follows easily from the denseness in L^1 of the set of C^∞ compactly supported functions.

Lemma 3.7. *If $a \in FL^1$, then $C_a u$ is a continuous function for every $u \in L^\infty$.*

As a slightly more sophisticated example, one could look at an operator of the following form or at the norm limit of a sequence of such operators.

Example 3.8. Put

$$K := \sum_{i=1}^{j} M_{b_i} C_{a_i} M_{c_i}, \tag{14}$$

where $b_i \in \mathrm{BC}$, $a_i \in FL^1$, $c_i \in L^\infty$ and $j \in \mathbf{N}$. For the condition that K maps L^∞ into BC, it is sufficient to impose continuity of the functions b_i in (14), whereas the functions c_i need not be continuous since their action is smoothed by the convolution thereafter. □

We also need the following simple auxiliary result.

Lemma 3.9. *Suppose that $A = I + K$ and that $K \in L(L^\infty)$ and $K(L^\infty) \subset \mathrm{BC}$. Abbreviate the restriction $A|_{\mathrm{BC}}$ by A_0. Then the following hold*:

a) *$Au \in \mathrm{BC}$ if and only if $u \in \mathrm{BC}$;*

b) *A is invertible on L^∞ if and only if A_0 is invertible on* BC. *In this case*

$$\|A_0^{-1}\|_{L(\mathrm{BC})} \le \|A^{-1}\|_{L(L^\infty)} \le 1 + \|A_0^{-1}\|_{L(\mathrm{BC})}\, \|K\|_{L(L^\infty)}. \tag{15}$$

c) *If A is a Fredholm operator on L^∞, then A_0 is Fredholm on* BC.

Proof. a) This is immediate from $Au = u + Ku$ and $Ku \in \mathrm{BC}$ for all $u \in L^\infty$.

b) If A is invertible on L^∞, then the invertibility of A_0 on BC and the first inequality in (15) follows from a).

Now let A_0 be invertible on BC. To see that A is injective on L^∞, suppose $Au = 0$ for $u \in L^\infty$. From $0 \in \mathrm{BC}$ and a) we get that $u \in \mathrm{BC}$ and hence $u = 0$ since A is injective on BC. Surjectivity of A on L^∞: Since A_0 is surjective on BC, for every $v \in L^\infty$ there is a $u \in \mathrm{BC}$ such that $A_0 u = Kv \in \mathrm{BC}$. Consequently,

$$A(v-u) = Av - A_0 u = v + Kv - Kv = v \tag{16}$$

holds, showing the surjectivity of A on L^∞. So A is invertible on L^∞, and, by (16), $A^{-1}v = v - u = v - A_0^{-1}Kv$ for all $v \in L^\infty$, and hence $A^{-1} = I - A_0^{-1}K$. This proves the second inequality in (15).

c) From a) we get that $\ker A \subset \mathrm{BC}$ since $0 \in \mathrm{BC}$. But this implies that

$$\ker A_0 = \ker A. \tag{17}$$

Another immediate consequence of a) is

$$A_0(\mathrm{BC}) = A(L^\infty) \cap \mathrm{BC}. \tag{18}$$

Finally, by (18), we have the following relation between factor spaces

$$\frac{\mathrm{BC}}{A_0(\mathrm{BC})} = \frac{\mathrm{BC}}{A(L^\infty) \cap \mathrm{BC}} \cong \frac{\mathrm{BC} + A(L^\infty)}{A(L^\infty)} \subset \frac{L^\infty}{A(L^\infty)}. \tag{19}$$

So, if A is Fredholm on L^∞, then (17), (19) and (18) show that also $\ker A_0$ and $\mathrm{BC}/A_0(\mathrm{BC})$ are finite-dimensional and $A_0(\mathrm{BC})$ is closed. ◻

Remark 3.10. a) The previous lemma clearly holds for arbitrary Banach spaces with one of them contained in the other in place of BC and L^∞.

b) If, moreover, K has a pre-adjoint operator on L^1, then an approximation argument as in the proof of Lemma 3.5 of [**4**] even shows that, in fact, (15) can be improved to the equality $\|A_0^{-1}\|_{L(\mathrm{BC})} = \|A^{-1}\|_{L(L^\infty)}$. □

This paper is concerned with the equation $Au = b$ with $A = I + K$, particularly with the case where $u \in L^\infty$ and $b \in \mathrm{BC}$ and the equation $Au = b$ is some integral equation

$$u(x) + \int_{\mathbf{R}^n} k(x,y)\, u(y)\, dy = b(x), \quad x \in \mathbf{R}^n. \tag{20}$$

In this case, by Lemma 3.9 a), we are looking for u in BC only.

In this setting, a popular approximation method which dates back at least to Atkinson [**5**] and Anselone and Sloan [**1**], is just to reduce the range of integration from $\mathbf{R}^n$ to the ball $|y| \le \tau$ for some $\tau > 0$. We call this procedure the *finite section method for* BC (short: BC-FSM). We are now looking for solutions $u_\tau \in \mathrm{BC}$ of

$$u_\tau(x) + \int_{|y|\le\tau} k(x,y)\, u_\tau(y)\, dy = b(x), \quad x \in \mathbf{R}^n \tag{21}$$

with $\tau > 0$, and hope that the sequence (u_τ) of solutions of (21) strictly converges to the solution u of (20) as $\tau \to \infty$.

This method (21) can be written as $A_\tau u_\tau = b$ with

$$A_\tau = I + KP_\tau. \tag{22}$$

As a consequence of Lemma 3.9 a) applied to A_τ, one also has

Corollary 3.11. *For every $\tau > 0$, it holds that $A_\tau u_\tau \in \mathrm{BC}$ if and only if $u_\tau \in \mathrm{BC}$.*

We say that a sequence of operators (A_τ) is *stable* if there exists a τ_0 such that all A_τ with $\tau > \tau_0$ are invertible and their inverses are uniformly bounded.

