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POINTWISE ERROR ESTIMATES FOR THE TRIGONOMETRIC COLLOCATION METHOD APPLIED TO SINGULAR INTEGRAL EQUATIONS AND PERIODIC PSEUDODIFFERENTIAL EQUATIONS

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Dedicated to the memory of Professor Dr. P. Henrici.

ABSTRACT. Pointwise rates of convergence for the collocation method applied to periodic singular integral equations and pseudodifferential equations are considered, using trigonometric polynomials of degree n as the space of trial functions. If the exact solution is in C^r, then the error in the maximum norm is shown to be $O(n^{-r}\log^2 n)$. This rate of convergence is almost optimal, since the error for the interpolant of the exact solution is $O(n^{-r}\log n)$ and for the best approximation is $O(n^{-r})$.

Introduction. This paper deals with the trigonometric collocation method and is a sequel to [**7**], which treated the trigonometric Galerkin method. We prove error estimates of the form

$$\|u_n - u\|_{C^s} \leq c(1/n)^{r-s}(\log n)^2\|u\|_{C^r}$$

for non-negative integers $s < r$, where u_n is the trigonometric polynomial of degree n obtained via collocation of a periodic singular integral equation or, more generally, of a periodic pseudodifferential equation, whose exact solution is u. For any periodic function $u \in C^r$, the error for the best approximation to u in the C^s norm by trigonometric polynomials of degree n is of order $(1/n)^{r-s}$. In this sense the error estimate above is less than optimal by a factor of $(\log n)^2$. Very recently, B. Silbermann (oral communication) has improved upon our result by showing that only one factor of $\log n$ is needed. This means that the

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pointwise rate of convergence for trigonometric collocation is the same as for trigonometric interpolation.

The book by Mikhlin and Prößdorf [**8**] contains detailed bibliographic information on numerical methods for singular integral equations. Of most relevance to the work presented here is the treatment of collocation methods due to Prößdorf and Silbermann [**16**], who obtained error estimates in L_p and Hölder norms. The difficulty in establishing sharp pointwise error estimates stems from the fact that singular integral operators and pseudodifferential operators fail to be bounded with respect to the maximum norms.

In the case of a single singular integral equation we perform all the analysis explicitly. Using Bessel potentials and the factorization of matrices, these results extend to general systems of pseudodifferential equations on closed smooth curves. The convergence results are valid for elliptic equations with vanishing left indices of the principal symbol $\sigma_0(x,-1)^{-1}\sigma_0(x,+1)$. A further generalization to equations with additional equilibrium conditions and new unknown parameters can be made without difficulties as in [**6**]. Here we omit these more technical details. The above class of equations includes boundary integral equations of various types, such as Fredholm integral equations of the first and second kind, Cauchy singular integral equations, hypersingular integral equations and elliptic integro-differential equations. All of these are used to solve many different problems in applications, some are listed in [**6**] and [**22**].

In connection with numerical integration, this method establishes a discrete version of the spectral method for these equations and is most efficient for smooth curves and data since the fast Fourier transform provides a simple, numerically stable and fast tool to handle the discrete equations for the Fourier coefficients. This idea goes back to P. Henrici who proposed a similar way in [**5**].

This paper is organised as follows. The collocation method is described briefly in §1, and then in §2 we gather together some results on approximation by trigonometric polynomials. As in [**16**], the error analysis for the collocation method relies on *factorization* of the equation's coefficients, a topic dealt with in §3. (Some of the material in §2 and §3 is discussed in greater detail in the paper [**7**] where the Galerkin method was treated.) For the Cauchy singular integral equations, the pointwise error estimates are proved in §4, by adapting the approaches used in [**7**] and [**16**]. Along the way, we also prove error estimates in the Hölder-Zygmund norm. In §5 and §6 we show how to extend the analysis, first to one pseudodifferential equation, and then to systems by using Bessel potentials. The necessary matrix factorization result is quoted.

Throughout the paper, c denotes a generic constant independent of n, u, u_n, not the same at each occurrence.

1. Collocation with trigonometric polynomials. Consider a singular integral equation or system of pseudodifferential equations

$$Au = f, \tag{1.1}$$

where u and f are 2π-periodic, complex-valued functions. We are interested in constructing approximations to the solution u.

Denote the trigonometric monomials by

$$e_l(t) := \exp[ilt], \qquad l \in \mathbf{Z},$$

and the space of trigonometric polynomials of degree n by

$$\mathcal{T}_n := \operatorname{span}\{e_l : |l| \le n\}, \qquad n \in \mathbf{N}_0.$$

Here, $\mathbf{N}_0 := \{0, 1, 2, \dots\}$ is the set of natural numbers including zero. When zero is excluded, we write $\mathbf{N} := \{1, 2, 3, \dots\}$. Since $\dim \mathcal{T}_n = 2n + 1$, define the equally-spaced collocation points by

$$t_{k,n} := 2\pi k/(2n+1), \qquad |k| \le n, \tag{1.2}$$

and seek $u_n \in \mathcal{T}_n$ satisfying

$$(Au_n)(t_{k,n}) = f(t_{k,n}), \qquad |k| \le n. \tag{1.3}$$

When it exists, the function u_n is said to be a trigonometric collocation solution of (1.1).

Denote the complex Fourier coefficients of a function f by

$$\hat{f}(l) := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ilt} f(t)\, dt, \qquad l \in \mathbf{Z},$$

then the Fourier series expansion of f is

$$f(t) \sim \sum_{l=-\infty}^{\infty} \hat{f}(l) e^{ilt}. \tag{1.4}$$

The collocation equations (1.3) hold if and only if the Fourier coefficients of u_n satisfy the $(2n+1) \times (2n+1)$ system of linear algebraic equations

$$\sum_{|l| \leq n} [(Ae_l)(t_{k,n})]\hat{u}_n(l) = f(t_{k,n}), \qquad |k| \leq n.$$

We shall see later that if A is invertible, then these equations are uniquely solvable for all n sufficiently large. For singular integral equations (1.1), it can further be shown that the l_2 condition number of the coefficient matrix is uniformly bounded as $n \to \infty$, see [**16**].

