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AN ITERATION PROCEDURE FOR A CLASS OF INTEGRODIFFERENTIAL EQUATIONS OF PARABOLIC TYPE

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ABSTRACT. This paper deals with a class of integrodifferential equations of parabolic type in which a function of the solution and its derivatives up to the second order with respect to the space variables is involved in a definite integral over the region. The problem can be applied to various models in physics and engineering. An iteration approach is used to establish the global solvability and stability for the problem. The technique is based on estimates of Green's function along with Gronwall's inequality.

1. Introduction. Let $T > 0$ and $Q_T = \Omega \times (0,T)$, where Ω is a bounded region in R^n with a smooth boundary $\partial\Omega$. Consider the following initial-boundary value problem:

$$(1.1) \qquad Lu = h(x,t) + \int_\Omega B(x,t,u,u_x,u_{xx})\,dx, \qquad in\ Q_T,$$

$$(1.2) \qquad u(x,t) = 0, \qquad (x,t) \in S_T = \partial\Omega \times (0,T),$$

$$(1.3) \qquad u(x,0) = u_0(x), \qquad x \in \bar{\Omega},$$

where

$$L = \frac{\partial}{\partial t} - \left[a_{ij}(x,t)\frac{\partial^2}{\partial x_i x_j} + b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t)\right]$$

is a parabolic operator with $a_{ij}\xi_i\xi_j \geq a_0|\xi|^2 (a_0 > 0)$ for $\xi \in R^n$, while $u_x = \{u_{x_i}; i = 1, 2, \cdots, n\}$ and $u_{xx} = \{u_{x_i x_j}; i,j = 1, 2, \cdots, n\}$.

Recently, much attention has been given to the study of integrodifferential equation of the following evolution type

$$Lu = \int_0^t A(x,t,u,u_x,u_{xx})\,dt,$$

where L is a parabolic operator. It represents a class of mathematical models which take into account the effect of the past history. Various

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approaches have been employed to study the well-posedness of the problems as well as their numerical solutions. The readers are referred to [**13**] for the derivation of the mathematical models and also [**3**], [**16**] and their references for the well-posedness and their numerical computations. There are, however, some physical, engineering and biological problems which are described by our equation (1.1) with the proper initial-boundary condition. M. Mimura and K. Ohara [**12**] in a population model considered the following equation

$$u_t = u_{xx} - \left[\int_{-1}^{1} K(x,y)u(y,t)dyu(x,t) \right]_x$$

with the proper initial-boundary condition. They discussed the existence of stationary solitary wave solutions. In [**6**], S.-I. Ei modified the above equation with a nonlinear perturbation and studied the well-posedness as well as some asymptotic behavior of the solution. More recently, A.P. Peirce, et al. [**14**] encountered the following equation (in real form)

$$u_t = \triangle u + \int_\Omega f(x,t)u(x,t)\,dx$$

when they investigated an optimal control problem in a quantum-mechanical system. As another example, note the following inverse problem (cf. [**1**], [**2**] and [**15**]) in which one needs to determine the heat source as well as the temperature distribution in a system, i.e., to find $(u(x,t), f(t))$ such that

$$Lu = f(t), \qquad in\ Q_T, \tag{1.4}$$

$$u(x,t) = g(x,t), \qquad on\ S_T, \tag{1.5}$$

$$u(x,0) = u_0(x), \qquad on\ \bar{\Omega}, \tag{1.6}$$

and an over-specified condition

$$\int_\Omega u(x,t)\,dx = h(t), \qquad on\ [0,T], \tag{1.7}$$

where L is the same operator as the one in (1.1).

Physically, the condition (1.7) means that a certain energy distribution over the region is specified in the system. By taking the derivative with respect to t in (1.7) and using the equation (1.4) the problem (1.4)-(1.7) is equivalent to the following one

$$Lu = \left(h'(t) - \int_\Omega L_0 u \, dx\right) \Big/ |\Omega|, \qquad \textit{in } Q_T,$$

where $|\Omega|$ is the measure of Ω and

$$L_0 = a_{ij}(x,t)\frac{\partial^2}{\partial x_i x_j} + b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t), \tag{1.8}$$

with the initial and boundary conditions (1.5) and (1.6). All these specific problems motivate us to investigate the general problem (1.1)-(1.3). Since a nonlocal integral term is involved in the equation (1.1) the maximum principle is no longer valid for our problem. The objective in this paper is to show the existence, uniqueness and the continuous dependence of the solution upon the known data. We will construct a successive approximation sequence of the solution by an iteration procedure and deduce a uniform bound for the sequence in the norm of a certain Banach space. The derivation of such a uniform bound is based on the singularity estimates for the Green's function of a parabolic operator along with Gronwall's inequality.