In accordance with the machinery presented in [**23, 36**], our strategy to study equation (20) and the stability of its approximation by (21) is to embed these into L^∞, where we can relate the BC-FSM (21) to the usual FSM

$$P_\tau A P_\tau u_\tau = P_\tau b \tag{23}$$

on L^∞. The study of the approximation method (23) for different classes of operators on both L^p and ℓ^p spaces started with work of Baxter [**7**] and Gohberg and Feldman [**19**] on Wiener-Hopf and Toeplitz operators in the early 1960s. For the state of the art on the FSM, see e.g., [36, 37].

Indeed, the applicabilities of these different methods turn out to be equivalent.

Proposition 3.12. *For the operator* $A = I + K$ *with* $K(L^\infty) \subset \mathrm{BC}$*, let*

$$A_\tau := I + KP_\tau \quad \textit{and} \quad A_{\lceil\tau\rfloor} := P_\tau A P_\tau + Q_\tau, \ \ \tau > 0.$$

Then the following statements are equivalent.

(i) *The* BC-FSM (A_τ) *alias* (21) *is applicable in* BC*.*

(ii) *The* BC-FSM (A_τ) *alias* (21) *is applicable in* L^∞*.*

(iii) *The FSM* $(A_{\lceil\tau\rfloor})$ *is applicable in* L^∞*.*

(iv) (A_τ) *is stable on* BC*.*

(v) (A_τ) *is stable on* L^∞*.*

(vi) $(A_{\lceil\tau\rfloor})$ *is stable on* L^∞*.*

Proof. The implication (i) $\Rightarrow$ (iv) is standard. The equivalence of (iv) and (v) follows from Lemma 3.9 b) applied to A_τ. The equivalence of (v) and (vi) was already pointed out in [**23**]. It comes from the following observation:

$$\begin{aligned} A_\tau = I + KP_\tau &= P_\tau + P_\tau K P_\tau + Q_\tau + Q_\tau K P_\tau \\ &= P_\tau A P_\tau + Q_\tau (I + Q_\tau K P_\tau) \\ &= (P_\tau A P_\tau + Q_\tau)(I + Q_\tau K P_\tau) \\ &= A_{\lceil\tau\rfloor}(I + Q_\tau K P_\tau), \end{aligned}$$

where the second factor $(I + Q_\tau K P_\tau)$ is always invertible with its inverse equal to $I - Q_\tau K P_\tau$, and hence $\|(I + Q_\tau K P_\tau)^{-1}\| \le 1 + \|K\|$ for all $\tau > 0$.

(v) $\Rightarrow$ (ii). Since (v) implies (vi), it also implies the invertibility of A on L^∞ by Theorem 4.2 in [**24**] ([**23**, Theorem 5.2]). But this, together with (v), implies (ii) by Theorem 1.44 in [**23**].

Finally, the implication (ii) $\Rightarrow$ (i) is trivial if we keep in mind Lemma 3.9 a) and Corollary 3.11, and the equivalence of (iii) and (vi) follows from Theorem 4.2 in [**24**]. □

For the study of property (iii) in Proposition 3.12 we have Theorem 4.2 in [**24**] (alias [**23**, Theorem 5.2]) involving limit operators of A, provided that, in addition, A is a rich operator. We will state the final result in subsection 5.2.

4. Properties of integral operators of the class $\mathcal{K}_f$. Recall that the aim of the paper is to study the operator $A = I + K$ where K is subject to (2)–(5) and that, for a given function $f \in \mathrm{BC}$, we denote the class of all these operators K by $\mathcal{K}_f$.

4.1 *The relationship between $\mathcal{K}_f$ and other classes of integral operators.* For technical reasons we find it convenient to embed the class $\mathcal{K}_f$ into a somewhat larger Banach algebra of integral operators. (It will turn out that this Banach algebra actually is not that much larger than $\mathcal{K}_f$). Therefore, given $f \in \mathrm{BC}$, put $f_- := \inf f$, $f_+ := \sup f$, and let R_f denote the set of all operators of the form

$$(Bu)(x) = \int_{\mathbf{R}^n} k\big(x-y, f(x), f(y)\big)\, u(y)\, dy, \quad x \in \mathbf{R}^n \tag{24}$$

with $k \in C(\mathbf{R}^n \times [f_-, f_+]^2)$ compactly supported. Moreover, put

$$\begin{aligned}
\widehat{\mathcal{B}} &:= \operatorname{clos\,span}\big\{M_b B M_c : b \in \mathrm{BC},\ B \in R_f,\ c \in L^\infty\big\},\\
\mathcal{B} &:= \operatorname{clos\,alg}\big\{M_b B M_c : b \in \mathrm{BC},\ B \in R_f,\ c \in L^\infty\big\},\\
\widehat{\mathcal{C}} &:= \operatorname{clos\,span}\big\{M_b C_a M_c : b \in \mathrm{BC},\ a \in FL^1,\ c \in L^\infty\big\},\\
\mathcal{C} &:= \operatorname{clos\,alg}\big\{M_b C_a M_c : b \in \mathrm{BC},\ a \in FL^1,\ c \in L^\infty\big\},\\
\mathcal{A} &:= \operatorname{clos\,alg}\big\{M_b, C_a : b \in L^\infty,\ a \in FL^1\big\}.
\end{aligned}$$

Remark 4.1. a) Here, $\operatorname{clos\,span} M$ denotes the closure in $L(\mathrm{BC})$ of the set of all finite sums of elements of $M \subset L(\mathrm{BC})$, and $\operatorname{clos\,alg} M$

denotes the closure in $L(\mathrm{BC})$ of the set of all finite sum-products of elements of M. So $\operatorname{clos\,span} M$ is the smallest closed subspace and $\operatorname{clos\,alg} M$ the smallest (not necessarily unital) Banach subalgebra of $L(\mathrm{BC})$ containing M. In both cases we say they are *generated by* M.

b) The following proposition shows that $\widehat{\mathcal{B}}$ and $\mathcal{B}$ do not depend on the function $f \in \mathrm{BC}$ which is why we omit f in their notations.

c) It is easily seen, see Example 3.8, that all operators in $\widehat{\mathcal{C}}$ map arbitrary elements from L^∞ into BC. Consequently, every $K \in \widehat{\mathcal{C}}$ is subject to the condition on K in subsection 3.2.

d) The linear space $\widehat{\mathcal{C}}$ is the closure of the set of operators considered in Example 3.8. The following proposition shows that this set already contains all of $\mathcal{K}_f$. More precisely, it coincides with the closure of $\mathcal{K}_f$ in the norm of $L(\mathrm{BC})$ and with the other spaces and algebras introduced above. □

Proposition 4.2. *The identity*

$$\operatorname{clos} \mathcal{K}_f = \widehat{\mathcal{B}} = \widehat{\mathcal{C}} = \mathcal{B} = \mathcal{C} \subset \mathcal{A}$$

holds.