2. Approximation Theory. Let C denote the set of continuous functions $f : \mathbf{R} \to \mathbf{C}$ which are 2π-periodic, that is,

$$f(x + 2\pi) = f(x), \qquad \text{for all } x \in \mathbf{R},$$

and equip C with the maximum norm

$$\|f\|_\infty := \max_{|x| \leq \pi} |f(x)|, \qquad f \in C.$$

Write $D = d/dx$, define

$$C^s := \{f \in C : D^j f \in C \quad \text{for } 0 \leq j \leq s\}, \qquad s \in \mathbf{N}_0,$$

and introduce the norm

$$\|f\|_{C^s} := \sum_{j=0}^{s} \|D^j f\|_\infty, \qquad s \in \mathbf{N}_0.$$

As usual, we put $C^\infty := \cap_{s=0}^\infty C^s$.

Define the Hölder-Zygmund seminorm

$$[f]^\alpha := \begin{cases} \sup_{h>0} \frac{\|\Delta_h f\|_\infty}{h^\alpha}, & 0 < \alpha < 1 \\ \sup_{h>0} \frac{\|\Delta_h^2 f\|_\infty}{h^\alpha}, & \alpha = 1, \end{cases}$$

where Δ_h is the forward difference operator

$$(\Delta_h f)(t) := f(t+h) - f(t),$$

and Δ_h^2 is the second forward difference operator

$$(\Delta_h^2 f)(t) := f(t+2h) - 2f(t+h) + f(t),$$

cf. [**23**, Vol. 1, p. 42-45]. Given $s > 0$, write

$$s = m + \alpha \qquad \text{where } m \in \mathbf{N}_0 \text{ and } 0 < \alpha \leq 1, \tag{2.1}$$

and define the periodic Hölder-Zygmund space

$$\mathcal{H}^s := \{f \in C^m : [D^m f]^\alpha < \infty\},$$

with the norm

$$\|f\|_{\mathcal{H}^s} := \|f\|_{C^m} + [D^m f]^\alpha,$$

see Triebel [**21**]. This function space can be characterized in terms of the error for best uniform approximation by trigonometric polynomials. Indeed, let

$$E_n(f) := \inf_{v \in \mathcal{T}_n} \|v - f\|_\infty,$$

then for all $s > 0$,

$$f \in \mathcal{H}^s \quad \Longleftrightarrow \quad E_n(f) = O(n^{-s}) \text{ as } n \to \infty;$$

see [**11**, p. 197, 201] or [**20**, p. 260, 333].

Let $\mathcal{P}_n$ be the orthogonal projection from L_2 onto $\mathcal{T}_n$, then $\mathcal{P}_n f$ is the best least-squares approximation to f and is also the n-th partial sum of the Fourier series (1.4), i.e.,

$$(\mathcal{P}_n f)(t) = \sum_{|l| \leq n} \hat{f}(l) e^{ilt}, \qquad n \in \mathbf{N}_0.$$

In terms of the Dirichlet kernel

$$D_n(t) := \sum_{|l| \leq n} e^{ilt} = \frac{\sin[(2n+1)t/2]}{\sin(t/2)},$$

we have

$$(\mathcal{P}_n f)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(t-x)f(x)\,dx. \tag{2.2}$$

For $f \in C$ and $n \in \mathbf{N}_0$, let $\mathcal{L}_n f$ be the trigonometric polynomial of degree n, which interpolates f at the points (1.2), so that $\mathcal{L}_n f \in \mathcal{T}_n$ and

$$(\mathcal{L}_n f)(t_{k,n}) = f(t_{k,n}) \qquad \text{for } |k| \le n. \tag{2.3}$$

The Dirichlet kernel satisfies

$$D_n(t_{k,n}) = \begin{cases} 2n+1, & \text{if } k = 0 (\operatorname{mod} 2n+1) \\ 0, & \text{if } k \ne 0 (\operatorname{mod} 2n+1); \end{cases}$$

therefore

$$(\mathcal{L}_n f)(t) = \frac{1}{2n+1}\sum_{|k|\le n} f(t_{k,n})D_n(t-t_{k,n}) \tag{2.4}$$

and

$$(\mathcal{L}_n f)\hat{}(l) = \frac{1}{2n+1}\sum_{|k|\le n} e^{-ilt_{k,n}} f(t_{k,n}), \qquad |l| \le n.$$

(Of course, $(\mathcal{L}_n f)\hat{}(l) = 0$ for $|l| \ge n+1$.)

The following properties of the projections $\mathcal{P}_n$ and $\mathcal{L}_n$ will be used later. Note that

$$\mathcal{P}_n^2 = \mathcal{P}_n, \mathcal{L}_n^2 = \mathcal{L}_n \text{ and } \mathcal{P}_n\mathcal{L}_n = \mathcal{L}_n, \mathcal{L}_n\mathcal{P}_n = \mathcal{P}_n.$$

From now on, we will always assume implicitly that $n \ge 2$, in order to ensure that $\log n$ is non zero.

THEOREM 2.1. *If $0 < s < r < \infty$ and $f \in \mathcal{H}^r$, then*

1. $\|(\mathcal{P}_n - I)f\|_{\mathcal{H}^s} \le c(1/n)^{r-s}(\log n)\|f\|_{\mathcal{H}^r}$,
2. $\|(\mathcal{L}_n - I)f\|_{\mathcal{H}^s} \le c(1/n)^{r-s}(\log n)\|f\|_{\mathcal{H}^r}$,

and, in both cases, the constant c depends only on the integer part of r.

