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WEAKLY SINGULAR INTEGRAL OPERATORS AS MAPPINGS BETWEEN FUNCTION SPACES

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ABSTRACT. Weakly singular integral operators K are investigated as mappings between function spaces of the Hilbert-Sobolev type defined on Riemannian manifolds M_n with boundary ∂M_n. The results obtained from this analysis are applied to the determination of function spaces for which the Fredholm integral equation of the first kind, $Ku = f$, admits solutions, and conditions on these function spaces are studied for which the boundary value problem $Ku = f$ in $M_n, u = g$ on ∂M_n has meaning.

1. Introduction. Let Ω_n be a bounded and open subset of $\mathbf{R}_n$, lying on one side of its boundary. The boundary of Ω_n, denoted by $\partial\Omega_n$, will be considered to be an infinitely differentiable manifold of dimension $n-1$.

Let K be the weakly singular integral operator K defined on the Sobolev space $H^s(\Omega_n), s \in \mathbf{R}$, by

$$(Ku)(x) = \int_{\Omega_n} k(x,y)u(y)dy,$$

where $0 \leq \alpha < n$ and $k(x,y) = 1/|x-y|^\alpha$.

The main purpose of this paper is to establish properties of weakly singular integral operators K as mappings between function spaces of the Hilbert-Sobolev type, and apply them to the study of the boundary value problem:

$$Ku = f \text{ in } \Omega_n \tag{1.1}$$

$$u = g \text{ on } \partial\Omega_n. \tag{1.2}$$

The action of K on certain subspaces H of $H^s(\Omega_n)$ is characterized, and these subspaces are shown to be mapped by K into $H^p(\Omega_n), q <$

$(n-\alpha)+s$. The function space of the boundary values of Ku for $u \in H$ is then determined.

The mapping properties of weakly singular integral operators K are shown to remain the same in the case when the function spaces are defined on Riemannian manifolds M_n with boundary ∂M_n, where M_n is assumed to be orientable, imbedded in $\mathbf{R}_{n+1}$, and homotopically equivalent to the unit ball D_n in $\mathbf{R}_n$ with homotopy equivalence $\phi \in C^\infty(D_n)$.

2. Preliminary results.

LEMMA 2.1. *The function* $u(\xi) = (1 + |\xi|^2)^{s/2} \in L^2(\mathbf{R}_n)$ *for all* $s < -n/2$.

PROOF.

$$||u||_{L^2(\mathbf{R}_n)} = \Big(\int_{\mathbf{R}_n} |u(\xi)|^2 d\xi\Big)^{1/2} = \Big(\int_{\mathbf{R}_n} |1 + |\xi|^2|^s d\xi\Big)^{1/2} < \infty$$

if and only if $s < -n/2$. □0

LEMMA 2.2. *Let* Δ *be the Laplace operator in* n *variables. The fundamental solution* γ *of the operator* $(1 - \Delta)$ *is in* $H^{s+2}(\mathbf{R}_n)$ *for all* $s < -n/2$.

PROOF. $(1 - \Delta)\gamma = \delta \Rightarrow (1 + |\xi|^2)\hat{\gamma} = 1 \Rightarrow \hat{\gamma} = 1/(1 + |\xi|^2)$. Lemma 2.1 $\Rightarrow (1 + |\xi|^2)^{(s+2)/2}\hat{\gamma} \in L^2(\mathbf{R}_n)$ for all $s < -n/2 \Rightarrow \gamma \in H^{s+2}(\mathbf{R}_n)$ for all $s < -n/2$. □0

The following result can be found in [**3**].

LEMMA 2.3. *If* $f \in H^s(\mathbf{R}_n)$ *and* $g \in H_0^t(\mathbf{R}_n)$, *where* s *and* t *are arbitrary real numbers, then* $f * g \in H^{s+t+(n/2)}_{\text{loc}}(\mathbf{R}_n)$.

LEMMA 2.4. *Let* $f(x) = |x|^{-\alpha}$ *where* $0 \le \alpha < n$ *and* $x \in \mathbf{R}_n$. *Then* $f \in H^s(\mathbf{R}_n)$ *for all* $s < (n/2) - \alpha$.

PROOF. $\hat{f}(\xi) = 2^{n-\alpha}\pi^{n/2}\Gamma < ((n-\alpha)/2)/\Gamma(\alpha/2))(1/|\xi|^{n-\alpha})$(see [2]). Then

$$(1+|\xi|^2)^{s/2} \cdot \hat{f}(\xi) = 2^{n-\alpha}\pi^{n/2}\frac{\Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)} \cdot (1+|\xi|^2)^{s/2} \cdot \frac{1}{|\xi|^{n-\alpha}}.$$

Lemma 2.1 implies $(1+|\xi|^2)^{s/2}.\hat{f}(\xi) \in L^2(\mathbf{R}_n)$ if $s - n + \alpha < -n/2$, i.e., $s < (n/2) - \alpha$. This implies $f \in H^s(\mathbf{R}_n)$ for all $s < (n/2) - \alpha$. □0

LEMMA 2.5. *Let*

$$\chi_{\Omega_n}(y) = \begin{cases} 1 \text{ if } y \in \overline{\Omega}_n \\ 0 \text{ if } y \in \mathbf{R}_n - \overline{\Omega}_n \end{cases}$$

Then $\chi_{\Omega_n} \in H^s(\mathbf{R}_n)$ *for all* $s < 1/2$.

PROOF. *Case* 1. Let $n = 2$ and let Ω_2 be the unit disc. Let $x = (x_1, x_2) \in \Omega_2$ and $\xi = (\xi_1, \xi_2) \in \mathbf{R}_2$. Then $\hat{\chi}_{\Omega_2}(\xi) = \int_{\Omega_2}\int e^{-i(\xi,x)}dx = \int_{\Omega_2}\int e^{-i[\xi_1x_1+\xi_2x_2]}dx_1dx_2$. Using polar coordinates: $x_1 = r\cos\theta, x_2 = r\sin\theta$, we have $\hat{\chi}_{\Omega_2}(\xi) = \int_0^{2\pi}\int_0^1 e^{-ir|\xi|\cos(\alpha-\theta)}$ $rdrd\theta$ where $\xi_1 = |\xi|\cos\alpha$ and $\xi_2 = |\xi|\sin\alpha$.