Proof. Clearly, $\widehat{\mathcal{C}} \subset \widehat{\mathcal{B}}$ since C_a with $a \in FL^1$ can be approximated in the operator norm by convolutions $B = C_{a'}$ with a continuous and compactly supported kernel. But these operators B are clearly in R_f.

For the reverse inclusion, $\widehat{\mathcal{B}} \subset \widehat{\mathcal{C}}$, it is sufficient to show that the generators of $\widehat{\mathcal{B}}$ are contained in $\widehat{\mathcal{C}}$. We will prove this by showing that $B \in \widehat{\mathcal{C}}$ for all $B \in R_f$. So let $k \in C(\mathbf{R}^n \times [f_-,\, f_+]^2)$ be compactly supported, and define B as in (24). To see that $B \in \widehat{\mathcal{C}}$, take $L \in \mathbf{N}$, choose $f_- = s_1 < s_2 < \cdots < s_{L-1} < s_L = f_+$ equidistant in $[f_-,\, f_+]$, and let φ_ξ denote the standard hat function for this mesh that is centered at s_ξ, i.e., $\varphi_\xi \in C([f_-,\, f_+])$ is a linear polynomial on each interval $[s_\eta, s_{\eta+1}]$, $\eta = 1, \ldots, L-1$, and $\varphi_\xi(s_\eta) = 1$ if $\xi = \eta$, and $= 0$ if $\xi \neq \eta$. Then, since k is uniformly continuous, its piecewise linear interpolations (with respect to the variables s and t),

$$k^{(L)}(r,s,t) := \sum_{\xi,\eta=1}^{L} k(r, s_\xi, s_\eta)\, \varphi_\xi(s)\, \varphi_\eta(t),$$
$$r \in \mathbf{R}^n, \quad s,t \in [f_-,\, f_+],$$

uniformly approximate k as $L \to \infty$, whence the corresponding integral operators with k replaced by $k^{(L)}$ in (24),

$$(B^{(L)}u)(x)=\int_{\mathbf{R}^n} \sum_{\xi,\eta=1}^{L} k\big(x-y, s_\xi, s_\eta\big)\, \varphi_\xi\big(f(x)\big)\, \varphi_\eta\big(f(y)\big)u(y)dy, \quad (25)$$

converge to B in the operator norm in $L(\mathrm{BC})$ as $L \to \infty$. But it is obvious from (25) that $B^{(L)} \in \widehat{\mathcal{C}}$, which proves that also $B \in \widehat{\mathcal{C}}$.

To see that $\mathcal{B} = \mathcal{C}$, it is sufficient to show that the generators of each of the algebras are contained in the other algebra. But this follows from $\widehat{\mathcal{B}} = \widehat{\mathcal{C}}$, which is already proven.

That $\mathcal{C}$ is contained in the Banach algebra $\mathcal{A}$ generated by L^1-convolutions and L^∞-multiplications, is obvious.

For the inclusion $\mathcal{C} \subset \widehat{\mathcal{C}}$ it is sufficient to show that $C_a M_b C_c \in \widehat{\mathcal{C}}$ for all $a, c \in FL^1$ and $b \in L^\infty$. So take an arbitrary $b \in L^\infty$ and let $a = F\kappa$ and $c = F\lambda$ with $\kappa, \lambda \in L^1$. Arguing as in the first paragraph of the proof, it is sufficient to consider the case where κ and λ are continuous and compactly supported, say $\kappa(x) = \lambda(x) = 0$ if $|x| > \ell$. It is now easily checked that

$$(C_a M_b C_c u)(x) = \int_{\mathbf{R}^n} k(x,y)u(y)\, dy, \quad x \in \mathbf{R}^n,$$

with

$$\begin{aligned} k(x,y) &= \int_{\mathbf{R}^n} \kappa(x-z)\, b(z)\, \lambda(z-y)\, dz \\ &= \int_{|t|\le\ell} \kappa(t)\, b(x-t)\, \lambda(x-t-y)\, dt. \end{aligned}$$

By taking a sufficiently fine partition into measurable subsets, $\{T_1, \dots, T_N\}$, of $\{t : |t| < \ell\}$, that is, a partition with $\max_i \operatorname{diam} T_i$ sufficiently small, and fixing $t_m \in T_m$ for $m = 1, \dots, N$, we can approximate $k(x,y)$ arbitrarily closely in the supremum norm on $\mathbf{R}^n \times \mathbf{R}^n$ by

$$k(x,y) = \sum_{m=1}^{N} \int_{T_m} \kappa(t)\, b(x-t)\, \lambda(x-t-y)\, dt \approx k_N(x,y)$$

where

$$k_N(x,y) := \sum_{m=1}^{N} \kappa(t_m)\,\lambda(x-t_m-y) \int_{T_m} b(x-t)\,dt \tag{26}$$
$$= \sum_{m=1}^{N} \kappa_m\,\lambda_m(x-y)\,b_m(x), \quad x,y \in \mathbf{R}^n,$$

with $\kappa_m = \kappa(t_m)$, $\lambda_m(x) = \lambda(x-t_m)$ and $b_m(x) = \int_{T_m} b(x-t)\,dt$, the latter depending continuously on x. In particular, choosing the partition so that $\max_i \operatorname{diam} T_i < 1/N$, and noting that $k(x,y) = k_N(x,y) = 0$ for $|x-y| > 2\ell$, we see that

$$\sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} |k(x,y) - k_N(x,y)|\,dy \longrightarrow 0 \ \text{ as } \ N \to \infty,$$

so that $C_a M_b C_c \in \widehat{\mathcal{C}}$.