PROOF. For the case $0 < s < r < 1$, Part 1 was first proved by Prößdorf in [**14**] and both parts can be found with proofs in the book by Prößdorf and Silbermann [**16**]. A detailed proof of Part 1 for the general case is given in [**7**]. Both parts of the theorem are also discussed by Prestin [**13**] for non-integer values of s and r. Here is the proof of Part 2.

Recall Bernstein's inequality [**23**, Vol. 2, p. 11]

$$\|Dg\|_\infty \le n\|g\|_\infty \qquad \text{for all } g \in \mathcal{T}_n; \tag{2.5}$$

we claim that if $0 < \alpha \le 1$, then

$$[g]^\alpha \le 2n^\alpha \|g\|_\infty \qquad \text{for all } g \in \mathcal{T}_n. \tag{2.6}$$

Suppose first that $0 < \alpha < 1$. On the one hand, if $h \ge n^{-1}$, then

$$\frac{\|\Delta_h g\|_\infty}{h^\alpha} \le \frac{2\|g\|_\infty}{h^\alpha} \le 2n^\alpha \|g\|_\infty,$$

and on the other hand, since $(\Delta_h g)(t) = h\,Dg(\xi)$ for some $\xi \in [t, t+h]$, if $h < n^{-1}$, then

$$\frac{\|\Delta_h g\|_\infty}{h^\alpha} \le h^{1-\alpha}\|Dg\|_\infty < (n^{-1})^{1-\alpha} n\|g\|_\infty = n^\alpha \|g\|_\infty.$$

Now suppose $\alpha = 1$. If $h \ge 2/n$, then

$$\|\Delta_h^2 g\|_\infty / h \le 4\|g\|_\infty / h \le 2n\|g\|_\infty.$$

For $h < 2/n$ use $\Delta_h^2 g(t) = h^2 D^2 g(\xi)/2$ for some $\xi \in [t, t+2h]$, then

$$\|\Delta_h^2 g\|_\infty / h \le h^{-1}(h^2/2)\|D^2 g\|_\infty \le (h/2)n^2\|g\|_\infty < n\|g\|_\infty.$$

This completes the proof of (2.6).

Let $0 < s < r < \infty$ and $f \in \mathcal{H}^r$. The triangle inequality implies

$$\|(\mathcal{L}_n - I)f\|_{\mathcal{H}^s} \le \|(\mathcal{P}_n - I)f\|_{\mathcal{H}^s} + \|(\mathcal{L}_n - \mathcal{P}_n)f\|_{\mathcal{H}^s},$$

and in view of Part 1, it suffices to estimate the $\mathcal{H}^s$-norm of

$$g_n := (\mathcal{L}_n - \mathcal{P}_n)f \in \mathcal{T}_n.$$

Let $s = m + \alpha$ as in (2.1), then (2.5) and (2.6) imply

$$\|g_n\|_{\mathcal{H}^s} \leq \Big(\sum_{j=1}^{m} n^j + 2n^s\Big)\|g_n\|_\infty \leq (s+2)n^s\|g_n\|_\infty.$$

The operator norms $\|\mathcal{P}_n\|_\infty$ and $\|\mathcal{L}_n\|_\infty$ of $\mathcal{P}_n, \mathcal{L}_n : C \to C$ generated by the maximum norm $\|\cdot\|_\infty$ on C satisfy the inequalities

$$\|\mathcal{P}_n\|_\infty \leq c\log n, \qquad \|\mathcal{L}_n\|_\infty \leq c\log n,$$

as can be verified by using the representations (2.2) and (2.4), see [**2**, p. 105] and [**10**, p. 390], respectively. Thus, the triangle inequality gives

$$\begin{aligned}\|g_n\|_\infty &\leq \|(\mathcal{L}_n - I)f\|_\infty + \|(\mathcal{P}_n - I)f\|_\infty \\ &\leq (1 + \|\mathcal{L}_n\|_\infty)E_n(f) + (1 + \|\mathcal{P}_n\|_\infty)E_n(f) \\ &\leq c\,(\log n)E_n(f),\end{aligned}$$

and so $\|g_n\|_{\mathcal{H}^s} \leq cn^s(\log n)E_n(f)$. The result now follows, because $E_n(f) \leq cn^{-r}\|f\|_{\mathcal{H}^r}$. □

3. Singular integral operators. In the usual way [**4**], the periodic singular integral operator A is written in the form

$$A = aP + bQ + K,$$

where P and Q are the complementary projection operators defined by

$$Pu(t) := \sum_{l\geq 0} \hat{u}(l)e^{ilt}, \qquad Qu(t) := \sum_{l\leq -1} \hat{u}(l)e^{ilt},$$

and where $K : \mathcal{H}^s \to \mathcal{H}^s$ is a compact linear operator for every $s > 0$. For simplicity, assume that the coefficients a and b belong to C^∞, then

$$A : \mathcal{H}^s \to \mathcal{H}^s, \qquad s > 0,$$

is a bounded linear operator. If the coefficients satisfy

$$|a(t)|^2 + |b(t)|^2 \neq 0 \qquad \text{for all } t \in \mathbf{R},$$

then A is said to be *elliptic* (or non-degenerate).

The function $t \to a(t)$ parametrizes a smooth, closed curve in the complex plane, whose winding number (about the origin) is denoted by

$$W(a) := (1/2\pi)[\arg a(t)]_{t=-\pi}^{\pi}.$$

Also, the kernel, image, cokernel and index of $A : \mathcal{H}^s \to \mathcal{H}^s$ are denoted by

$$\begin{aligned}
\ker(A) &:= \{u \in \mathcal{H}^s : Au = 0\},\\
\operatorname{im}(A) &:= \{f \in \mathcal{H}^s : f = Au \text{ for some } u \in \mathcal{H}^s\},\\
\operatorname{coker}(A) &:= \mathcal{H}^s/\overline{\operatorname{im}(A)},\\
\operatorname{ind}(A) &:= \dim\ker(A) - \dim\operatorname{coker}(A).
\end{aligned}$$

The index theorem for singular integral operators states that A is a Fredholm operator if and only if A is elliptic, in which case

$$\operatorname{ind}(A) = W(b) - W(a). \tag{3.1}$$

Essentially, this is a classical result of F. Noether [**12**]. For the case $0 < s < 1$, a proof may be found in the well-known book by Muskhelishvili [**9**, p. 143], and modern treatments are given by Gohberg and Krupnik [**4**, p. 196] and by Mikhlin and Prößdorf [**8**, p. 83]. In [**7**], we have given a fairly self-contained proof for the general case $0 < s < \infty$.