Consider the generating function for Bessel's functions of integral order (see [5]):

$$e^{(z/2)(t-1/t)} = \sum_{m=-\infty}^{\infty} J_m(z)t^m.$$

Letting $t = e^{-i(\theta+\pi/2)}$, we obtain

$$e^{-iz\sin(\theta+\pi/2)} = e^{-iz}\cos\theta = \sum_{m=-\infty}^{\infty} J_m(z)e^{-im(\theta+\pi/2)}.$$

Using this and Fubini's theorem, we get

$$\begin{aligned}\hat{\chi}_{\Omega_2}(\xi) &= \int_0^1 \Big(\sum_{m=-\infty}^{\infty}\int_0^{2\pi} J_m(r|\xi|)e^{-im[\alpha-\theta+\pi/2]}d\theta\Big)rdr \\ &= \int_0^1 \Big(\sum_{m=-\infty}^{\infty} J_m(r|\xi|)\int_0^{2\pi} e^{-im[\alpha-\theta+\pi/2]}d\theta\Big)rdr \\ &= 2\pi\int_0^1 J_0(r|\xi|)rdr.\end{aligned}$$

The last equality follows since $\int_0^{2\pi} e^{-im[\alpha-\theta+\pi/2]}d\theta = 0$ for all $m \neq 0$. Letting $y = r|\xi|$, we get $rdr = ydy/|\xi|^2$, and hence

$$2\pi \int_0^1 J_0(r|\xi|)rdr = \frac{2\pi}{|\xi|^2}\int_0^{|\xi|} J_0(y)ydy.$$

From the recursion formula $J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$, we have

$$\begin{aligned} J_1'(x) &= J_0(x) - \frac{1}{x}J_1(x) \\ &\Rightarrow J_1'(x) + \frac{1}{x}J_1(x) = J_0(x) \\ &\Rightarrow xJ_1'(x) + J_1(x) = xJ_0(x) \\ &\Rightarrow (xJ_1(x))' = xJ_1'(x) + J_1(x) = xJ_0(x) \end{aligned}$$

Hence,

$$\frac{2\pi}{|\xi|^2}\int_0^{|\xi|} J_0(y)\cdot ydy = \frac{2\pi}{|\xi|^2}\Big(|\xi|\cdot J_1(|\xi|)\Big)$$

$$= \frac{2\pi}{|\xi|}\cdot J_1(|\xi|) = \frac{2\pi}{|\xi|}\Big(\frac{\sqrt{2}}{\sqrt{\pi}\sqrt{|\xi|}}\Big)\Big(\cos\Big(|\xi| - \frac{\pi}{2} - \frac{\pi}{4}\Big) + 0(|\xi|^{-1})\Big).$$

Using the symbol "$\sim$" to denote asymptotic behavior in the variable ξ, we have

$$(1+|\xi|^2)^{s/2}\hat{\chi}_{\Omega_2}(\xi) \sim (1+|\xi|^2)^{s/2}.\frac{1}{|\xi|^{3/2}}.$$

Then Lemma 2.1 $\Rightarrow$ $(1+|\xi|^2)^{s/2}\hat{\chi}_{\Omega_2}(\xi) \in L^2(\mathbf{R}_2)$ for all s such that $s - 3/2 < -1$, and this implies $\chi_{\Omega_2}(y) \in H^s(\mathbf{R}_2)$ for all $s < 1/2$.

Case 2. Let Ω_2 be any bounded, open subset of $\mathbf{R}_2$ with boundary $\partial\Omega_2$.

By enclosing $\partial\Omega_2$ between two circles and using Case 1, we arrive at the same result.

Case 3. Let Ω_n be a bounded and open subset of $\mathbf{R}_n$, lying on one side of its boundary $\partial\Omega_n$, an infinitely differentiable manifold of dimension $n-1$.

The asymptotic behavior of $\hat{\chi}_{\Omega_n}(\xi)$ in the variable ξ is given by $\hat{\chi}_{\Omega_n}(\xi) \sim 1/|\xi|^{(n+1)/2}$. This implies $\chi_{\Omega_n} \in H^s(\mathbf{R}_n)$ for all s such that $s - ((n+1)/2) < -n/2$, i.e., for all s such that $s < 1/2$. □0

LEMMA 2.6. *Let* $v_p(y) = \chi_{\Omega_n}(y).e^{<p,y>}$. *Then:*

(i) $v_p \in H^s(\mathbf{R}_n)$ *for all* $s < 1/2$,

(ii) $v_p \in N(1-\Delta) \cap H^s(\mathbf{R}_n)$ for all $s < 1/2$ and if $|p|^2 = 1$.

PROOF.

(i).

$$\begin{aligned}\hat{v}_p(\xi) &= \int_{\mathbf{R}_n} v_p(y) e^{-i<y,\xi>} dy = \int_{\mathbf{R}_n} \chi_{\Omega_n}(y) e^{<y,p>} e^{<y,i\xi>} dy \\ &= \int_{R_n} \chi_{\Omega_n}(y) e^{<y,p+i\xi>} dy = \int_{R_n} \chi_{\Omega_n}(y) e^{<y,i(\xi-ip)>} dy \\ &= \int_{\mathrm{R}_n} \chi_{\Omega_n}(y) e^{-i<y,\xi-ip>} dy = \hat{\chi}_{\Omega_n}(\xi - ip).\end{aligned}$$

Therefore,

$$\hat{v}_p(\xi)(1+|\xi|^2)^{s/2} = \hat{\chi}_{\Omega_n}(\xi - ip)(1+|\xi|^2)^{s/2} \sim 1/|\xi - ip|^{(n+1)/2}. \\ (1+|\xi|^2)^{s/2}.$$

Lemma 2.1 $\Rightarrow$ $\hat{v}_p(\xi) \cdot (1+|\xi|^2)^{s/2} \in L^2(\mathbf{R}_n)$ for all s such that $s - ((n+1)/2) < -n/2$, i.e., for all $s < 1/2$. Hence, $v_p \in H^s(\mathbf{R}_n)$ for all $s < 1/2$.