The inclusion $\widehat{\mathcal{B}} \subset \operatorname{clos}\mathcal{K}_f$ is also obvious since (4) and (5) hold if $b_i \in \mathrm{BC}$, $c_i \in L^\infty$ and k_i is compactly supported and continuous on all of $\mathbf{R}^n \times [f_-,\, f_+]^2$.

So it remains to show that $\operatorname{clos}\mathcal{K}_f \subset \widehat{\mathcal{B}}$. This clearly follows if we show that $\mathcal{K}_f \subset \widehat{\mathcal{B}}$. So let $K \in \mathcal{K}_f$ be arbitrary, that means K is an integral operator of the form (2) with a kernel $k(\cdot,\cdot)$ subject to (3), (4) and (5). For every $\ell \in \mathbf{N}$, let $p_\ell : [0,\infty) \to [0,1]$ denote a continuous function with support in $[1/(2\ell),\, 2\ell]$ which is identically equal to 1 on $[1/\ell,\, \ell]$. Then, for $i = 1, \dots, j$,

$$k_i^{(\ell)}(r,s,t) := p_\ell(|r|)\ k_i(r,s,t), \quad r \in \mathbf{R}^n,\ s,t \in [f_-,\, f_+],$$

is compactly supported and continuous on $\mathbf{R}^n \times [f_-,\, f_+]^2$, whence $B_i^{(\ell)} \in R_f$ with

$$(B_i^{(\ell)}u)(x) := \int_{\mathbf{R}^n} k_i^{(\ell)}\big(x-y, f(x), f(y)\big)\,u(y)\ dy, \ \ x \in \mathbf{R}^n,$$

for all $u \in \mathrm{BC}$. Now put

$$k^{(\ell)}(x,y) := \sum_{i=1}^{j} b_i(x)\,k_i^{(\ell)}\big(x-y, f(x), f(y)\big)\,c_i(y)$$
$$= p_\ell(|x-y|)\ k(x,y),$$

and let $K^{(\ell)}$ denote the operator (2) with k replaced by $k^{(\ell)}$; that is,

$$K^{(\ell)} = \sum_{i=1}^{j} M_{b_i} B_i^{(\ell)} M_{c_i}, \tag{27}$$

which is clearly in $\widehat{\mathcal{B}}$. It remains to show that $K^{(\ell)} \rightrightarrows K$ as $\ell \to \infty$. Therefore, note that

$$\begin{aligned}
\|K - K^{(\ell)}\| &\leq \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \left| k(x,y) - k^{(l)}(x,y) \right| \, dy \\
&= \sup_{x\in\mathbf{R}^n} \int_{\mathbf{R}^n} \left| \left(1 - p_\ell(|x-y|)\right) \, k(x,y) \right| \, dy \\
&\leq \sup_{x\in\mathbf{R}^n} \int_{|x-y|<1/\ell} |k(x,y)| \, dy \\
&\quad + \sup_{x\in\mathbf{R}^n} \int_{|x-y|>\ell} |k(x,y)| \, dy \\
&\leq \sup_{x\in\mathbf{R}^n} \int_{|x-y|<1/\ell} |\kappa(x-y)| \, dy \\
&\quad + \sup_{x\in\mathbf{R}^n} \int_{|x-y|>\ell} |\kappa(x-y)| \, dy \\
&= \int_{|z|<1/\ell} |\kappa(z)| \, dz + \int_{|z|>\ell} |\kappa(z)| \, dz
\end{aligned}$$

with $\kappa \in L^1$ from (5). But clearly, this goes to zero as $\ell \to \infty$. ◻

4.2. *The limit operators of integral operators in $\mathcal{K}_f$.* In order to apply our results on Fredholmness and the finite section method to $A = I+K$, we need to know about the limit operators of A, which, clearly, reduces to finding the limit operators of $K \in \mathcal{K}_f$. But before we start looking for these limit operators, we single out a subclass $\mathcal{K}_f^\$$ of $\mathcal{K}_f$, all elements of which are rich operators. So, this time, given $f \in$ BUC, let $\mathcal{K}_f^\$$ denote the set of all operators K subject to (2)–(5), for some $j \in \mathbf{N}$ and $\kappa \in L^1$, with $b_i \in$ BUC and $c_i \in L^\infty_{\mathrm{SC\$}}$ for $i = 1, \ldots, j$, and let

$$\begin{aligned}
\widehat{\mathcal{B}_\$} &:= \operatorname{clos\,span}\left\{M_b B M_c : b \in \mathrm{BUC},\ B \in R_f,\ c \in L^\infty_{\mathrm{SC\$}}\right\}, \\
\mathcal{B}_\$ &:= \operatorname{clos\,alg}\left\{M_b B M_c : b \in \mathrm{BUC},\ B \in R_f,\ c \in L^\infty_{\mathrm{SC\$}}\right\}, \\
\widehat{\mathcal{C}_\$} &:= \operatorname{clos\,span}\left\{M_b C_a M_c : b \in \mathrm{BUC},\ a \in FL^1,\ c \in L^\infty_{\mathrm{SC\$}}\right\}, \\
\mathcal{C}_\$ &:= \operatorname{clos\,alg}\left\{M_b C_a M_c : b \in \mathrm{BUC},\ a \in FL^1,\ c \in L^\infty_{\mathrm{SC\$}}\right\}
\end{aligned}$$

denote the rich counterparts of $\widehat{\mathcal{B}}$, $\mathcal{B}$, $\widehat{\mathcal{C}}$ and $\mathcal{C}$. Moreover, put

$$\mathcal{A}'_\$:= \operatorname{clos\,alg}\left\{M_b,\, C_a M_c : b \in L^\infty_\$,\ a \in FL^1,\ c \in L^\infty_{\mathrm{SC}\$}\right\}.$$

Recall that, by [**25**, Theorem 3.9], $\mathrm{BC} \cap L^\infty_\$ = \mathrm{BUC}$, and that $C_a M_c$ is rich for all $a \in FL^1$ and $c \in L^\infty_{\mathrm{SC}\$}$, whence every operator in $\mathcal{A}'_\$$ is rich. Then the following "rich version" of Proposition 4.2 holds.