Let

$$\begin{aligned}
C_+^\infty &:= \{f \in C^\infty : \hat{f}(l) = 0 \text{ for all } l \le 0\}\\
C_-^\infty &:= \{f \in C^\infty : \hat{f}(l) = 0 \text{ for all } l \ge 0\},
\end{aligned}$$

then the pointwise multiplication operators associated with functions $a_\pm \in C_\pm^\infty$ satisfy

$$\begin{aligned}
&Pa_+P = a_+P, \qquad Qa_+Q = Qa_+, \qquad Qa_+P = 0,\\
&Pa_-P = Pa_-, \qquad Qa_-Q = a_-Q, \qquad Pa_-Q = 0,
\end{aligned} \tag{3.2}$$

cf. [**3**, p. 126] or [**7**, Theorem 3.5]. Given $a \in C^\infty$, a representation

$$a(t) = a_+(t)e^{i\kappa t}a_-(t), \qquad t \in \mathbf{R},$$

is said to be a *factorization* of a if

$$\kappa \in \mathbf{Z}, \qquad a_\pm \in C_\pm^\infty, \qquad 1/a_\pm \in C_\pm^\infty.$$

The integer κ is uniquely determined by the function a since $\kappa = W(a)$, and if $a(t) = \tilde{a}_+(t)e^{i\kappa t}\tilde{a}_-(t)$ is another factorization, then $\tilde{a}_+/a_+ = a_-/\tilde{a}_- =$ constant. A factorization of $a \in C^\infty$ exists if and only if $a(t) \neq 0$ for all t. These facts are all elementary, see [**7**, Theorems 3.4 and 3.5], but it is worth noting that the existence of factorizations of continuous functions and of matrix-valued functions is more difficult to establish; see [**3**], [**4**] or [**8**].

Suppose that $A = aP + bQ + K$ is elliptic with $W(a) = W(b)$, then the formula (3.1) shows that $\operatorname{ind}(A) = 0$. Furthermore, $W(b^{-1}a) = W(a) - W(b) = 0$, so there exists a factorization

$$b^{-1}a = \rho_+\rho_-, \qquad \rho_\pm \in C^\infty_\pm.$$

Define the operators

$$M := b\rho_+, \qquad N := P\rho_- + Q\rho_+^{-1},$$

then, by using the identities (3.2) and the fact that $P + Q = I$, it is easy to see that

$$M^{-1} := \rho_+^{-1}b^{-1}, \qquad N^{-1} := P\rho_-^{-1} + Q\rho_+.$$

Let $[\cdot,\cdot]$ be the usual commutator bracket, and define

$$T := M^{-1}K + [\rho_-, P] + [\rho_+^{-1}, Q], \tag{3.3}$$

then elementary algebra gives

$$A = M(N + T). \tag{3.4}$$

This representation is the basis for the stability analysis of the next section.

4. Error estimates. It is clear from (1.3) and (2.3) that the trigonometric collocation method can be thought of as the projection method determined by the interpolation operator $\mathcal{L}_n$. In other words, the functions u and u_n from §1 satisfy

$$Au = f, \qquad \mathcal{L}_nAu_n = \mathcal{L}_nf, \qquad \mathcal{L}_nu_n = u_n. \tag{4.1}$$

Suppose that $A = aP + bQ + K$ is a periodic singular integral operator, and consider the following assumptions.

A1 The singular integral operator A is elliptic, with $W(a) = W(b)$ and $\ker(A) = \{0\}$.

A2 There is an $\varepsilon > 0$ such that $K : \mathcal{H}^s \to H^{s+\varepsilon}$ is bounded for every $s > 0$.

A3 There is an $\varepsilon > 0$ such that $K : C \to \mathcal{H}^\varepsilon$ is bounded.

It is clear from the index formula (3.1) that the condition **A1** is necessary and sufficient for the invertibility of the operator $A : \mathcal{H}^s \to \mathcal{H}^s (s > 0)$. Prößdorf [**15**] has shown, for the case $K = 0$, that **A1** is also necessary and sufficient for $u_n \to u$ in $\mathcal{H}^s$ whenever $f \in \mathcal{H}^r$ and $0 < s < r < 1$. The assumptions **A2** and **A3** are of a technical nature. Both will be satisfied if, for example, K is a periodic integral operator with a weakly singular kernel function. In applications, it often happens that K is a logarithmic convolution, or that K has a C^∞ kernel function.

When **A1** is not satisfied, it may nevertheless be possible to obtain approximate solutions of $Au = f$ by the following modification due to Mikhlin and Prößdorf [**8**, p. 442-443].

With $\kappa := W(a) - W(b) \neq 0$ introduce the new operator

$$C := e_{-\kappa} aP + bQ + K(e_{-\kappa} P + Q),$$

whose coefficients have equal winding numbers. Now let $v_n = \sum_{|l| \leq n} \xi_l e_l$ denote the approximate solution of the equation $Cv = f$ obtained by applying the collocation method. Then the sequence

$$u_n := \sum_{l=0}^{n} \xi_l e_{l-\kappa} + \sum_{l=-n}^{-1} \xi_l e_l$$

converges to a solution u of the original equation $Au = f$ provided a solution u exists.