The solution for the problem (1.1)-(1.3) is defined as follows.

DEFINITION. A function $(u(x,t) \in C(\bar{Q}_T)$ is called a strong solution of the problem (1.1)-(1.3) if $u(x,t) \in C^{1+1,0+1}(\bar{Q}_T)$ satisfies the equation (1.1) almost everywhere and the initial-boundary conditions (1.2)-(1.3). If $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$ satisfies (1.1)-(1.3) in the classical sense, then $u(x,t)$ is a classical solution of the problem (1.1)-(1.3).

The paper is organized as follows. In §2, a uniform bound is derived for the successive approximation solutions and the existence theorem is established. The uniqueness and continuous dependence are proved in §3. In §4 an example is given to illustrate the application of the theory.

For convenience, the following notations will be used throughout this paper.

For a vector $\xi = \{\xi_1, \cdots, \xi_k\} \in R^k$,

$$|\xi| = \sum_{i=1}^{k} |\xi_i|.$$

$C^{\alpha,\alpha/2}(\bar{Q}_T), C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$, etc., are the standard Banach spaces defined as those in Friedman's book [**7**]. For $1 \le p \le +\infty$, note the following:

$$W_p^2(\Omega) = \{v(x) : \|v\|_{W_p^2(\Omega)} < +\infty\}$$

and

$$W_p^{2,1}(Q_T) = \{u(x,t) : \|u\|_{W_p^{2,1}(Q_T)} < +\infty\},$$

where

$$\|v\|_{W_p^2(\Omega)} = \left[\int_\Omega (|v|^p + |v_x|^p + |v_{xx}|^p)\, dx\right]^{1/p}$$

and

$$\|u\|_{W_p^{2,1}(Q_T)} = \left[\int_{Q_T} (|u|^p + |u_x|^p + |u_{xx}|^p + |u_t|^p)\, dx\right]^{1/p}.$$

2. The Existence of the Solution. Begin with the basic assumptions.

H(1). The functions $a_{ij}(x,t), b_i(x,t), c(x,t)$ and $h(x,t)$ are $C^{\alpha,\alpha/2}(\bar{Q}_T)$ with $a_{ij}\xi_i\xi_j \ge a_0|\xi|^2 (a_0 > 0)$ for $\xi \in R^n$.

H(2). The function $B(x,t,u,p,r)$ is twice differentiable with respect to all its arguments. Moreover,

$$|B(x,t,u,p,r)| \le C_0(1 + |u| + |p| + |r|) \tag{2.1}$$

for $(x,t,u,p,r) \in \bar{Q}_T \times R^3$.

H(3). The function $u_0(x) \in C^{2+\alpha}(\bar{\Omega})$. The consistency conditions $u_0(x) = 0$ *on* $\partial\Omega$ and $L_0 u_0(x) + h(x,0) = 0$ on $\bar{\Omega}$ hold.

Remark. We can relax the condition (2.1) to the following $H^*(2)$ and establish the solvability globally under certain restrictions on the known data (See Theorem 2.2 for details).

H*(2): The function $B(x,t,u,p,r)$ is twice differentiable with respect to all its arguments. Moreover,

$$|B(x,t,u,p,r)| \le C_0(1 + |u|^s + |p|^s + |r|^s) \tag{2.2}$$

for $(x,t,u,p,r) \in \bar{Q}_T \times R^3$ and s is an arbitrary positive integer.

Without loss of generality, assume

$$\int_\Omega B(x,t,0,0,0)\,dx = 0.$$

Now construct a successive sequence by an iteration procedure. Take $u_0(x,t) = u_0(x)$ and define $u_{k+1}(x,t)$ as a solution of the following parabolic problem:

$$Lu = F(x,t), \qquad in\ Q_T \tag{2.3}$$

$$u(x,t) = 0, \qquad on\ S_T, \tag{2.4}$$

$$u(x,0) = u_0(x), \qquad x \in \bar{\Omega}, \tag{2.5}$$

where the operator L is the one in (1.1) and

$$F(x,t) = h(x,t) + \int_\Omega B(x,t,u_k,u_{kx},u_{kxx})\,dx,\ (x,t) \in \bar{Q}_T,\ k = 0,1,\cdots.$$

By the hypotheses H(1)-H(3), the problem (2.3)-(2.5) possesses a unique solution $u_{k+1}(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T), k = 0,1,\cdots$. To obtain the existence, it can be shown that the sequence $\{u_k(x,t)\}$ is covergent to a solution. The following lemma is stated without its proof. The reader is referred to the paper [**16**] for details.