Proposition 4.3. *If $f \in \mathrm{BUC}$, then it holds that*

$$\operatorname{clos} \mathcal{K}^\$_f = \widehat{\mathcal{B}_\$} = \widehat{\mathcal{C}}_\$ = \mathcal{B}_\$ = \mathcal{C}_\$ \subset \mathcal{A}'_\$.$$

In particular, every $K \in \mathcal{K}^\$_f$ is rich.

Proof. All we have to check is that the arguments we made in the proof of Proposition 4.2 preserve membership of b and c in BUC and $L^\infty_{\mathrm{SC}\$}$, respectively. In only two of these arguments are there multiplications by b and c involved at all.

The first one is the proof of the inclusion $\widehat{\mathcal{B}} \subset \widehat{\mathcal{C}}$. In this argument, we show that every $B \in R_f$ is contained in $\widehat{\mathcal{C}}$. But in fact, this construction even yields $B \in \widehat{\mathcal{C}}_\$$, which can be seen as follows. $B \in R_f$ is approximated in the operator norm by the operators $B^{(L)}$ from (25). Since the Courant hats φ_ξ and φ_η are in BUC and also $f \in \mathrm{BUC}$, we get $\varphi_\xi \circ f \in \mathrm{BUC}$ and $\varphi_\eta \circ f \in \mathrm{BUC} \subset L^\infty_{\mathrm{SC}\$}$. So $B^{(L)} \in \widehat{\mathcal{C}}_\$$, and hence $B \in \widehat{\mathcal{C}}_\$$.

The second argument involving multiplication operators is the proof of the inclusion $\mathcal{C} \subset \widehat{\mathcal{C}}$. But also at this point it is easily seen that the functions $b_m(x) = \int_{T_m} b(x-t)\,dt$ that are invoked in (26) are in fact in BUC, whence $\mathcal{C}_\$ \subset \widehat{\mathcal{C}}_\$$. ◻

Now we are ready to say something about the limit operators of $K \in \mathcal{K}^\$_f$. Not surprisingly, the key to these operators is the behavior of the surface function f and of the multipliers b_i and c_i at infinity. We will show that every limit operator K_h of K is of the same form (2) but with f, b_i and c_i replaced by $f^{(h)}$, $b_i^{(h)}$ and $\tilde{c}_i^{(h)}$, respectively, in (3), where we use the notations introduced in and right after (12) and (13). We will even formulate and prove the analogous result for operators in $\mathcal{B}_\$$. The key step to this result is the following lemma.

Lemma 4.4. *Let $B \in R_f$, that is, B is of the form (24) with a compactly supported kernel function $k \in C(\mathbf{R}^n \times [f_-,\, f_+]^2)$, and let $c \in L^\infty_{\mathrm{SC\$}}$. If a sequence $h = (h_m) \subset \mathbf{Z}^n$ tends to infinity and the functions $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, then the limit operator $(BM_c)_h$ exists and is the integral operator*

$$\Big((BM_c)_h u\Big)(x) = \int_{\mathbf{R}^n} k\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}^{(h)}(y)\, u(y)\, dy, \qquad x \in \mathbf{R}^n. \tag{28}$$

Proof. Choose $\ell > 0$ large enough that $k(r,s,t) = 0$ for all $r \in \mathbf{R}^n$ with $|r| \geq \ell$ and all $s,t \in [f_-,\, f_+]$. Now take a sequence $h = (h_m) \subset \mathbf{Z}^n$ such that the functions $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, i.e., such that

$$\big\|\, f|_{h_m+U} - f^{(h)}|_U \big\|_\infty \longrightarrow 0 \quad \text{and} \quad \big\|\, c|_{h_m+U} - \tilde{c}^{(h)}|_U \big\|_1 \longrightarrow 0 \tag{29}$$

as $m \to \infty$ for every compact set $U \subset \mathbf{R}^n$, which is possible since $f \in \mathrm{BUC} \subset L^\infty_\$$ and $c \in L^\infty_{\mathrm{SC\$}}$, see formulas (12), (13) and the surrounding text. Then it is easily seen that

$$\begin{aligned}(V_{-h_m} BM_c V_{h_m} u)(x) \\ = \int_{\mathbf{R}^n} k\big(x-y, f(x+h_m), f(y+h_m)\big)\, c(y+h_m)\ u(y)\, dy\end{aligned}$$

for all $x \in \mathbf{R}^n$ and $u \in \mathrm{BC}$. Abbreviating $A_m := V_{-h_m} BM_c V_{h_m} - (BM_c)_h$, we get that $(A_m u)(x) = \int_{\mathbf{R}^n} d_m(x,y)\, u(y)\, dy$, where

$$\begin{aligned}|d_m(x,y)| &= \Big|\, k\big(x-y, f(x+h_m), f(y+h_m)\big)\, c(y+h_m) \\ &\qquad -\ k\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}^{(h)}(y)\, \Big| \\ &\leq \Big|\, k\big(x-y, f(x+h_m), f(y+h_m)\big) \\ &\qquad \Big| -k\big(x-y, f^{(h)}(x), f^{(h)}(y)\big) \Big| \cdot \|c\|_\infty \\ &\quad + \|k\|_\infty \cdot \Big|\, c(y+h_m) - \tilde{c}^{(h)}(y) \Big|\end{aligned} \tag{30}$$

for all $x, y \in \mathbf{R}^n$ and $m \in \mathbf{N}$. Moreover, $d_m(x,y) = 0$ if $|x-y| \geq l$.