The following lemma is an immediate consequence of the fact that, for any $a \in C^\infty$, the commutators $[a, P]$ and $[a, Q]$ are periodic integral operators with C^∞ kernel functions.

LEMMA 4.1. *Let T be the operator defined by* (3.3).

1. If **A2** *holds, then* $T : \mathcal{H}^s \to \mathcal{H}^{s+\varepsilon}$ *is bounded for every* $s > 0$.

2. If **A2** *and* **A3** *hold, then* $T : C^s \to \mathcal{H}^{s+\varepsilon}$ *is bounded for every* $s \in \mathbf{N}_0$.

In the proof of the next theorem, as in the book by Prößdorf and Silbermann [**16**, p. 99], the following identities play a crucial role.

LEMMA 4.2. *For any* $a \in C^\infty$ *and* $a_\pm \in C^\infty_\pm$,

$$\mathcal{L}_n a \mathcal{L}_n = \mathcal{L}_n a, \qquad \mathcal{L}_n(Pa_- + Qa_+)\mathcal{L}_n = (Pa_- + Qa_+)\mathcal{L}_n.$$

PROOF. It is obvious from the definition of $\mathcal{L}_n$ that $\mathcal{L}_n a \mathcal{L}_n = \mathcal{L}_n a$. To prove the second identity, note that Gohberg and Fel'dman show in [**3**, p.71] the identities

$$\begin{aligned} \mathcal{P}_n(a_+P + a_-Q)\mathcal{P}_n &= \mathcal{P}_n(a_+P + a_-Q), \\ \mathcal{P}_n(Pa_- + Qa_+)\mathcal{P}_n &= (Pa_- + Qa_+)\mathcal{P}_n. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{L}_n(Pa_- + Qa_+)\mathcal{L}_n &= \mathcal{L}_n(Pa_- + Qa_+)\mathcal{P}_n\mathcal{L}_n \\ &= \mathcal{L}_n\mathcal{P}_n(Pa_- + Qa_+)\mathcal{P}_n\mathcal{L}_n \\ &= \mathcal{P}_n(Pa_- + Qa_+)\mathcal{P}_n\mathcal{L}_n \\ &= (Pa_- + Qa_+)\mathcal{P}_n\mathcal{L}_n \\ &= (Pa_- + Qa_+)\mathcal{L}_n. \ \square \end{aligned}$$

THEOREM 4.3. *Suppose* $0 < s < r < \infty$. *If* **A1** *and* **A2** *are satisfied, then for all n sufficiently large, there exists a unique collocation solution* u_n, *and*

$$\|u_n - u\|_{\mathcal{H}^s} \le c(1/n)^{r-s}(\log n)\|u\|_{\mathcal{H}^r}.$$

PROOF. Equations (3.4) and (4.1) imply

$$M(N+T)u = f, \tag{4.2}$$

$$\mathcal{L}_n M(N+T)u_n = \mathcal{L}_n f, \tag{4.3}$$

and Lemma 4.2 implies

$$(4.4) \qquad \mathcal{L}_n M^{\pm 1}\mathcal{L}_n = \mathcal{L}_n M^{\pm 1}, \qquad \mathcal{L}_n N^{\pm 1}\mathcal{L}_n = N^{\pm 1}\mathcal{L}_n,$$

therefore we may proceed as in [**16**, Chapter 4.3] and [**7**, Theorem 4.3]. Indeed, (4.4) shows that $\mathcal{L}_n M^{-1}\mathcal{L}_n$ is the inverse of the finite-dimensional operator $\mathcal{L}_n M \mathcal{L}_n : \mathcal{T}_n \to \mathcal{T}_n$, hence by multiplying (4.3) on the left by $\mathcal{L}_n M^{-1}$, one obtains

$$\mathcal{L}_n(N+T)u_n = \mathcal{L}_n M^{-1} f.$$

Furthermore, $\mathcal{L}_n N u_n = \mathcal{L}_n N \mathcal{L}_n u_n = N\mathcal{L}_n u_n$, so

$$(N + \mathcal{L}_n T)u_n = \mathcal{L}_n M^{-1} f,$$

and in view of (4.2), it follows that

$$(4.5) \qquad (N + \mathcal{L}_n T)(u_n - u) = (\mathcal{L}_n - I)Nu.$$

Part 1 of Lemma 4.1 and Part 2 of Theorem 2.1 together imply that the operator norm $\|\mathcal{L}_n T - T\|_{\mathcal{H}^s \to \mathcal{H}^s}$ tends to zero as $n \to \infty$. Moreover, the operator $N+T : \mathcal{H}^s \to \mathcal{H}^s$ is invertible because $\operatorname{ind}(N+T) = 0$ and $\ker(N+T) = \ker(A) = \{0\}$, so we conclude that for all n sufficiently large, the perturbed operator $N + \mathcal{L}_n T$ has an inverse satisfying the uniform bound

$$\|(N + \mathcal{L}_n T)^{-1}\|_{\mathcal{H}^s \to \mathcal{H}^s} \le c.$$

The result is now an immediate consequence of (4.5) and Part 2 of Theorem 2.1, bearing in mind that $N : \mathcal{H}^r \to \mathcal{H}^r$ is bounded. □

It is not possible to replace $\mathcal{H}^s$ by C^s in the proof above, since $(N+T)^{-1}$ fails to be bounded on C^s (unless $a = b$). The way around this difficulty is to multiply (4.5) on the left by N^{-1}, and obtain a new equation for the error, namely,

$$(4.6) \qquad (I + N^{-1}\mathcal{L}_n T)(u_n - u) = N^{-1}(\mathcal{L}_n - I)Nu.$$

Here, the operator on the left is a perturbation of $I + N^{-1}T$, which does possess a bounded inverse on C^s.