LEMMA 2.1. *Let $G(x,y;t,\tau)$ be a Green's function associated with the parabolic operator L and the first boundary problem. Then there exist two constants $\beta \in (0,1)$ and C which depend on the coefficients of L and the boundary of the domain Ω such that*

$$\left|\int_\Omega G_{xx}(x,y;t,\tau)\,dy\right| \le C(t-\tau)^{-\beta}, \qquad t > \tau.$$

REMARK. It is known ([**11**], page 413) that for the Green's function G, the estimate

$$|G_{xx}(x,y;t,\tau)| \le C(t-\tau)^{-(n+2)/2}\exp\left\{-\frac{c|x-y|^2}{t-\tau}\right\}, \qquad t > \tau$$

holds. It follows that

$$\int_\Omega |G_{xx}(x,y;t,\tau)|\,dy \le C(t-\tau)^{-1}, \qquad t>\tau.$$

Therefore, the result in Lemma 2.1 improves the singularity estimate for the Green's function. A statement of the generalized Gronwall inequality which is useful is:

LEMMA 2.2. *Let $F(t)$ be a nondecreasing function and $\gamma \in (0,1)$. If*

$$y(t) \le F(t) + \int_0^t y(\tau)/(|t-\tau|^\gamma)\,d\tau,$$

then

$$y(t) \le CF(t),$$

where C depends only on γ and the upper bound of T.

The proof can be carried out by an elementary iteration (cf. [9], Lemma 7.1.1). Next, define a function $W_k(t)$ by

$$\begin{aligned} W_k(t) &= \|u_k(\cdot,t)\|_{L^\infty(\Omega)} + \|u_{kx}(\cdot,t)\|_{L^\infty(\Omega)} + \|u_{kxx}(\cdot,t)\|_{L^\infty(\Omega)} \\ &= \|u_k(\cdot,t)\|_{W^2_\infty(\Omega)} \end{aligned}$$

for $t\in[0,T], k=0,1,\ldots$.

LEMMA 2.3. *There exists a constant C depending only upon the operator L and the domain Q_T such that*

$$W_k(t) \le C, \qquad t\in[0,T]. \tag{2.6}$$

PROOF. By Green's representation, we see that

(2.7)

$$\begin{aligned} u_k(x,t) &= \int_\Omega G(x,y;t,0)u_0(y)\,dy \\ &\quad + \int_0^t \int_\Omega \Big\{G(x,y;t,\tau)\Big[\int_\Omega B(\cdots)\,dz + h(y,\tau)\Big]\Big\}\,dyd\tau \\ &= H(x,t) + \int_0^t \int_\Omega \Big\{G(x,y;t,\tau)\Big[\int_\Omega B(\cdots)\,dz\Big]\Big\}\,dyd\tau \\ &= H(x,t) + \int_0^t \Big\{\Big[\int_\Omega G(x,y;t,\tau)\,dy\Big]\Big[\int_\Omega B(\cdots)\,dz\Big]\Big\}\,d\tau \end{aligned}$$

where

$$H(x,t) = \int_\Omega G(x,y;t,0)u_0(y)\,dy + \int_0^t \int_\Omega G(x,y;t,\tau)h(y,\tau)\,dyd\tau, \qquad (x,t) \in \bar{Q}_T$$

and $B(\cdots) = B(z,\tau,u_{k-1}(z,\tau),\ u_{k-1}(z,\tau)_x,\ u_{k-1}(z,\tau)_{xx})$.