Now take an arbitrary $\tau > 0$, and denote by U and V the balls around the origin with radius $\tau + \ell$ and τ, respectively. Then, by (30),

$$\begin{aligned}\|P_\tau A_m\| &= \operatorname{ess\,sup}_{x\in V} \int_{\mathbf{R}^n} |d_m(x,y)|\,dy \\ &= \operatorname{ess\,sup}_{x\in V} \int_{U} |d_m(x,y)|\,dy \longrightarrow 0\end{aligned}$$

as $m \to \infty$ since (29) holds and k is uniformly continuous. Analogously,

$$\begin{aligned}\|A_m P_\tau\| &= \operatorname{ess\,sup}_{x\in \mathbf{R}^n} \int_{V} |d_m(x,y)|\,dy \\ &= \operatorname{ess\,sup}_{x\in U} \int_{V} |d_m(x,y)|\,dy \longrightarrow 0\end{aligned}$$

as $m \to \infty$. This proves that $(BM_c)_h$ from (28) is indeed the limit operator of BM_c with respect to the sequence $h = (h_m)$. □

Proposition 4.5. a) *Let $K = M_b B M_c$ with $b \in \mathrm{BUC}$, $B \in R_f$ and $c \in L^\infty_{\mathrm{SC\$}}$. If $h = (h_m) \subset \mathbf{Z}^n$ tends to infinity and all functions $b^{(h)}$, $f^{(h)}$ and $\tilde{c}^{(h)}$ exist, then the limit operator K_h exists and is the integral operator*

$$(K_h u)(x) = \int_{\mathbf{R}^n} b^{(h)}(x)\, k\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}^{(h)}(y)\, u(y)\, dy, \quad x \in \mathbf{R}^n. \tag{31}$$

b) *Every limit operator of $K = M_b B M_c$ with $b \in \mathrm{BUC}$, $B \in R_f$ and $c \in L^\infty_{\mathrm{SC\$}}$ is of this form* (31).

c) *Formula* (31) *for the limit operators of the generators of the Banach algebra $\mathcal{B}_\$$ determines all limit operators of every operator $K \in \mathcal{B}_\$$ in the sense of Lemma* 3.4. *In particular, all limit operators K_h of $K \in \mathcal{K}^\$_f \subset \mathcal{B}_\$$, i.e., K given by* (2)–(5), *with $j \in \mathbf{N}$, $b_i \in \mathrm{BUC}$ and $c_i \in L^\infty_{\mathrm{SC\$}}$ for $i = 1, \ldots, j$, are of the same form* (2) *with k replaced by*

$$\hat{k}^{(h)}(x,y) = \sum_{i=1}^{j} b_i^{(h)}(x)\, k_i\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}_i^{(h)}(y). \tag{32}$$

Proof. a) From basic properties of limit operators, see Lemma 3.4 b), we get that K_h exists and is equal to $(M_b)_h(BM_c)_h$ which is exactly (31) by Lemma 4.4.

b) Suppose $g \subset \mathbf{Z}^n$ is a sequence tending to infinity that leads to a limit operator K_g of K. Since $b, f \in L^\infty_\$$ and $c \in L^\infty_{\mathrm{SC}\$}$, there is a subsequence h of g such that the functions $b^{(h)}$, $f^{(h)}$ and $\tilde{c}^{(h)}$ exist. But then we are in the situation of a), and the limit operator K_h of K exists and is equal to (31). Since h is a subsequence of g, we have $K_g = K_h$.

c) If $K \in \mathcal{B}_\$$, then K is the norm limit of a sequence of finite sum-products of operators of the form M_bBM_c with $b \in \mathrm{BUC}$, $B \in R_f$ and $c \in L^\infty_{\mathrm{SC}\$}$. Enumerate these operators of the form M_bBM_c (the ones that K decomposes to) by K_ι with $\iota \in J$, where J is an at most countable index set. Now if $g \subset \mathbf{Z}^n$ is any sequence going to infinity such that K_g exists, then, since all operators $K_\iota \in \mathcal{B}_\$$ are rich by Proposition 4.3, we can, by a Cantor diagonal argument, pass to an infinite subsequence h of g such that all the limit operators $(K_\iota)_h$ with $\iota \in J$ exist. Then, by Lemma 3.4, the limit operator K_h exists and is composed from the limit operators $(K_\iota)_h$, given by (31), in the natural way. But since $h \subset g$, this limit operator K_h equals K_g.

The formula for the limit operators of $K \in \mathcal{K}_f^\$$ follows from the approximation of K by (27) for which we explicitly know the limit operators. □

Example 4.6. Suppose $K \in \mathcal{K}_f^\$$ where the surface function f and the functions b_i and c_i are all slowly oscillating. Let $h \subset \mathbf{Z}^n$ be a sequence tending to infinity such that $b_i^{(h)}$, $f^{(h)}$ and $\tilde{c}_i^{(h)}$ exist, otherwise pass to a subsequence of h with this property which is always possible.

From Example 3.3 we know that all of $b_i^{(h)}$, $f^{(h)}$ and $\tilde{c}_i^{(h)} = c_i^{(h)}$ are constant. Then, by Proposition 4.5 c), the limit operator K_h is the integral operator with kernel function

$$\hat{k}^{(h)}(x,y) = \sum_{i=1}^{j} b_i^{(h)}\ \tilde{c}_i^{(h)}\ k_i\big(x-y, f_h^{(h)}, f_h^{(h)}\big), \quad x, y \in \mathbf{R}^n \tag{33}$$

which is just a pure operator of convolution by $\hat{\kappa}^{(h)} \in L^1$ with

$$\hat{\kappa}^{(h)}(x-y) = \hat{k}^{(h)}(x,y) \tag{34}$$

for all $x, y \in \mathbf{R}^n$. □

5. The main results. The explicit formula (32) for the limit operators of K, given by (2)–(5), together with our results on Fredholmness and the finite section method in terms of limit operators of A, give us the desired criteria. These criteria are particularly explicit if all of the functions b_i, c_i and f are slowly oscillating, as in Example 4.6.

• In this case, Theorem 3.5 says that A is Fredholm if the Fourier transforms $F\hat{\kappa}_h$ of $\hat{\kappa}_h$ from (34) all stay away from the point -1, and Theorem 3.6 says that this is a necessary condition for invertibility.

• Moreover, it will turn out that the BC-FSM is applicable to A if and only if A is invertible and all functions $F\hat{\kappa}_h$ stay away from -1 and have winding number zero with respect to -1.

Here are the results in the more general case, for $f \in \mathrm{BUC}$ and $K \in \mathcal{K}_f^{\$}$, so that K is given by (2)–(5), for some $j \in \mathbf{N}$ with $b_i \in \mathrm{BUC}$ and $c_i \in L^\infty_{\mathrm{SC\$}}$ for $i = 1, \dots, j$.