THEOREM 4.4. *If $s \in \mathbf{N}_0$ and $s < r < \infty$, then for all n sufficiently large*

$$\|u_n - u\|_{C^s} \leq c(1/n)^{r-s}(\log n)^2\|u\|_{\mathcal{H}^r}.$$

PROOF. The inclusion $\mathcal{H}^{s+\varepsilon} \subset C^s$ is compact, so by Part 2 of Lemma 4.1, the linear operator $N^{-1}T : C^s \to C^s$ is compact. It is easy to see that $\ker(I + N^{-1}T) = 0$. Hence, $I + N^{-1}T : C^s \to C^s$ is invertible. Using Lemma 4.1 again, together with Part 2 of Theorem 2.1, we find that $\|N^{-1}\mathcal{L}_nT - N^{-1}T\|_{C^s \to C^s}$ tends to zero as $n \to \infty$ since

$$\begin{aligned}\|(N^{-1}\mathcal{L}_nT - N^{-1}T)v\|_{C^s} &\leq c\|(\mathcal{L}_n - I)Tv\|_{\mathcal{H}^{s+\varepsilon/2}} \\ &\leq cn^{-\varepsilon/2}\log n\|Tv\|_{\mathcal{H}^{s+\varepsilon}} \\ &\leq cn^{-\varepsilon/2}\log n\|v\|_{C^s}.\end{aligned}$$

Consequently, for all n sufficiently large, the inverse of $I + N^{-1}\mathcal{L}_nT$ exists and satisfies the uniform bound

$$\|(I + N^{-1}\mathcal{L}_nT)^{-1}\|_{C^s \to C^s} \leq c.$$

Hence, by (4.6), the error in the C^s norm satisfies

$$\|u_n - u\|_{C^s} \leq c\|N^{-1}(\mathcal{L}_n - I)N\|_{C^s},$$

and it only remains to estimate the right hand side.

By considering the periodic Hilbert transform as in [**7**, Lemmas 4.6, 4.7], it is not difficult to show that N^{-1} satisfies the estimate

$$\|N^{-1}v\|_{C^s} \leq c\varepsilon^{-1}\|v\|_{\mathcal{H}^{s+\varepsilon}}, \tag{4.7}$$

where the constant c is independent of $\varepsilon \in (0,1]$. Using Part 2 of Theorem 2.1,

$$\begin{aligned}\|N^{-1}(\mathcal{L}_n - I)Nu\|_{C^s} &\leq c\varepsilon^{-1}\|(\mathcal{L}_n - I)Nu\|_{\mathcal{H}^{s+\varepsilon}} \\ &\leq c\varepsilon^{-1}(1/n)^{r-(s+\varepsilon)}(\log n)\|Nu\|_{\mathcal{H}^r} \\ &\leq c\,(1/n)^{r-s}(n^\varepsilon/\varepsilon)(\log n)\|u\|_{\mathcal{H}^r},\end{aligned}$$

and if we choose $\varepsilon = 1/\log n$, then $n^\varepsilon/\varepsilon = e\log n$. □

In conclusion, note that if $r \in \mathbf{N}$, then $\|u\|_{\mathcal{H}^r} \leq c\|u\|_{C^r}$, and the error estimate given in the Introduction follows at once from the theorem above. Furthermore, we point out that if $a = b$ (in other words, if $A = aI + K$ is not really a *singular* integral operator) then it is easy to see that the pointwise error estimate for the trigonometric collocation solution involves only one factor of $\log n$, rather than two such factors.

5. Periodic pseudodifferential equations. We consider equations with operators

$$Bv = (aP + bQ + K)\Lambda^\beta v = f \tag{5.1}$$

where the operator Λ^β denotes the Bessel potential operator of order $\beta \in \mathbf{R}$, given by

$$\Lambda^\beta e_l = |l + \delta_{0l}|^\beta e_l \tag{5.2}$$

and corresponding continuous extensions. Here δ_{0l} denotes the Kronecker symbol. For $\beta = 0, B$ is just a singular integral operator.

In view of Agranovich's Theorem [**1**], [**17**], any one-dimensional pseudodifferential operator of order β acting on periodic functions (or functions defined on a closed curve) can be written in the form (5.1). If B_0 is the principal part of B with

$$B_0 u(x) = \sum_{k \in \mathbf{Z}} \sigma_0(x, k)\hat{u}(k)e^{ikx},$$

then the principal symbol $\sigma_0(x, k)$ of B is 2π-periodic in x and satisfies

$$\sigma_0(x, k) = \begin{cases} |k|^\beta \sigma_0(x, k/|k|), & k \neq 0, \\ 1, & k = 0, \end{cases}$$

and

$$a(x) := \sigma_0(x, +1), \qquad b(x) := \sigma_0(x, -1). \tag{5.3}$$

Therefore we can write

$$B_0 = (aP + bQ)\Lambda^\beta.$$

The following mapping property of Λ^β allows the extension of the former results to (5.1), see [**18**, p.149]. In particular, for $s > 0$, $s - \beta > 0$, the mapping $\Lambda^\beta : \mathcal{H}^s \to \mathcal{H}^{s-\beta}$ is an isomorphism. Hence, as in [**7**] the definition of the spaces $\mathcal{H}^s$ can be extended to arbitrary $s \in \mathbf{R}$ by defining $\mathcal{H}^s$ to be the set of all periodic distributions f satisfying $\Lambda^{-\beta} f \in \mathcal{H}^{s+\beta}$ for some (and hence all) β with $s + \beta > 0$, and then equipping $\mathcal{H}^s$ with the norm

$$\|f\|_{\mathcal{H}^s} := \|\Lambda^{-\beta} f\|_{\mathcal{H}^{s+\beta}}. \tag{5.4}$$

Different choices of β lead to equivalent norms. Note that the operator Λ^β commutes with $\mathcal{P}_n$, with P and with Q.