(ii). This is immediate from (i) and because $|p|^2 = 1 \Rightarrow v_p \in N(1-\Delta)$. □0

LEMMA 2.7. *Let f be defined by* $f(x) = |x|^{-\alpha}, 0 \leq \alpha < n$, *for* $x \in \mathbf{R}_n$. *Let* $v_p(y) = \chi_{\Omega_n}(y) \cdot e^{<p,y>}$. *If* $w_p(x) = f * v_p$ *then* $w_p \in H^s(\mathbf{R}_n)$ *for all* $s < (n-\alpha) + (1/2)$.

PROOF. $\hat{w}_p(\xi) = \hat{f}(\xi) \cdot \hat{v}_p(\xi) = \hat{f}(\xi) \cdot \hat{\chi}_{\Omega_n}(\xi - ip)$. Lemmas 2.4 and 2.5 imply

$$\hat{w}_p(\xi) = \hat{f}(\xi).\hat{\chi}_{\Omega_n}(\xi - ip) \sim \frac{1}{|\xi|^{n-\alpha}} \cdot \frac{1}{|\xi - ip|^{(n+1)/2}}.$$

Then Lemma 2.1 implies

$$(1 + |\xi|^2)^{s/2} \cdot \hat{w}_p(\xi) \in L^2(\mathbf{R}_n)$$

for all s such that $s - n + \alpha - ((n+1)/2 < -n/2$, i.e., for all $s < (n - \alpha) + (1/2)$. □

3. Singular integral operators on subspaces of $H^s(\Omega_n)$.

LEMMA 3.1. *Let $v_p(y) = \chi_{\Omega_n}(y) \cdot e^{<p,y>}$. The operator K maps $v_p \in N(1 - \Delta) \cap H^s(\mathbf{R}_n), s < 1/2$ and $|p|^2 = 1$, into $Kv_p \in H^q(\mathbf{R}_n)$ for all $q < (n - \alpha) + (1/2)$.*

PROOF. Let $w_p = Kv_p$. Then $w_p(x) = (Ke^{<p,y>})(x) = \int_{\Omega_n} f(x - y)e^{<p,y>}dy = f * v_p$ implies, by Lemma 2.7, that $w_p \in H^q(\mathbf{R}_n)$ for all $q < (n - \alpha) + (1/2)$. □

LEMMA 3.2. *If $u \in H_0^s(\Omega_n)$, then $Ku \in H_{\text{loc}}^q(\mathbf{R}_n)$* for all $q < (n - \alpha) + s$.

PROOF. Since $u \in H_0^s(\Omega_n)$, we can write $Ku = f * u$. Lemma 2.4 implies $f \in H^t(\mathbf{R}_n)$ for all $t < (n/2) - \alpha$. Lemma 2.3 implies $Ku = f * u \in H_{\text{loc}}^{s+t+(n/2)}(\mathbf{R}_n)$ for all $t < (n/2) - \alpha$, i.e. , $Ku \in H_{\text{loc}}^q(\mathbf{R}_n)$ for all q such that $q = s + t + (n/2) < s + ((n/2) - \alpha) + (n/2) = (n - \alpha) + s$. □

In order to analyze the existence and nature of the boundary values of the image of $u \in H^s(\Omega_n)$ under the operator K, the action of K on certain subspaces $\mathcal{H}$ of $H^s(\Omega_n)$ is studied, and these subspaces are shown to be mapped by K into $H^q(\Omega_n), q < (n - \alpha) + s$. The function space of the boundary values of Ku for $u \in H$ is then determined.

A few terms needed in the sequel are introduced in the following definitions.

DEFINITION 3.1. Consider the equation $(1-\Delta)u = u_0$. If $u \in N(1-\Delta)$ in $\overline{\Omega}_n^c$ takes on boundary values $\phi = (\gamma_0 u, \gamma_1 u)^T = (u|_{\partial\Omega_n}, -\frac{\partial u}{\partial n}|_{\partial\Omega_n})^T = (\phi_1, -\phi_2)^T$ on $\partial\Omega_n$, then u is said to be outgoing with respect to $\partial\Omega_n$ if $u = 1/2L(\phi)$, where $L(\phi)$ is expressed in terms of the single layer operator S and the double layer operator D by $L(\phi) = D\phi_1 - S\phi_2$.

DEFINITION 3.2. Let u be a function that takes on boundary values $\phi = (u_+|_{\partial\Omega_n}, -\frac{\partial u}{\partial n}_+|_{\partial\Omega_n})^T$ on $\partial\Omega_n$. Then u is said to be incoming with respect to $\partial\Omega_n$, if $u \in N(1-\Delta)$ in Ω_n and $u = -1/2L(\phi)$.

DEFINITION 3.3. Let the operator Γ be defined by $\Gamma = \begin{pmatrix} K & Q \\ -\tilde{Q} & -K^T \end{pmatrix}$ where, for $\gamma(x) = (\Gamma(n/2)/\pi^{n/2}) \cdot (e^{ik|x|}/|x|^{(n-1)/2})$,

$$K\theta = \text{P.V.} \int_{\partial\Omega_n} \frac{\partial}{\partial n_y} \gamma(x-y)\theta(y)d\omega_y,$$

$$K^T\theta = \text{P.V.} \int_{\partial\Omega_n} \frac{\partial}{\partial n_x} \gamma(x-y)\theta(y)d\omega_y,$$

$$Q\theta = \text{P.V.} \int_{\partial\Omega_n} \gamma(x-y)\theta(y)d\omega_y,$$

$$\tilde{Q}\theta = \text{P.V.} \int_{\partial\Omega_n} \frac{\partial^2}{\partial n_x \partial n_y} \gamma(x-y)\theta(y)d\omega_y$$

For the properties of the above operators, see [**4**]. The proof of the following lemma can be found in [**3**].

LEMMA 3.3.

(i) *u is outgoing in Ω_n with respect to $\partial\Omega_n$ if and only if the boundary values ϕ satisfy $(-I+\Gamma)\phi = 0$.*

(ii) *u is incoming with respect to $\partial\Omega_n$ if and only if $(I+\Gamma)\phi = 0$.*

(iii) *if u takes on boundary ϕ where $u \in N(1-\Delta)$, then* $\phi = 1/2(I+\Gamma)\phi + 1/2(I-\Gamma)\phi$.

DEFINITION 3.4. $u \in N(1-\Delta)\cap H^s(\Omega_n)$ will be said to have smooth boundary values if the image of u under the trace operators γ_j is in $H^\infty(\partial\Omega_n) = \cap_{s\in\mathbf{R}} H^s(\partial\Omega_n) \subset \mathbf{C}^\infty(\partial\Omega_n)$.