Differentiate (2.7) with respect to x to obtain

$$u_{kx}(x,t) = H_x(x,t) + \int_0^t \Big\{\Big[\int_\Omega G_x(x,y;t,\tau)\,dy\Big]\Big[\int_\Omega B(\cdots)\,dz\Big]\Big\}d\tau,$$

and

$$u_{kxx}(x,t) = H_{xx}(x,t) + \int_0^t \Big\{\Big[\int_\Omega G_{xx}(x,y;t,\tau)\,dy\Big]\Big[\int_\Omega B(\cdots)\,dz\Big]\Big\}\,d\tau.$$

Hence,

$$|W_k(t)| \leq \|H(\cdot,t)\|_{W^2_\infty(\Omega)} + \int_0^t \Big\{\Big[\Big|\int_\Omega G\,dy\Big| + \Big|\int_\Omega G_x\,dy\Big| + \Big|\int_\Omega G_{xx}\,dy\Big|\Big]\Big[\int_\Omega B(\cdots)\,dz\Big]\Big\}\,d\tau.$$

It is clear by H(1) and H(3) that $\|H(\cdot,t)\|_{W^2_\infty(\Omega)}$ is uniformly bounded. Moreover, one has the following estimates of Green's function that

$$|G(x,y;t,\tau)| \leq C(t-\tau)^{-n/2}\exp\Big\{-\frac{c|x-y|^2}{t-\tau}\Big\}$$

and

$$|G(x,y;t,\tau)_x| \leq C(t-\tau)^{-(n+1)/2}\exp\Big\{-\frac{c|x-y|^2}{t-\tau}\Big\}.$$

Since

$$\begin{aligned}\int_\Omega (t-\tau)^{-n/2}\exp\Big\{-\frac{c|x-y|^2}{t-\tau}\Big\}\,dy &\leq \int_{R^n} (t-\tau)^{-n/2}\exp\Big\{-\frac{c|x-y|^2}{t-\tau}\Big\}\,dy \\ &\leq c^{-n/2}\int_{R^n} \exp\{-\sigma^2\}\,d\sigma \\ &= \pi c^{-n/2},\end{aligned}$$

it follows that

$$\int_{\Omega}(|G|+|G_x|)\,dy \le C[1+(t-\tau)^{-1/2}], \qquad t>\tau. \tag{2.8}$$

By Lemma 2.1,

$$\left|\int_{\Omega} G_{xx}(x,y;t,\tau)\,dy\right| \le C(t-\tau)^{-\beta}, \qquad t>\tau. \tag{2.9}$$

Combine the estimates (2.8) and (2.9) to obtain

$$|W_k(t)| \le C + C\int_0^t \left\{ b(t-\tau)\Big[\int_{\Omega} B(\cdots)\Big]\right\} d\tau,$$

where $b(t-\tau) = 1+(t-\tau)^{-1/2}+(t-\tau)^{-\beta}$, $0\le\tau<t$.

From the condition H(2), one has the estimate

$$\begin{aligned}\left|\int_{\Omega} B(\cdots)\,dz\right| &\le C[1+\|u_{k-1}(\cdot,\tau)\|_{W^2_\infty(\Omega)}]\\ &\le C[1+W_{k-1}(\tau)].\end{aligned}$$

Therefore, we obtain

$$W_k(t) \le C + C\int_0^t b(t-\tau)W_{k-1}d\tau.$$

Let

$$S_k(t) = 1/(k+1)\sum_{i=0}^{k} W_i(t), \qquad t\in[0,T].$$

Then

$$\begin{aligned} S_k(t) &\le C + \int_0^t b(t-\tau)\Big(\frac{1}{k+1}\Big)\sum_{i=0}^{k-1} W_i(\tau)\,d\tau\\ &\le C + \int_0^t b(t-\tau)S_k(\tau)\,d\tau.\end{aligned}$$

Since $\beta\in(0,1)$ and then

$$\int_0^t b(t-\tau)\,d\tau \le C,$$

the result of Lemma 2.2 implies that

$$S_k(t) \le C,$$

where the constant C is independent of k.

It follows that

$$W_k(t) \le C, \qquad t \in [0,T], k = 0,1,\ldots. \square$$

The condition H(2) is essential to carry out the above estimate. The same estimate (2.6) can now be derived under the assumption $\mathbf{H}^*(\mathbf{2})$ and an additional smallness condition on the known data. First, an elementary lemma is given:

LEMMA 2.4. *Let* $f(z) = A_0 z^s - z + A_1$, $z \in [0, +\infty), s > 1$, *where* A_0 *and* A_1 *are two positive constants. If*

$$A_1 < \left[\frac{s-1}{s}\right]\left[\frac{1}{A_0 s}\right]^{1/(s-1)},$$

then $f(z)$ *has two distinct positive roots.*

PROOF. It is clear that $f(z)$ has only one minimum value which is attained at point $z = z_0 = [1/A_0 s]^{1/(s-1)}$. Note that $f(0) = A_1 > 0, f(+\infty) = +\infty$ and

$$\begin{aligned} f(z_0) &= A_0(1/A_0 s)^{1/(s-1)} - (1/A_0 s)^{1/(s-1)} + A_1 \\ &= (1/A_0 s)^{1/(s-1)}[(1-s)/s)] + A_1 \\ &< 0, \end{aligned}$$

provided that

$$A_1 < [(s-1)/s][1/A_0 s]^{1/(s-1)}. \square$$

LEMMA 2.5. *Under the assumptions* $\mathbf{H(1)}$, $\mathbf{H}^*(\mathbf{2})$ *and* $\mathbf{H(3)}$, *there exists a constant* C *which depends only on the known data such that*

$$\|u(\cdot,t)\|_{W^2_\infty(\Omega)} \le C,$$

provided that $\|u_0(x)\|_{W^2_\infty(\Omega)}+\max_{0\le t\le T}\|h(\cdot,t)\|_{W^2_\infty(\Omega)}$ *is small enough.*

PROOF. Use the same notations as those in Lemma 2.3. By the same calculation as those in Lemma 2.3, one reaches

$$W_k(t)\le\|H(\cdot,t)\|_{W^2_\infty(\Omega)}+C\int_0^t\left\{b(t-\tau)\left[\int_\Omega B(\cdots)\,dz\right]\right\}d\tau,\quad t\in[0,T].$$

The assumption $\mathbf{H}^*(\mathbf{2})$ yields

$$\left|\int_\Omega B(\cdots)\,dz\right|\le C[1+W_{k-1}(\tau)^s].$$

It follows that

$$S_k(t)\le\|H(\cdot,t)\|_{W^2_\infty(\Omega)}+C\int_0^t b(t-\tau)S_k(\tau)^s\,d\tau.$$

If we define

$$S_k^*(t)=\sup_{0\le\xi\le t}S_k(\xi),\qquad t\in[0,T],$$

then

$$S_k^*(t)\le\sup_{0\le\xi\le t}\|H(\cdot,\xi)\|_{W^2_\infty(\Omega)}+CS_k^*(t)^s\sup_{0\le\xi\le t}\int_0^\xi b(t-\tau)\,d\tau,\quad t\in[0,T].$$

Note that

$$\begin{aligned}\int_0^\xi b(\xi-\tau)\,d\tau&=\xi-\left[2\xi^{1/2}+\frac{1}{1-\beta}\xi^{1-\beta}\right].\\&\le C(T).\end{aligned}$$

We obtain

$$S_k^*(t)\le\sup_{0\le\xi\le T}\|H(\cdot,\xi)\|_{W^2_\infty(\Omega)}+CC(T)S_k^*(t)^s,\qquad t\in[0,T].$$

Let

$$\begin{aligned}f(S_k^*(t))&=CC(T)S_k^*(t)^s-S_k^*(t)+\sup_{0\le\xi\le T}\|H(\cdot,\xi)\|_{W^2_\infty(\Omega)}\\&\equiv A_0S_k^*(t)^s-S_k^*(t)+A_1.\end{aligned}$$

Case 1: If $A_1 = \sup_{0\leq\xi\leq T} \|H(\cdot,\xi)\|_{W^2_{+\infty}(\Omega)} \equiv 0$, then we have the unique solution $u(x,t) = 0$ by the results in §3. The result of Lemma 2.5 is automatically true.

Case 2: If $A_1 = \sup_{0\leq\xi\leq T} \|H(\cdot,\xi)\|_{W^2_{\infty}(\Omega)} > 0$.

Then $f(S_k^*(T))$ has two distinct positive roots m and M provided that

$$\sup_{0\leq\xi\leq T} \|H(\cdot,\xi)\|_{W^2_{+\infty}(\Omega)} < \frac{s-1}{s}\left[\frac{1}{CC(T)s}\right]^{1/(s-1)}.$$

It is clear that the constants m and M are independent of k and m is positive. Observing that

$$0 \leq S_k^*(0) = \|H(\cdot,0)\|_{W^2_{\infty}(\Omega)} \leq A_1.$$

But from the hypothesis of $\mathbf{H}^*(\mathbf{2})$ and Lemma 2.4,

$$\begin{aligned} A_1 &\leq \frac{s-1}{s}\left[\frac{1}{A_0 s}\right]^{\frac{1}{s-1}} \\ &\leq \left[\frac{1}{A_0 s}\right]^{\frac{1}{s-1}} \end{aligned}$$

which is the location of the minimum of $f(S^*(t))$. Furthermore, $f(A_1) = A_0 A_1^s > 0$. Thus A_1 and hence $S^*(0) < m$, the smallest root. The continuity of $S_k^*(t)$ guarantees that

$$S_k^*(t) \leq m,$$

for all $t \in [0,T]$, giving the uniform boundedness of $W_k(t)$ in $[0,T]$.