Just as for the case of a singular integral operator ($\beta = 0$), we say that B is *elliptic* if the functions (5.3) satisfy

$$a(x) \neq 0 \quad \text{and} \quad b(x) \neq 0 \qquad \text{for all } x \in \mathbf{R},$$

but note that $K : \mathcal{H}^s \to \mathcal{H}^s$ is now assumed to be compact for all $s \in \mathbf{R}$. Obviously, when B is elliptic, the mapping

$$B : \mathcal{H}^s \to \mathcal{H}^{s-\beta}, \qquad s \in \mathbf{R},$$

is an isomorphism if and only if $W(a) = W(b)$ and $\ker(B) = \{0\}$.

Let us write $A = aP + bQ + K$, then the equation

$$Bv = A\Lambda^\beta v = f \tag{5.5}$$

is equivalent to

$$Au = f \qquad \text{with } u = \Lambda^\beta v.$$

The trigonometric collocation method for (5.5) reads as to find $v_n \in \mathcal{T}_n$ satisfying

$$\mathcal{L}_n B v_n = \mathcal{L}_n f \tag{5.6}$$

and is equivalent to finding

$$u_n = \Lambda^\beta v_n \in \mathcal{T}_n \tag{5.7}$$

satisfying

$$\mathcal{L}_n A u_n = \mathcal{L}_n f, \tag{5.8}$$

i.e., (1.2). In the usual way, (5.6) and (5.8) lead to systems of equations for the Fourier coefficients of v_n and u_n, respectively. The two systems are related by a simple rescaling of the unknowns, as can be seen from (5.7).

Let us formulate assumptions for B corresponding to the earlier assumptions for A.

B1 The pseudodifferential operator B, given in the form (5.1), is elliptic with $W(a) = W(b)$ and $\ker(B) = \{0\}$.

B2 There is an $\varepsilon > 0$ such that K is a pseudodifferential operator of order $-\varepsilon$.

Assumption **B2** implies that K satisfies **A2** and **A3** from §4, because K must be a periodic integral operator whose kernel behaves along the diagonal $x = t$ like $|x-t|^{\varepsilon-1}$ if $0 < \varepsilon < 1$, and like $\log|x-t|$ if $\varepsilon = 1$; see [**19**, p.40].

THEOREM 5.1. *Suppose $\beta < s < r < \infty$. If B satisfies* **B1** *and* **B2**, *then for all n sufficiently large, there exists a unique collocation solution v_n of* (5.6), *and*

$$\|v_n - v\|_{\mathcal{H}^s} \le c(1/n)^{r-s}(\log n)\|v\|_{\mathcal{H}^r}.$$

When $s \in \mathbf{N}_0$,

$$\|v_n - v\|_{C^s} \le c(1/n)^{r-s}(\log n)^2\|v\|_{\mathcal{H}^r}.$$

PROOF. By using Theorem 4.3 and the relations (5.4) through (5.8) we find

$$\begin{aligned}\|v_n - v\|_{\mathcal{H}^s} &= \|\Lambda^{-\beta}(u_n - u)\|_{\mathcal{H}^s}\\ &= \|u_n - u\|_{\mathcal{H}^{s-\beta}}\\ &\le c\,(1/n)^{(r-\beta)-(s-\beta)}\log n\|u\|_{\mathcal{H}^{r-\beta}}\\ &= c\,(1/n)^{(r-s)}\log n\|\Lambda^\beta v\|_{\mathcal{H}^{r-\beta}}\\ &= c\,(1/n)^{(r-s)}\log n\|v\|_{\mathcal{H}^r},\end{aligned}$$

which is the first estimate.

For proving the pointwise estimate we introduce the operators

$$\tilde{N} = N\Lambda^\beta,\ \tilde{T} = T\Lambda^\beta$$

and find the equation

$$(I + \tilde{N}^{-1}\mathcal{L}_n\tilde{T})(v_n - v) = \tilde{N}^{-1}(\mathcal{L}_n - I)\tilde{N}v$$

as in the proof of Theorem 4.3. Assumption **B2** together with the definition (3.3) of T implies that $\tilde{T} : \mathcal{H}^s \to \mathcal{H}^{s-\beta+\varepsilon}$ is continuous with $\varepsilon > 0$. Hence, $\tilde{N}^{-1}\tilde{T} = \Lambda^{-\beta}N^{-1}\tilde{T} : C^s \to C^s$ is a compact operator since $\mathcal{H}^{s+\delta}$ is compactly imbedded into C^s for any $\delta > 0$. As in the proof of Theorem 4.4 we have

$$\|(\tilde{N}^{-1}\mathcal{L}_n\tilde{T} - \tilde{N}^{-1}\tilde{T})v\|_{C^s} \leq cn^{-\varepsilon/2}\log n\|u\|_{C^s}$$

and, for n sufficiently large, the inverse of $I + \tilde{N}^{-1}\mathcal{L}_n\tilde{T}$ exists and is uniformly bounded,

$$\|(I + \tilde{N}^{-1}\mathcal{L}_n\tilde{T})^{-1}\|_{C^s \to C^s} \leq c.$$

Consequently,

$$\begin{aligned} \|v_n - v\|_{C^s} &\leq c\|\tilde{N}^{-1}(\mathcal{L}_n - I)\tilde{N}v\|_{C^s} \\ &\leq c\|N^{-1}\Lambda^{-\beta}(\mathcal{L}_n - I)\tilde{N}v\|_{C^s} \\ &\quad + c\|(N^{-1}\Lambda^{-\beta} - \Lambda^{-\beta}N^{-1})(\mathcal{L}_n - I)\tilde{N}v\|_{C^s}. \end{aligned} \tag{5.9}$$

For the first term on the right hand side we use (4.7) and Part 2 of Theorem 2.1 obtaining

$$\begin{aligned} \|N^{-1}\Lambda^{-\beta}(\mathcal{L}_n - I)\tilde{N}v\|_{C^s} &\leq c\varepsilon^{-1}\|\Lambda^{-\beta}(\mathcal{L}_n - I)\tilde{N}v\|_{\mathcal{H}^{s+\varepsilon}} \\ &\leq c\varepsilon^{-1}n^{s-r+\varepsilon}\log n\|\tilde{N}v\|_{\mathcal{H}^{r-\beta}} \\ &\leq c\varepsilon^{-1}n^{s-r+\varepsilon}\log n\|v\|_{\mathcal{H}^r}. \end{aligned}$$