LEMMA 3.4. *If K is defined on $H^s(\Omega_n)$ by $(Ku)(x) = \int_{\Omega_n} k(x,y)u(y)dy$, then the transpose of K, denoted K^T, is defined on $(H^q(\Omega_n))'$ by*

$$(K^T v)(x) = \int_{\Omega_n} \overline{k(y,x)} v(y)dy, \text{ where } q < (n-\alpha)+s.$$

Furthermore, $K^T : H_0^\sigma(\Omega_n) \to H_{loc}^{\sigma'}(R_n)$ for all $\sigma' < \sigma + n - \alpha$.

PROOF. Let $u \in H^s(\Omega_n)$ be such that $Ku \in H^q(\Omega_n), q < (n-\alpha)+s$. Let $v \in (H^q(\Omega_n))' = H_0^{-q}(\Omega_n)$. Then

$$\begin{aligned}
\langle K^T v, u\rangle_{\Omega_n} = \langle v, Ku\rangle_{\Omega_n} &= \int_{\Omega_n}\int_{\Omega_n} \overline{k(x,y)}\,\overline{u(y)} v(x)dxdy \\
&= \int_{\Omega_n} \overline{u(y)}\Big(\int_{\Omega_n} \overline{k(x,y)} v(x)dx\Big)dy \\
&= \int_{\Omega_n} \overline{u(x)}\Big(\int_{\Omega_n} \overline{k(y,x)} v(y)dy\Big)dx \\
&\Rightarrow (K^T v)(x) = \int_{\Omega_n} \overline{k(y,x)} v(y)dy.
\end{aligned}$$

In our case, $k(x,y) = 1/|x-y|^\alpha = \overline{k(y,x)}$. Hence Lemma 3.2 $\Rightarrow K^T : H_0^\sigma(\Omega_n) \to H_{\text{loc}}^{\sigma'}(\mathbf{R}_n)$ for all $\sigma' < \sigma + n - \alpha$. □

REMARK. As a matter of convenience, the following notation will be used: $\langle f, g\rangle_{\Omega_n} = \int_{\Omega_n} f(x)g(x)d\omega(x)$, where ω is a measure on Ω_n, shall indicate the inner product of two elements f and g in some Hilbert function space defined on the set Ω_n.

$B^s, s \in \mathbf{R}$, shall denote the cross-product space $H^{s-1/2}(\partial\Omega_n) \times H^{s-3/2}(\partial\Omega_n)$ of Sobolev spaces defined on the boundary $\partial\Omega_n$ of Ω_n.

B^s_- shall denote the space of boundary values of incoming functions taking on boundary values in B^s. Likewise, B^s_+ shall denote the space of boundary values of outgoing functions taking on boundary values in B^s.

THEOREM 3.1. *Let $u \in N(1-\Delta) \cap H^s(\Omega_n)$, $s \in \mathbf{R}$, be incoming with respect to $\partial\Omega_n$. Then $Ku \in H^q(\Omega_n)$ for all $q < (n-\alpha) + s$.*

PROOF. Let ϕ denote the boundary values of u on $\partial\Omega_n$. From Lemma 3.4, we have $K^T : H_0^\sigma(\Omega_n) \to H_{\text{loc}}^{\sigma'}(\mathbf{R}_n)$ for all $\sigma' < \sigma + n - \alpha$. Let $p \in H_0^{-q}(\Omega_n)$ where $-q > (\alpha - n) - s$. Then $K^T p \in H_{\text{loc}}^{\sigma'}(\mathbf{R}_n)$ for all $\sigma' < (-q) + n - \alpha$.

In particular, $-s < (-q) + n - \alpha$ implies

$$K^T p \in H_{\text{loc}}^{-s}(\mathbf{R}_n). \tag{3.1}$$

Let $\eta = \phi K^T p$ where $\phi = 1$ on $\overline{\Omega}_n$ and $\phi \in C_0^\infty(\mathbf{R}_n)$. Then

$$\eta \in H_0^{-s}(\mathbf{R}_n). \tag{3.2}$$

Let γ be the fundamental solution of $(1-\Delta)$ such that $\hat{\gamma}(\xi) = 1/1+|\xi|^2$. We then have

$$\gamma \in H^t(\mathbf{R}_n) \text{ for all } t < 2 - (n/2). \tag{3.3}$$

From (3.2), (3.3), and Lemma 2.3, we obtain

$$\gamma * \eta \in H_{\text{loc}}^{t-s+(n/2)}(\mathbf{R}_n) \quad \text{for all } t < 2 - (n/2)$$

and

$$\gamma * \eta \in H_{\text{loc}}^{2-s}(R_n). \tag{3.4}$$

Let $v = \gamma^* \eta$ and let ψ denote the boundary values of v on $\partial\Omega_n$. Since $u \in N(1-\Delta)$ is incoming with respect to $\partial\Omega_n$, we have $(I+\Gamma)\phi = 0$. Hence $\phi = 1/2(I+\Gamma)\phi + 1/2(I-\Gamma)\phi = 1/2(I-\Gamma)\phi$ and $[\phi, \psi] = [1/2(I-\Gamma)\phi, \psi] = [\phi, 1/2(I+\Gamma)\psi]$. Therefore $\psi = 1/2(I+\Gamma)\psi$. This implies $v = \gamma * \eta$ is outgoing in $\overline{\Omega}_n^c$ with respect to $\partial\Omega_n$ since

$1/2(-I+\Gamma)\psi = 0$. Hence, $v \in N(1-\Delta) \cap H^{2-s}_{\text{loc}}(\overline{\Omega}^c_n)$ takes on boundary values $\psi \in B^{2-s}_+ = H^{(3/2)-s}(\partial\Omega_n) \times H^{(1/2)-s}(\partial\Omega_n)$ by the trace theorem (see [**7**, pp. 41-43] or [**1**, pp. 189-200]).

Case 1. $s > 1/2$. The trace theorem implies $\phi \in B^s = H^{s-1/2}(\partial\Omega_n)\times H^{s-3/2}(\partial\Omega_n)$. We also have $(\Delta-1)u = 0$ and $(\Delta-1)v = \eta$ in Ω_n. Then $\langle \Delta v, u\rangle_{\Omega_n} - \langle u, v\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n}$ and $\langle \Delta u, v\rangle_{\Omega_n} - \langle u, v\rangle_{\Omega_n} = \langle 0, v\rangle_{\Omega_n} = 0$ implies $\langle \Delta v, u\rangle_{\Omega_n} - \langle \Delta u, v\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n}$.