Using Lemma 2.3 and Lemma 2.5 the following theorems can be established.

THEOREM 2.1. *Under the assumption H(1)-H(3), if*

$$B(x,t,u,u_x,u_{xx}) = d_{ij}(x,t,u,u_x)u_{x_ix_j} + e(x,t,u,u_x),$$

then the problem (1.1)-(1.3) has at least one strong solution.

PROOF. From the estimate (2.6) and the equation (2.3), one immediately obtains for any k

$$\|u_{kt}(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \qquad t \in [0,T].$$

By compactness argument, we have a subsequence (still denoted by $\{u_k(x,t)\}$ for convenience) such that

$$u_k(x,t) \to u(x,t) \in C^{1+1,0+1}(\bar{Q}_T)$$

and

$$u_{kx_i}(x,t) \to u_{x_i}(x,t) \in C^{0+1,1/2}(\bar{Q}_T)$$

uniformly as $k \to +\infty$.

$$u_{kx_ix_j}(x,t) \to u_{x_ix_j}(x,t) \in L^\infty(\bar{Q}_T)$$

and

$$u_{kt}(x,t) \to u_t(x,t)$$

in the sense of *-weak.

By a standard argument [**7**], $u(x,t)$ satisfies the equation (1.1) almost everywhere in the region Q_T and satisfies the initial-boundary conditions (1.2)-(1.3) in the classical sense. i.e. $u(x,t)$ is a strong solution of the problem (1.1)-(1.3). □

Similarily,

THEOREM 2.2. *Under the assumption* **H(1)**, **H*****(2)** *and* **H(3)**, *if*

$$B(x,t,u,u_x,u_{xx}) = d_{ij}(x,t,u,u_x)u_{x_ix_j} + e(x,t,u,u_x),$$

then the problem (1.2)-(1.3) *has at least one strong solution provided that* $\|u_0(x)\|_{W^2_\infty(\Omega)} + \max_{0\le t\le T}\|h(x,t)\|_{W^2_\infty(\Omega)}$ *is sufficiently small.*

THEOREM 2.3. *Under the assumptions* **H(1)-H(3)**, *if*

$$B(x,t,u,u_x,u_{xx}) = d_{ij}(x,t,u)u_{x_ix_j} + e(x,t,u,u_x),$$

then the problem (1.1)-(1.3) *has at least one classical solution.*

PROOF. Perform the integration by parts to see that

$$\begin{aligned}
&\int_\Omega B(x,t,u,u_x,u_{xx})\,dx \\
&\quad \equiv \int_\Omega [d_{ij}(x,t,u)u_{x_ix_j} + e(x,t,u,u_x)]\,dx \\
&\quad = -\int_\Omega d_{ijx_i}(x,t,u)u_{x_j}dx + \int_{\partial\Omega} d_{ij}(x,t,u)u_{x_j}N_i\,ds \\
&\qquad + \int_\Omega e(x,t,u,u_x)\,dx
\end{aligned}$$

Since $u(x,t) \in C^{1+1,0+1}(\bar{Q}_T)$, the function in the right side of the above equality is in $C^{0+1,1/2}(\bar{Q}_T)$. Then Schauder theory yields that $u(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)$. i.e. $u(x,t)$ is a classical solution. □

THEOREM 2.4. *Under the assumptions* $\mathbf{H(1)}$, $\mathbf{H}^*\mathbf{(2)}$ *and* $\mathbf{H(3)}$, *if*

$$B(x,t,u,u_x,u_{xx}) = d_{ij}(x,t,u)u_{x_ix_j} + e(x,t,u,u_x),$$

then the problem (1.1)-(1.3) *has at least one classical solution, provided that* $\|u_0(x)\|_{W^2_\infty(\Omega)} + \max_{0\le t\le T}\|h(x,t)\|_{W^2_\infty(\Omega)}$ *is sufficiently small.*

REMARK 1. All the results are still true if Equation (1.1) comes in the form

$$Lu = h(x,t) + g(x,t)\int_\Omega B(x,t,u,u_x,u_{xx})\,dx,$$

provided that $g(x,t)$ is smooth on $\bar{Q}_T$.