5.1. *Fredholmness and invertibility.* Let $f \in \mathrm{BUC}$ and $K \in \mathcal{K}_f^{\$}$. From Proposition 4.3 we know that $K \in \mathcal{A}'_{\$}$. By Lemma 3.4 a), all limit operators of $A = I + K$ are of the form $A_h = I + K_h$, i.e., by Proposition 4.5 c),

$$\begin{aligned} (A_h u)(x) \\ = u(x) + \int_{\mathbf{R}^n} \sum_{i=1}^{j} b_i^{(h)}(x)\, k_i\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}_i^{(h)}(y)\, u(y)\, dy \end{aligned} \tag{35}$$

for $u \in \mathrm{BC}$ and $x \in \mathbf{R}^n$. Now we apply Theorems 3.5 and 3.6 to our operator $A = I + K$.

Theorem 5.1. *If $f \in$ BUC and $K \in \mathcal{K}_f^{\$}$, then the following statements hold.*

(i) *If A is invertible on* BC, *then all limit operators* (35) *are invertible on L^∞.*

(ii) *If all limit operators* (35) *are invertible on* L^∞, *then* A *is Fredholm on* BC.

Proof. We just have to check that K is subject to all conditions in Theorem 3.5. Since, by Proposition 4.3, $\mathcal{K}_f^\$ \subset \mathcal{A}'_\$$, K is rich. By Proposition 4.2, we have $K \in \mathcal{C}$. But since the generators of $\mathcal{C}$ are band-dominated and the set of band-dominated operators is a Banach algebra, we get that all elements of $\mathcal{C}$, including K, are band-dominated. Moreover, every operator in $\mathcal{C}$ is locally compact since L^1-convolution operators are locally compact and multiplication operators commute with P_τ for all $\tau > 0$. ◻

5.2. *The* BC-FSM. Since $K \in \mathcal{K}_f$ maps L^∞ into BC, see Remark 4.1 c), we can, by Proposition 3.12, study the applicability of the BC-FSM (21) for $A = I + K$ by passing to its FSM (23) on L^∞ instead. This method is studied, for the case $n = 1$, in Theorem 4.2 in [**24**]. So let us restrict ourselves to operators on the axis, $n = 1$. By [**24**, Theorem 4.2], we have to look at all operators of the form

$$QV_{-\tau}A_hV_\tau Q + P \quad \text{with} \quad A_h \in \sigma^{\mathrm{op}}_+(A) \tag{36}$$

and

$$PV_{-\tau}A_hV_\tau P + Q \quad \text{with} \quad A_h \in \sigma^{\mathrm{op}}_-(A) \tag{37}$$

with $\tau \in \mathbf{R}$, where $P = P_{[0,+\infty)}$, $Q = I-P$, and $\sigma^{\mathrm{op}}_\pm(A)$ refers to the set of limit operators A_h of A with $h_k \to \pm\infty$, respectively. The operator (36) is invertible if and only if the operator $QV_{-\tau}A_hV_\tau Q$, mapping u to

$$\begin{aligned} u(x) + \int_{-\infty}^0 \sum_{i=1}^j & b_i^{(h)}(x+\tau) \\ & \times k_i\big(x-y, f^{(h)}(x+\tau), f^{(h)}(y+\tau)\big)\, \tilde{c}_i^{(h)}(y+\tau)\, u(y)\, dy \end{aligned} \tag{38}$$

for $x < 0$, is invertible on the negative half axis, or, equivalently, if the operator

$$V_\tau(QV_{-\tau}A_hV_\tau Q)V_{-\tau} = (V_\tau QV_{-\tau})A_h(V_\tau QV_{-\tau}) = P_{(-\infty,\tau]}A_hP_{(-\infty,\tau]},$$

mapping u to

$$u(x)+\int_{-\infty}^{\tau}\sum_{i=1}^{j} b_i^{(h)}(x)\, k_i\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}_i^{(h)}(y)\, u(y)\, dy, \quad x<\tau,$$

is invertible on the half axis $(-\infty,\tau)$, for the corresponding sequence h leading to a limit operator at plus infinity.

Completely analogously, the operator (37) is invertible if and only if the operator $PV_{-\tau}A_hV_\tau P$, mapping u to

$$u(x)+\int_{0}^{+\infty}\sum_{i=1}^{j} b_i^{(h)}(x+\tau)\, k_i\big(x-y, f^{(h)}(x+\tau), f^{(h)}(y+\tau)\big) \times \tilde{c}_i^{(h)}(y+\tau)\, u(y)\, dy$$

with $x>0$ is invertible on the positive half axis, or, equivalently, if the operator that maps u to

$$u(x)+\int_{\tau}^{+\infty}\sum_{i=1}^{j} b_i^{(h)}(x)\, k_i\big(x-y, f^{(h)}(x), f^{(h)}(y)\big)\, \tilde{c}_i^{(h)}(y)\, u(y)\, dy, \ x>\tau$$

is invertible on the half axis $(\tau,+\infty)$, for the corresponding sequence h leading to a limit operator at minus infinity.

Combining this with our previous results, we get the following theorem. For brevity we will say that a set $\{A_\tau\}_{\tau\in\mathbf{R}}$ of operators is *uniformly invertible* if all A_τ are invertible and their inverses are uniformly bounded, and we call it *essentially invertible* if almost all, i.e., with exceptions in an index set of measure zero, A_τ are invertible and their inverses are uniformly bounded.