Green's theorem implies $\langle u, \eta\rangle_{\Omega_n} = \int_{\partial\Omega_n}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)d\omega$, i.e., $\langle u, \eta\rangle_{\Omega_n} = \int_{\partial\Omega_n}(\phi_1\psi_2 - \psi_1\phi_2)d\omega$. We use the notation $[\phi, \psi]$ to denote the functional on $\partial\Omega_n$ defined by the boundary integral. Then $\langle Ku, p\rangle_{\Omega_n} = \langle u, K^T p\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n} = [\phi, \psi]$. By transposition (see [**6**, p. 164] or [**7**, p. 166]), $Ku \in (H^{-q}(\Omega_n))' = H^q(\Omega_n), q < (n-\alpha)+s$.

Case 2. $s \le 1/2$. If ϕ is smooth, then we have $\langle Ku, p\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n} = [\phi, \psi]$, and, by transposition, $Ku \in (H_0^{-q}(\Omega_n))' = H^q(\Omega_n), q < (n-\alpha)+s$.

If ϕ is not smooth, let ϕ_ν be a sequence of smooth boundary values of $u_\nu \in N(1-\Delta)\cap H^s(\Omega_n)$ converging to ϕ in the Hilbert space B^s, and where u_ν converges to u in $H^s(\Omega_n)$. Such a sequence of smooth boundary values ϕ_ν exists since $H^\infty(\partial\Omega_n)$ is dense in $H^s(\Omega_n)$ and B^∞ is dense in B^s.

We have $(\Delta-1)u_\nu = 0$ and $(\Delta-1)v = \eta$ in Ω_n. Green's theorem implies $\langle u_\nu, \eta\rangle_{\Omega_n} = [\phi_\nu, \psi]$. But $[\phi_\nu, \psi] \to [\phi, \psi]$ and $\langle u_\nu, \eta\rangle_{\Omega_\eta} \to \langle u, \eta\rangle_{\Omega_n}$ as $\nu \to +\infty$ implies $\langle u, \eta\rangle_{\Omega_n} = [\phi, \psi]$.

Hence, $\langle Ku, p\rangle_{\Omega_n} = \langle u, K^T p\rangle_{\Omega_n} = \langle u, \eta\rangle_{\Omega_n} = [\phi, \psi]$. By transposition, $Ku \in (H_0^{-q}(\Omega_n))' = H^q(\Omega_n), q < (n-\alpha)+s$. □

THEOREM 3.2. *Let $u \in N(1-\Delta)\cap H^s(\Omega_n)$ be incoming with respect to $\partial\Omega_n$. Let ϕ denote the boundary values of u on $\partial\Omega_n$, and let ψ denote the boundary values of Ku on $\partial\Omega_n$. Then $\psi \in B^q$, for all $q < (n-\alpha)+s$.*

PROOF. By Theorem 3.1, we have that $Ku \in H^q(\Omega_n)$ for all $q < (n-\alpha)+s$.

Case 1. If $q > 1/2$, then the trace theorem implies Ku has boundary values $\psi \in B^q$.

Case 2. If $q \leq 1/2$, then $2 - q \geq 3/2$. By definition, there exists $v \in H^q_{\mathrm{loc}}(\mathbf{R}_n)$ such that $Ku = r_{\Omega_n} v$. If γ is the fundamental solution of $(1 - \Delta)$ such that $\gamma \in H^t(\mathbf{R}_n)$ for all $t < 2 - (n/2)$, then there exists $w \in H_0^{q-2}(\mathbf{R}_n)$ such that $v = \gamma * w$.

Let ψ_1 denote the boundary values of $\gamma * w$. Let $\phi \in N(1 - \Delta) \cap H^{2-q}(\Omega_n)$ take on boundary values χ. Since $2 - q \geq 3/2$, the trace theorem implies $\chi \in B^{2-q}$. Since $(\Delta - 1)v = w$ and $(\Delta - 1)\phi = 0$ on Ω_n, an application of Green's theorem yields

$$\langle \phi, w|_{\Omega_n} \rangle = [\chi, \psi_1], \tag{3.5}$$

where $\phi \in N(1 - \Delta) \cap H^{2-q}(\Omega_n), w \in H_0^{q-2}(\mathbf{R}_n), \chi \in B^{2-q}$, and ψ_1 denotes the boundary values of $\gamma * w$ on $\partial\Omega_n$. From (3.5), we have that $\psi_1 \in B^q$. But v takes on boundary values ψ_1, and since Ku and v take on the same boundary values on $\partial\Omega_n$, we have that $\psi = \psi_1 \in B^q$. □

THEOREM 3.3. *Let* $H^{-q}_{\overline{\Omega}_n}(\mathbf{R}_n) = \{f/f \in H^{-q}(\mathbf{R}_n), f$ *with support in* $\overline{\Omega}_n\}$. *Then* $K^T : H^{-q}_{\overline{\Omega}_n}(\mathbf{R}_n) \to (H^s(\Omega_n))'$ *for all* s *such that* $-s < -q + (n - \alpha)$.

PROOF. Let $u \in N(1 - \Delta) \cap H^s(\Omega_n)$. Using the same notation and terminology as in the proof of Theorem 3.1, and using the fact that there exists a $w \in H^q(\mathbf{R}_n)$ such that $Ku = r_{\Omega_n} w$, we have

$$\begin{aligned} \langle Ku, p\rangle_{\Omega_n} &= \langle r_{\Omega_n} w, p\rangle_{\Omega_n} = \langle w, r^T_{\Omega_n} p\rangle \mathbf{R}_n \\ &= \langle Ku, r^T_{\Omega_n} p\rangle_{\overline{\Omega}_n} = \langle u, K^T r^T_{\Omega_n} p\rangle_{\overline{\Omega}_n}, \end{aligned} \tag{3.6}$$

where $r^T_{\Omega_n}$ is an isomorphism of $H_0^{-q}(\Omega_n) = (H^q(\Omega_n))'$ onto $H^{-q}_{\Omega_n}(\mathbf{R}_n)$, i.e., $r^T_{\Omega_n} : H_0^{-q}(\Omega_n) \to H^{-q}_{\overline{\Omega}_n}(\mathbf{R}_n)$ (see [**7**]). But (3.6) implies $K^T r^T_{\Omega_n} p H_0^{-s}(\Omega_n) = (H^s(\Omega_n))'$. We conclude that $K^T : H^{-q}_{\overline{\Omega}_n}(\mathbf{R}_n) \to H_0^{-s}(\Omega_n)$ where $q < (n - \alpha) + s$, or equivalently, $-s < -q + (n - \alpha)$. □