REMARK 2. If the function $B(x,t,u,p,r)$ is independent of the variable r, the global existence of the solution for the following equation can be established.

$$Lu = F\Big(x,t,u,u_x,\int_0^t A(x,\tau,u,u_x,u_{xx})\,d\tau, \int_\Omega B(y,t,u,u_x)\,dy\Big)$$

and the proper initial-boundary condition, provided that the hypothesis **H(2)** (or **H***(**2**) with small data) holds for both the functions A and B.

3. The Continuous Dependence and Uniqueness. In this section, the continuous dependence of the classical solution upon the known functions is shown and then the uniqueness is obtained as a direct corollary of it. The technique is similar to those employed in §2.

THEOREM 3.1. *Let $u(x,t)$ and $\bar{u}(x,t)$ be the classical solutions of (1.1)-(1.3) corresponding to the known functions $(u_0(x), h(x,t))$ and $(\bar{u}_0(x), \bar{h}(x,t))$ which satisfy the hypothesis* **H(3)**. *Then*

$$\begin{aligned}&\|u(x,t)-\bar{u}(x,t)\|_{C^{2+\alpha,1+\alpha/2}(\bar{Q}_T)}\\ &\leq C\Big[\|u_0(x)-\bar{u}_0(x)\|_{C^{2+\alpha}(\bar{\Omega})}+\|h(x,t)-\bar{h}(x,t)\|_{C^{\alpha,\alpha/2}(\bar{Q}_T)}\Big],\end{aligned}$$

where the constant C depends only on the known data.

PROOF. Let $V(x,t) = u(x,t) - \bar{u}(x,t), (x,t) \in Q_T$. It is clear that $V(x,t)$ is the unique solution of the following problem:

$$\begin{aligned}LV &= \int_\Omega [A_{ij}(x,t)V_{x_ix_j} + B_i(x,t)V_{x_i} + C(x,t)V]\,dx\\ &\quad + h(x,t) - \bar{h}(x,t), \qquad \text{in } Q_T,\\ V(x,t) &= 0, \qquad \text{on } S_T,\\ V(x,0) &= u_0(x) - \bar{u}_0(x), \qquad \text{on } \Omega,\end{aligned}$$

where

$$\begin{aligned}A_{ij}(x,t) &= \int_0^1 B_{r_{ij}}(x,t,u,u_x,\theta u_{x_ix_j} + (1-\theta)\bar{u}_{x_ix_j})\,d\theta,\\ B_i(x,t) &= \int_0^1 B_{p_i}(x,t,u,\theta u_{x_i} + (1-\theta)\bar{u}_{x_i},\bar{u}_{xx})\,d\theta,\\ C(x,t) &= \int_0^1 B_u(x,t,\theta u + (1-\theta)\bar{u},\bar{u}_x,\bar{u}_{xx})\,d\theta.\end{aligned}$$

By an analogous technique to those used in the proof of Lemma 2.3,

$$W(t) \leq F(t) + C\int_0^t b(t-\tau)W(\tau)\,d\tau,$$

where

$$W(t) = \|u(\cdot,t) - \bar{u}(\cdot,t)\|_{W^2_\infty(\Omega)},$$

and

$$F(t) = \|u_0 - \bar{u}_0\|_{W^2_\infty(\Omega)} + \|h(\cdot,t) - \bar{h}(\cdot,t)\|_{L^\infty(\Omega)}.$$

Let $F^*(t) = \sup_{0\leq\xi\leq t} F(\xi), t \in [0,T]$.

If $F(t)$ is replaced by $F^*(t)$ in the above inequality, then Gronwall's inequality and Schauder theory imply the desired result. □

COROLLARY. *The problem* (1.1)-(1.3) *has at most one classical solution.*

4. Applications. Recently, several authors considered the following parabolic equation with nonlocal boundary condition:

$$Lu = h(x,t), \qquad in\ Q_T,$$

$$u(x,t) = \int_\Omega f(y)u(y,t)\,dy, \qquad on\ S_T,$$

$$u(x,0) = u_0(x), \qquad on\ \bar{\Omega}.$$

The problem arises from the quasi-static theory of thermoelasticity. Day [**4**] and [**5**] studied the monotonic decay property of the solution in one space dimension. Friedman [**8**] and Kawohl [**10**] generalized results into the n-dimensional case by employing the classical maximum principle. In [**8**], the existence and uniqueness of the solution are also established for the spatially dependent boundary condition

$$u(x,t) = \int_\Omega f(x,y)u(y,t)\,dy \text{ with } \theta(x) = \int_\Omega |f(x,y)|\,dy < 1$$

which is essential in the proof. Now transfer this problem into the form of our problem and obtain the existence, uniqueness and continuous dependence without the above restriction.