Theorem 5.2. *If* $f\in$ BUC *and* $K\in\mathcal{K}_f^{\$}$, *then the* BC-FSM *is applicable to* $A=I+K$ *if and only if*

- A *is invertible on* L^∞,
- *for every sequence* h *leading to* $+\infty$ *for which the limit operator* A_h *exists, the set of operators* $\{QV_{-\tau}A_hV_\tau Q\}_{\tau\in\mathbf{R}}=\{u\mapsto(38)\}_{\tau\in\mathbf{R}}$ *is essentially invertible on* $L^\infty(-\infty,0)$, *and*

• *for every sequence h leading to $-\infty$ for which the limit operator A_h exists, the set of operators $\{PV_{-\tau}A_hV_\tau P\}_{\tau\in\mathbf{R}} = \{u \mapsto (39)\}_{\tau\in\mathbf{R}}$ is essentially invertible on $L^\infty(0,+\infty)$.*

Proof. Combine Proposition 3.12 above and Theorem 4.2 in [**24**]. □

Remark 5.3. a) Both the operators $QV_{-\tau}A_hV_\tau Q$ and $PV_{-\tau}A_hV_\tau P$ depend continuously, with respect to the operator norm on $L^\infty(-\infty,0)$ and $L^\infty(0,+\infty)$, respectively, on $\tau \in \mathbf{R}$. This implies that each 'essentially invertible' can be replaced by 'uniformly invertible' in the above theorem. We conjecture that, using the generalized collective compactness results of [**10, 16**], the words 'essentially invertible' can also be replaced by 'elementwise invertible' in Theorem 5.2.

b) If, as in Example 4.6, all of f, b_i and c_i are slowly oscillating, then we have $A_h = I + C_{F\hat{\kappa}^{(h)}}$ with $\hat{\kappa}^{(h)}$ as introduced in Example 4.6. In this case, by Theorem 3.5, A is Fredholm if -1 is not in the spectrum of any $C_{F\hat{\kappa}^{(h)}}$, that is, all the (closed, connected) curves $F\hat{\kappa}^{(h)}(\dot{\mathbf{R}}) \subset \mathcal{C}$ stay away from the point -1. (Here $\dot{\mathbf{R}}$ stands for the one-point compactification $\mathbf{R}\cup\{\infty\}$ of the real line. Note that $F\hat{\kappa}^{(h)}(\infty) = 0$, by the Riemann-Lebesgue lemma.) Moreover, the BC-FSM is applicable to A if and only if A is invertible and all curves $F\hat{\kappa}^{(h)}(\dot{\mathbf{R}})$, in addition to staying away from -1, have winding number zero with respect to this point.

c) In some cases, see Example 5.4 below, the functions $\hat{k}^{(h)}(x,y)$ from (33) in Example 4.6 even depend on $|x-y|$ only, which shows that the same is true for $\hat{\kappa}^{(h)}(x-y) := \hat{k}^{(h)}(x,y)$ then. If we then look at the applicability of the BC-FSM for $n=1$, we get the following interesting result. The invertibility of A already implies the applicability of the finite section method. Indeed, if A is invertible, then, all limit operators A_h are invertible, which shows that all functions $F\hat{\kappa}_h$ stay away from the point -1. But from $F\hat{\kappa}^{(h)}(z) = F\hat{\kappa}^{(h)}(-z)$ for all $z \in \mathbf{R}$ we get that the point $F\hat{\kappa}^{(h)}(z)$ traces the same curve (just in opposite directions) for $z < 0$ and for $z > 0$. So the winding number of the curve $F\hat{\kappa}^{(h)}(\dot{\mathbf{R}})$ around -1 is automatically zero. □

Example 5.4. Let us come back to Example 2.1 where, as we found out earlier, $n = 1$, $j = 2$, $b_1 \equiv -1/\pi$, $c_1 = f'$, $b_2 \equiv 1/\pi$, $c_2 \equiv 1$,

$$k_1(r,s,t) = \frac{r}{r^2 + (s-t)^2} - \frac{r}{r^2 + (s+t)^2}$$

and

$$k_2(r,s,t) = \frac{s-t}{r^2 + (s-t)^2} + \frac{s+t}{r^2 + (s+t)^2}.$$

In addition, suppose that $f'(x) \to 0$ as $x \to \infty$. Then, by Lemma 3.12 b) in [**25**], all of b_1, b_2, c_1, c_2 and f are slowly oscillating, and, for every sequence h leading to infinity such that the strict limit $f^{(h)}$ exists, we have that $b_1^{(h)} \equiv -1/\pi$, $c_1^{(h)} \equiv 0$, $b_2^{(h)} \equiv 1/\pi$, $c_2^{(h)} \equiv 1$ and $f^{(h)} \geq f_- > 0$ is a constant function, whence

$$\begin{aligned} \hat{k}^{(h)}(x,y) &= \frac{1}{\pi}\left(\frac{f^{(h)} - f^{(h)}}{(x-y)^2 + (f^{(h)} - f^{(h)})^2} + \frac{f^{(h)} + f^{(h)}}{(x-y)^2 + (f^{(h)} + f^{(h)})^2}\right) \\ &= \frac{2f^{(h)}}{\pi}\,\frac{1}{(x-y)^2 + 4(f^{(h)})^2} =: \hat{\kappa}^{(h)}(x-y), \quad x, y \in \mathbf{R}^n, \end{aligned}$$

where $f^{(h)}$ is an accumulation value of f at infinity.

Now it remains to check the function values of the Fourier transform $F\hat{\kappa}^{(h)}$. A little exercise in contour integration shows that $F\hat{\kappa}^{(h)}(z) = \exp(-2f^{(h)}|z|)$ for $z \in \mathbf{R}$, cf. Remark 5.3 d). So $F\hat{\kappa}^{(h)}(\dot{\mathbf{R}})$ stays away from -1 and has winding number zero.

Consequently, by our criteria derived earlier, we get that A is Fredholm and that the finite section method is applicable if and only if A is invertible.

As discussed in [**33**], by other, somewhat related arguments, it can, in fact, be shown that A is invertible, even when f is not slowly oscillating. Precisely, injectivity of A can be established via applications of the maximum principle to the associated BVP, and then limit operator-type arguments can be used to establish surjectivity.

We note also that the modified version of the finite section method proposed in [**11**] could be applied in this case. (This method first approximates the actual surface function f by a function f_τ for which f'_τ is compactly supported and $f_\tau(s) = f(s)$ for $|s| \leq \tau - \tau^*$, and then applies the finite section method (7). Here $\tau^* \in (0, \tau)$ is some parameter whose value is fixed independently of τ, for all τ sufficiently large.) For this modified version the arguments of [**11**] and the invertibility of A establish applicability even when f is not slowly oscillating, provided τ^* is chosen large enough. □

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