COROLLARY. *K maps $H^s_{\overline{\Omega}_n}(\mathbf{R}_n)$ into $H_0^q(\Omega_n)$ for all $q < (n - \alpha) + s$.*

Of importance in the study of the boundary value problems for differential operators defined on a domain Ω is the determination of spaces of functions defined on the boundary of Ω containing the traces $\gamma_0 u = u|_{\partial\Omega}$ of functions u in $H^s(\Omega)$. The problem of characterizing the image of $H^s(\Omega)$ under the operator $\gamma_0 : u \to u|_{\partial\Omega}$ has been studied by many authors; for example Lions [**6**]. This idea is extended to the boundary value problem (1.1), (1.2) for the weakly singular integral operator K, where the Fredholm integral equation of the first kind

$$Ku = f \tag{3.7}$$

is considered.

The problem of existence of solutions of (3.7) is considered by viewing the operator K as a mapping between function spaces, and the results of this paper show that if equation (3.7) has a solution u for given $f \in H^q(\Omega_n), q < (n-\alpha)+s$, then the solution u must be in $H^s(\Omega_n)$.

In addition, the boundary value problem (1.1), (1.2) has meaning in the following sense.

If (1.1) has solutions for $f \in \mathcal{F} = H^q(\Omega_n), q < (n-\alpha)+s$, then the set of solutions

$$\mathcal{U} = \{u = K^{-1}f + u_0 : Ku_0 = 0\} \subset H^s(\Omega_n)$$

must take on boundary values in the set

$$\begin{aligned}\mathcal{G} = \{g = g_1 + g_0 : g_1 &\text{is the boundary value of } K^{-1}f \\ &\text{and } g_0 \text{ is the boundary value of } u_0 \text{ on } \partial\Omega_n\} \\ &\subset H^{s-(1/2)}(\partial\Omega_n).\end{aligned}$$

Hence, if (1.1), (1.2) is to have a solution $u \in \mathcal{U}$ for given $f \in \mathcal{F}$, then $g \in \mathcal{G}$.

On the other hand, if we let $g \in \mathcal{G} = H^{s-(1/2)}(\partial\Omega_n)$ and require (1.2) to be satisfied by solutions of (1.1) which are in the set
$\mathcal{U} = \{u_\alpha : Ku_\alpha = f, u_\alpha$ takes on the boundary value g, and is in some indexing set $A\} \subset H^s(\Omega_n)$, then $F = \{f : f = \sum_{\alpha\in A} a_\alpha K u_\alpha,\ \sum_{\alpha \in A} a_\alpha = 1\} \subset H^q(\Omega_n),\ q < (n-\alpha)+s$.

Hence, if (1.1), (1.2) is to have a solution $u \in \mathcal{U}$ for given $g \in \mathcal{G}$, then $f \in \mathcal{F}$.

These results regarding the Fredholm integral equation of the first kind can be extended to function spaces defined on Riemannian manifolds M_n with boundary ∂M_n.

4. Singular integral operators on subspaces of $\mathbf{H^s(M_n)}$. Let M_n denote a Riemannian manifold of dimension n with boundary ∂M_n, assumed to be an infinitely differentiable manifold of dimension $n-1$.

Let $\mathcal{A}$ be a complete atlas of M_n consisting of the collection of local charts (also called local coordinate systems) (U_α, ϕ_α) on M_n, where α is in some indexing set A. If $p \in U_\alpha$ and $\phi_\alpha(p) = (x_1(p), \ldots, x_n(p)) \in \mathbf{R}_n$, then the open set U_α will be called a coordinate patch or coordinate neighborhood of p and the numbers $x_i(p), 1 \leq i \leq n$, will be called local coordinates of p. The mapping $\phi_\alpha : p \in U_\alpha \to (x_1(p), \ldots, x_n(p))$ will in general be denoted by $(x_1, \ldots, x_n)$.

We assume M_n is orientable, i.e., we can find a collection of local charts $(\mathcal{U}_\alpha, \phi_\alpha)$ such that $\{U_\alpha\}_{\alpha \in A}$ is a covering of M_n and such that for any $\alpha, \beta, \in A$, the mapping $\phi_\beta \cdot \phi_\alpha^{-1}$ has strictly positive Jacobian determinant in its domain of definition $\phi_\alpha(U_\alpha \cap U_\beta)$.

DEFINITION 4.1. Let M_n be an orientable Riemannian manifold of dimension n, imbedded in $\mathbf{R}_{n+1}$ and homotopically equivalent to the unit ball $D_n = \{\xi \in \mathbf{R}_n : |\xi| < 1\}$. Let ∂M_n denote the boundary of M_n, assumed to be a C^∞-manifold of dimension $n-1$, and let ∂D_n denote the boundary of D_n. If ϕ is a homotopy equivalence of M_n with D_n, then we define the function space $H^s(M_n), s \in \mathbf{R}$, by

$$u \in H^s(M_n) \text{ if and only if } u \cdot \phi^{-1} \in H^s(\phi(M_n)) = H^s(D_n).$$

DEFINITION 4.2. Let $M_n, \partial M_n, D_n$ and ∂D_n be manifolds satisfying the same conditions as in Definition 4.1. By means of the Riemannian metric on the Riemannian manifold M_n, we can obtain a distance function p between two points of M_n in such a way that p is metrically equivalent to the usual Euclidean distance function $d(x,y) = |x-y|$.

We define the weakly singular integral operator K on $H^s(M_n), s \in \mathbf{R}$, by

$$(Ku)(x) = \int_{M_n} \frac{1}{[p(x,y)]^\alpha} u(y) dm_y,$$

where $0 \leq \alpha < n$, and m is a measure on M_n which gives the surface area, volume element, etc. (depending on the appropriate dimension), locally equivalent to Lebesgue measure.