For the second term on the right side of (5.9) we use the fact that the commutator $[N^{-1}, \Lambda^{-\beta}]$ is a pseudodifferential operator of order $-\beta-\delta$ with some $\delta > 0$ since $N^{-1}, \Lambda^{-\beta}$ and $\Lambda^{-\beta}N^{-1}$ have the same principal symbol (see [**19**, p.46, Theorem 4.4]) and we obtain

$$\begin{aligned} &\|(N^{-1}\Lambda^{-\beta} - \Lambda^{-\beta}N^{-1})(\mathcal{L}_n - I)\tilde{N}v\|_{C^s} \\ &\leq c\|[N^{-1}, \Lambda^{-\beta}](\mathcal{L}_n - I)\tilde{N}v\|_{\mathcal{H}^{s+\delta}} \\ &\leq c\|(\mathcal{L}_n - I)\tilde{N}v\|_{\mathcal{H}^{s-\beta}} \\ &\leq cn^{s-r}\log n\|v\|_{\mathcal{H}^r}. \end{aligned}$$

Both inequalities together yield

$$\|v_n - v\|_{C^s} \leq c\varepsilon^{-1} n^{s-r+\varepsilon} \log n \|v\|_{\mathcal{H}^r}$$

and the choice $\varepsilon = 1/\log n$ gives the desired estimate. □

6. Systems of pseudodifferential equations. As in [**7**] following the arguments by Prößdorf and Silbermann [**16**, Chap. 4, §3, §9 and 1.2], the analysis of the previous paragraphs can be extended to systems of singular integral equations and, further, to elliptic systems of pseudodifferential equations on Γ. As indicated in [**22**, §4], and using the results of [**1**], [**17**], every system of pseudodifferential equations on Γ can be written in the form

$$(6.1)\qquad Bv = (\sigma_0(x,+1)P + \sigma_0(x,-1)Q + K)\Lambda^\beta v = f,$$

where σ_0 is the principal symbol of B, a matrix valued function associated with the 2π-periodic parametrization, and where $\beta = (\beta_1, \cdots, \beta_L) \in \mathbf{R}^L$ is a suitable vector of orders. Λ^β is defined by the diagonal matrix of operators

$$\Lambda^\beta = (\delta_{jk}\Lambda^{\beta_j}),$$

$j, k = 1, \cdots, L, \delta_{jk}$ the Kronecker symbol, K satisfies assumptions **A2** and **A3**. Equations of the type (6.1) include a large class of integro-differential equations [**6**] and boundary integral equations of various types resulting from the reduction to the boundary Γ of elliptic two-dimensional boundary value problems [**22**]. Now, with

$$u_n = \Lambda^\beta v_n,$$

the trigonometric collocation method for (6.1) reads as to find $v_n \in \mathcal{T}_n^L$ satisfying

$$(6.2)\qquad Bv_n(t_{k,n}) = f(t_{k,n}), \qquad |k| \leq n.$$

Then (6.2) is equivalent to

$$(6.3)\qquad \mathcal{L}_n A u_n = \mathcal{L}_n f$$

with the operator

$$A = \sigma_0(x, +1)P + \sigma_0(x, -1)Q + K \tag{6.4}$$

which defines a classical system of singular integral equations. For the application of the analysis of §4 to (6.3), assumption **A1** or **B1**, respectively, is to be replaced by corresponding properties for the system (6.4). Following Prößdorf and Silbermann [**16**, p.137], we assume:

B1' $\det \sigma_0(x, \pm 1) \neq 0$ for all x and the left indices of $\sigma_0(x, -1)^{-1}\sigma_0(x, +1)$ are all equal to zero, $\ker(B) = \{0\}$,

and **B2**. Assumption **B1**' guarantees the existence of the *matrix factorization*

$$\sigma_0(x, -1)^{-1}\sigma_0(x, +1) = \rho_+\rho_-$$

with $\rho_\pm \in (C^\infty_\mp)^{L\times L}$, having all the desired properties needed in §3 and §4. In [**6**] one finds some stronger but simpler conditions for σ_0 implying **B1**'.

THEOREM 6.1. *Suppose* $0 < s < r < \infty$. *If* B *satisfies* **B1**' *and* **B2**, *then for all* n *sufficiently large, there exists a unique collocation solution* v_n *of* (6.2), *and*

$$\begin{aligned} &\|v_n - v\|_{\mathcal{H}^{\beta+s}_L} \le c\,(1/n)^{r-s}(\log n)\|v\|_{\mathcal{H}^{\beta+r}_L} \text{ and} \\ &\|v_n - v\|_{C^{\beta+s}_L} \le c'(1/n)^{r-s}(\log n)^2\|v\|_{\mathcal{H}^{\beta+r}_L}. \end{aligned} \tag{6.5}$$

Here we denote $\mathcal{H}^{\beta+s}_L = \prod_{j=1}^L \mathcal{H}^{\beta_j+s}$ which is equipped with the norm

$$\|g\|_{\mathcal{H}^{\beta+s}_L} = \sum_{j=1}^L \|g_j\|_{\mathcal{H}^{\beta_j+s}}.$$

Correspondingly, we define $C^{\beta+s}_L$ and identify its components with non integer $\beta_j + s$ with $\mathcal{H}^{\beta_j+s}$.

As was indicated above, the proof follows as before if the function factorizations are replaced by the matrix factorizations; we omit these details.

The results in Theorem 6.1 can also be extended to systems which generalize (6.1) by incorporating additional compatibility conditions and unknown scalars as in [**6**].

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