Let

$$V(x,t) = u(x,t) - \int_\Omega f(x)u(x,t)\,dx, \ (x,t) \in \bar{Q}_T.$$

By a direct calculation,

$$u(x,t) = V(x,t) + \frac{\int_\Omega f(x)V(x,t)\,dx}{1 - \int_\Omega f(x)\,dx},$$

provided that $1 - \int_\Omega f(x)\,dx \neq 0$.

Moreover, $V(x,t)$ satisfies

$$LV = h(x,t) - \int_\Omega f(x)[h(x,t) + L_0 u]\,dx + c(x,t)\int_\Omega f(x)u\,dx, \ \textit{in } Q_T$$

$$V(x,t) = 0, \qquad \textit{on } S_T,$$

$$V(x,0) = u_0(x) - \int_\Omega f(x)u_0(x)\,dx, \qquad \textit{on } \bar{\Omega},$$

where L_0 is the same as (1.8).

Note that

$$L_0 u = L_0 V + c(x,t)\frac{\int_\Omega f(x)V(x,t)\,dx}{1 - \int_\Omega f(x)\,dx}.$$

Hence we have the existence, uniqueness and continuous dependence of the solution upon the known data from our previous results in this paper.

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REFERENCES

1. John R. Cannon and H.M. Yin, *Nonlinear non-classical parabolic equations*, to appear in J. Differential Equations.

2. ——— and D. Zachmann, *Parameter determination in parabolic partial differential equations from overspecified boundary data*, Int. J. Engng Sci., **20** (1982), 779-788.

3. G. Da Proto and M. Iannelli, *Existence and regularity for a class of integrodifferencial equations of parabolic type*, J. Math. Anal. Appl., **112** (1985), 36-55.

4. W.A. Day, *Extensions of a property of the heat equation to linear thermoelasticity and other theories*, Quar. Appl. Math., **40** (1982), 319-330.

5. ———, *A decreasing property of solution of parabolic equations with applications to thermoelasticity*, Quar. Appl. Math. **41** (1983), 468-475.

6. S.-I. Ei, *The effect of non-local convection on reaction-diffusion equations*, Hiroshima Math. J., **17** (1987), 281-307.

7. A. Friedman, *Partial differential equations of parabolic type*, Prentice-Hall, Englewood Cliffs, 1964.

8. ———, *Monotonic decay of solutions of parabolic equations with nonlocal boundary conditions*, Quar. Appl. Math., **44** (1986), 401-407.

9. Dan Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, Vol. 840, Spring-Verlag, 1981.

10. B. Kawohl, *Remarks on a paper by W.A. Day on a maximum principle under nonlocal boundary conditions*, Quar. Appl. Math., **44** (1986), 751-752.

11. O.A. Ladyzenskaja, V.A. Solonikov and N.N. Ural'ceva, *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Vol. 23, Providence, R.I., 1964.

12. M. Mimura and K. Ohara, *Standing wave solutions for a Fisher type equation with a nonlocal convection*, Hiroshima Math. J., **16** (1985), 33-50.

13. J.A. Nohel, *Nonlinear Volterra equations for heat flow in materials with memory*, Eds. T.L. Herdman, H.W. Stech and S.M. Rankin, III, Lecture Notes in Pure and Applied Mathematics, Vol. 67 (1981), 3-82.

14. A.P. Peirce, M.A. Dahleh and H. Rabitz, *Optical control of quantum-mechanical systems: Existence, numerical approximation and applications*, Physical Review A, **37** (1988), 4950-4964.

15. A.I. Prilepko and D.G. Orlovskii, *Determination of the evolution parameter of an equation and inverse problems of mathematical physics I*, Differentscial'nye Uravneniya, **21** (1985), 119-129, II, **21** (1985), 694-701.

16. E.G. Yanik and G. Fairweather, *Finite element methods for parabolic and hyperbolic partial integrodifferential equations*, Nonlinear Analysis, Theory, Method and Application, **12** (1988), 785-809.

17. H.M. Yin, *A uniqueness theorem for a class of nonclassical parabolic equations*, to appear in Applicable Analysis.

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