REMARK. In the sequel, we shall use $(Ku)(x) = \int_{M_n} \frac{1}{|x-y|^\alpha} u(y) dm_y$, since the action of K on $H^s(M_n)$ is unchanged as a mapping between function spaces due to the equivalence of the metrics p and d.

Let ϕ be a homotopy equivalence of D_n with M_n such that $\phi \in C^\infty(D_n)$. Denote by $(x_1, \dots, x_n)$ the coordinates in M_n and by $(\xi_1, \dots, \xi_n)$ the coordinates in K_n. If $y = (y_1, \dots, y_n) \in M_n$, then there exists $\eta = (\eta_1, \dots, \eta_n) \in D_n$ such that $\phi(\eta) = y$, i.e.,

$$\begin{cases} y_1 = \phi_1(\eta_1, \dots, \eta_n) \\ y_n = \phi_n(\eta_1, \dots, \eta_n) \end{cases}$$

We use ϕ to perform a change of variables in the integral as follows:

$$\begin{aligned} (Ku)(x) &= \int_{M_n} \frac{1}{|x-y|^\alpha} u(y) dm_y \\ &= \int_{D_n} \frac{1}{|\phi(\xi) - \phi(\eta)|^\alpha} (u \cdot \phi)(\eta) |J_\phi(\eta)| d\omega_\eta, \end{aligned}$$

where ω is Lebesgue measure on D_n and

$$J_\phi(\eta) = \det \begin{pmatrix} \frac{\partial \phi_1}{\partial \xi_1}\Big|_{\xi=\eta} & \cdots & \frac{\partial \phi_1}{\partial \xi_n}\Big|_{\xi=\eta} \\ \vdots & & \vdots \\ \frac{\partial \phi_n}{\partial \xi_1}\Big|_{\xi=\eta} & \cdots & \frac{\partial \phi_n}{\partial \xi_n}\Big|_{\xi=\eta} \end{pmatrix}$$

By Definition 4.1, $u \in H^s(M_n)$ if and only if $u \cdot \phi \in H^s(\phi^{-1}(M_n)) = H^s(D_n)$.

We now let

(i) $q(\xi, \eta) = |J_\phi(\eta)| \frac{|\xi-\eta|^\alpha}{|\phi(\xi)-\phi(\eta)|^\alpha}$,

(ii) $v = u \cdot \phi$,

(iii) $(Qv)(\xi) = \int_{D_n} \frac{q(\xi,\eta)}{|\xi-\eta|^\alpha} v(\eta) d\omega_\eta$.

We then have $(Ku)(x) = (Qv)(\xi)$, where $\phi(\xi) = x$.

LEMMA 4.1. *If M_n is orientable, then* $q(\xi,\eta) = |J_\phi(\eta)|\frac{|\xi-\eta|^\alpha}{|\phi(\xi)-\phi(\eta)|^\alpha}$ *is bounded and smooth for all* $(\xi,\eta) \in D_n \times D_n$.

PROOF. Denote by $(x_1,\ldots,x_n)$ the coordinates in M_n and by $(\xi_1,\ldots,\xi_n)$ the coordinates in D_n.

If $x = (x_1,\ldots,x_n) \in M_n$ and $\xi = (\xi_1,\ldots,\xi_n) \in D_n$, by $\phi(\xi) = x$ we mean

$$\begin{cases} x_1 = \phi_1(\xi_1,\cdots,\xi_n) \\ x_n = \phi_n(\xi_1,\cdots,\xi_n) \end{cases}$$

Since M_n is orientable, we have that ϕ has strictly positive Jacobian determinant in its domain of definition $\phi^{-1}(M_n) = D_n$, i.e.,

$$J_\phi(\xi) = \det\begin{pmatrix} \frac{\partial\phi_1}{\partial\xi_1} & \cdots & \frac{\partial\phi_1}{\partial\xi_n} \\ \vdots & & \vdots \\ \frac{\partial\phi_n}{\partial\xi_1} & \cdots & \frac{\partial\phi_n}{\partial\xi_n} \end{pmatrix} > 0 \text{ for all } \xi \in D_n$$

and hence

$$J_\phi(\eta) = \det\begin{pmatrix} \frac{\partial\phi_1}{\partial\xi_1}|_{\xi=\eta} & \cdots & \frac{\partial\phi_1}{\partial\xi_n}|_{\xi=\eta} \\ \vdots & & \vdots \\ \frac{\partial\phi_n}{\partial\xi_1}|_{\xi=\eta} & \cdots & \frac{\partial\phi_n}{\partial\xi_n}|_{\xi=\eta} \end{pmatrix} > 0$$

Denote by (J) the Jacobian matrix corresponding to the above Jacobian determinant $J_\phi(\eta)$.

To prove that $q(\xi,\eta)$ is bounded for all $(\xi,\eta) \in D_n \times D_n$, we need only show that $|\phi(\xi)-\phi(\eta)|/|\xi-\eta|$ is bounded away from zero, i.e., $|\phi(\xi)-\phi(\eta)|/|\xi-\eta| > 0$. Using Taylor's theorem for several variables, we have $\phi(\xi)-\phi(\eta) = \sum_{k=1}^{\infty}\frac{1}{k!}((\xi-\eta)\cdot\nabla)^k(\phi(\eta)$ for all ξ in a neighborhood of η, say $\eta_\delta = \{\xi \in D_n : |\xi-\eta| < \delta\}$. Writing out explicitly the first term of the expansion we have

$$\begin{aligned}\phi(\xi)-\phi(\eta) &= (\xi_1-\eta_1)\frac{\partial\phi}{\partial\eta_1}\Big|_{\xi=\eta}+(\xi_2-\eta_2)\frac{\partial\phi}{\partial\xi_2}\Big|_{\xi=\eta}\\ &\quad+\cdots+(\xi_\eta-\eta_n)\frac{\partial\phi}{\partial\xi_n}\Big|_{\xi=n}+\sum_{k=2}^{\infty}\frac{1}{k}((\xi-\eta).\nabla)^k\phi(\eta)\\ &= (\xi_1-\eta_1)\left(\frac{\partial\phi_1}{\partial\xi_1}\Big|_{\xi=\eta},\cdots,\frac{\partial\phi_n}{\partial\xi_1}\Big|_{\xi=\eta}\right)\\ &\quad+\cdots+(\xi_n-\eta_n)\left(\frac{\partial\phi_1}{\partial\xi_n}\,|_{\xi=\eta},\cdots,\frac{\partial\phi_n}{\partial\xi_n}\,|_{\xi=\eta}\right)\\ &\quad+\sum_{k=2}^{\infty}\frac{1}{k!}((\xi-\eta)\cdot\nabla)^k\phi(\eta)\\ &= (J)(\xi-\eta)+\sum_{k=2}^{\infty}\frac{1}{k!}((\xi-\eta)\cdot\nabla)^k\phi(\eta),\end{aligned}$$

where

$$(J)=\begin{pmatrix}\frac{\partial\phi_1}{\partial\xi_1}\,|_{\xi=\eta} & \cdots & \frac{\partial\phi_1}{\partial\xi_n}\,|_{\xi=\eta}\\ \vdots & & \vdots\\ \frac{\partial\phi_n}{\partial\xi_1}\,|_{\xi=\eta} & \cdots & \frac{\partial\phi_n}{\partial\xi_n}\,|_{\xi=\eta}\end{pmatrix}$$

$$(J)(\xi-\eta)=\begin{pmatrix}\frac{\partial\phi_1}{\partial\xi_1}\,|_{\xi=\eta} & \cdots & \frac{\partial\phi_1}{\partial\xi_n}\,|_{\xi=\eta}\\ \vdots & & \vdots\\ \frac{\partial\phi_n}{\partial\xi_1}\,|_{\xi=\eta} & \cdots & \frac{\partial\phi_n}{\partial\xi_n}\,|_{\xi=\eta}\end{pmatrix}\begin{pmatrix}\xi_1-\eta_1\\ \xi_n-\eta_n\end{pmatrix}$$

In fact, if we use multi-index notation, the Taylor series expansion takes the form $\phi(\xi)-\phi(\eta)=\sum_{|\alpha|\geq 1}\frac{D^\alpha\phi(\eta)}{\alpha!}(\xi-\eta)^\alpha=\sum_{|\alpha|=1}\frac{D^\alpha\phi(\eta)}{\alpha!}(\xi-\eta)^\alpha$ $+o(\xi-\eta|^2)$.

Hence $|\phi(\xi)-\phi(\eta)|/|\xi-\eta| = |(J)(\xi-\eta)/|\xi-\eta|+o(|\xi-\eta|)|$ for all $\xi\in\eta_\delta$. But $\det(J)\neq 0\Rightarrow (J)(\xi-\eta)/|\xi-\eta|\neq\vec{0}$ since $(\xi-\eta)/|\xi-\eta|$ is a unit vector. Then $|\phi(\xi)-\phi(\eta)|/|\xi-\eta|>0$ for all $\xi\in\eta_\delta=\{\xi\in D_n:|\xi-\eta|<\delta\}$. Obviously, $|\phi(\xi)-\phi(\eta)|/|\xi-\eta|>0$ for all $\xi\in\{\xi\in D_n:|\xi-\eta|\geq\delta\}$.

Since $|\phi(\xi)-\phi(\eta)|/|\xi-\eta|$ is bounded away from zero for all $(\xi,\eta)\in D_n\times D_n$, we conclude that $q(\xi,\eta)$ is bounded for all $(\xi,\eta)\in D_n\times D_n$, and also smooth since $\phi\in C^\infty(D_n)$.

THEOREM 4.1. *Let $u \in H^s(D_n), s \in \mathbf{R}$, be such that $Ku \in H^q(D_n)$ where*

$$(Ku)(\xi) = \int_{D_n} \frac{1}{|\xi - \eta|^\alpha} u(\eta) d\omega_\eta \text{ and } q < n - \alpha + s, \quad 0 \leq \alpha < n.$$

Let $(Qu)(\xi) \int_{D_n} \frac{q(\xi,\eta)}{|\xi-\eta|^\alpha} u(\eta) d\omega_\eta$ where $q(\xi,\eta)$ is bounded for all $\xi, \eta) \in D_n \times D_n$. Then $Qu \in H^q(D_n)$.

PROOF. Let $v = Ku$ and $w = Qu$. We know by hypothesis that $u \in H^s(D_n)$ implies $v = Ku \in H^q(D_n)$. We suppose $w \in \mathcal{H}(D_n)$. We wish to show $w = Qu \in H^q(D_n)$, i.e., $\mathcal{H}(D_n) \subset H^q(D_n)$.

Suppose not. Then, for all $M > 0$ and for all $w \in \mathcal{H}(D_n)$, there exists $v' \in (H^q(D_n))'$ such that $|\langle w, v' \rangle_{D_2}| > M$. We will obtain a contradiction by showing that there exists $M_1 < 0$ such that $|\langle w, v' \rangle| \leq M_1$ for all $w \in \mathcal{H}(D_n)$ and for every $v' \in (H^q(D_n))'$.

Note that $|\langle w, v' \rangle| = \Big| \int_{D_n} v'(\xi).w(\xi) d\omega_\xi| = | \int_{D_n} v'(\xi) \int_{D_n} \frac{q(\xi,\eta)}{|\xi-\eta|^\alpha} u(\eta) d\omega_\eta d\omega_\xi|$. Let $|q(\xi,\eta)| \leq A$ for every $(\xi,\eta) \in D_n \times D_n$. Then $|\langle w, v' \rangle| \leq A|\langle v, v' \rangle| \leq AN$ for some $N > 0$. Hence we can choose $M_1 = AN$ to get the contradiction and conclude that $\mathcal{H}(D_n) \subset H^q(D_n)$, i.e.,

$$Qu \in H^q(D_n) \text{ if } Ku \in H^q(D_n).$$

COROLLARY. *Let $\mathcal{H}^s(M_n) = \{u \in H^s(M_n) : u \circ \phi \in H^s(D_n) \cap N(1 - \Delta)$ is incoming with respect to $\partial D_n\}$. Then $Ku \in H^q(M_n)$ for all $q < n - \alpha + s$.*

REMARK. The problem of investigating the action of weakly singular integral operators on function spaces of a Riemannian manifold with boundary has in effect been reduced to a problem already investigated in the previous sections, namely the action of weakly singular integral operators as mappings between Sobolev spaces on Euclidean manifolds.

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