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EXISTENCE AND CONVERGENCE RESULTS FOR INTEGRAL INCLUSIONS IN BANACH SPACES

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ABSTRACT. The existence of solutions is established for multivalued Volterra integral equations (integral inclusions) defined in a separable Banach space and governed by convex and nonconvex orientor fields. Also we prove a convergence result for such integral inclusions. In doing that we obtain some new interesting results about multifunctions, including a new set valued version of Fatou's lemma.

Introduction-preliminaries. Several problems in applied mathematics (control theory, mathematical economics, mechanics, etc.) involve various types of ambiguity, indeterminacy, or uncertainty (which in particular includes the impossibility of a comprehensive description of the dynamics of the system under consideration). This leads to mathematical models that involve differential and integral inclusions. In recent years the study of this more general class of equations has received considerable attention and many mathematicians have contributed interesting results, mostly in the direction of differential inclusions.

The main purpose of the present paper is to study the problem of existence of solutions for Volterra type integral inclusions defined in a separable Banach space. We prove two existence theorems; one for convex orientor fields and the other for nonconvex ones. Then we present a convergence result for the family of integral inclusions that we consider. The convergence property is one of the most important properties in differential and integral equations. As was shown by Strauss-Yorke [**20**], much of the fundamental theory of ordinary differential equations follows directly from a convergence theorem. In the process of obtaining that convergence result we also prove a multivalued version of Fatou's lemma that generalizes earlier results of Artstein [**1**] and Schmeidler [**19**] and which is interesting in its own because of its important potential applications in control theory and mathematical economics.

Let $(\Omega, \sum)$ be a measurable space and X a separable Banach space.

Throughout this paper we will use the following notations:

$$P_{f(c)}(X) = \{A \subseteq X : \text{nonempty, closed, (convex)}\}$$

$$P_{(w)k(c)}(X) = \{A \subseteq X : \text{nonempty, (w)-compact, (convex)}\}.$$

for $A \in 2^X \backslash \{\emptyset\}$ we set $|A| = \sup_{x \in A} ||x||$(norm of A), $\sigma_A(x^*) = \sup_{x \in A}(x^*, x), x^* \in X^*$ (the support function of A) and $d_A(z) = \inf_{x \in A} ||z - x||, z \in X$ (the distance function from A). Also if $B, C \in 2^X \backslash \{\emptyset\}$ the excess of B over C is defined as $h^*(B, C) = \sup_{y \in B} d_C(y)$.

A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if it satisfies any of the following three equivalent conditions:

i) for all $x \in X, \omega \to d_{F(\omega)}(x)$ is measurable,

ii) there exist $\{f_n(\cdot)\}_{n \geq 1}$ measurable selectors of $F(\cdot)$ s.t., for all $\omega \in \Omega, F(\omega) = \mathrm{cl}\{f_n(\omega)\}_{n \geq 1}$ (Castaing representation of $F(\cdot)$),

iii) $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \sum$ for all $U \subseteq X$ open.

For more details concerning measurable multifunctions the reader can consult any of the following three excellent references: Castaing-Valadier [**4**], Rockafellar [**16**], Wagner [**21**].

Let $F : \Omega \to 2^X \backslash \{\emptyset\}$ be a multifunction. We introduce the set

$$S_F^1 = \{f(\cdot) \in L_X^1(\Omega) : f(\omega) \in F(\omega) \mu - a.e.\}.$$

If $GrF = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \sum \times B(X)$, the S_F^1 is nonempty if and ony if $\inf_{x \in F(\omega)} ||x|| \in L^1(\Omega)$. Also if $F(\cdot)$ is closed valued, then it is easy to see that S_F^1 if a strongly closed subset of the Lebesgue-Bochner space $L_X^1(\Omega)$. Having this set, we can now define an integral for the multifunction $F(\cdot)$. So we set

$$\int_\Omega F(\omega) d\mu(\omega) = \left\{ \int_\Omega f(\omega) d\mu(\omega) : f(\cdot) \in S_F^1 \right\},$$

where $\int_\Omega f(\omega) d\mu(\omega)$ is understood as a Bochner integral. This set valued integral was first introduced by Aumann [**3**] as the natural generalization of the integral of a point valued function and of the Minkowski sum of sets.

We will say that a multifunction $F : \Omega \to P_f(X)$ is integrably bounded if it is measurable and $|F(\cdot)| \in L^1(\Omega)$.

Suppose that Y, Z are Hausdorff topological spaces and $F : Y \to 2^Z \backslash \{\emptyset\}$. We say that $F(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c)) if, for all $U \subseteq Z$ open, we have that $\{y \in Y : F(y) \subset U\}$ (resp. $\{y \in Y : F(y) \cap U \neq \emptyset\}$) is open too. If Z is a metric space and $F_i : Y \to 2^Z \backslash \{\emptyset\}, i \in I$, we will say that the family $\{F_i(\cdot)\}_{i \in I}$ is equi-h*-u.s.c. at y if, for every $\varepsilon > 0$, there exists a neighborhood V of y such that, for $z \in V, h^*(F_i(z), F_i(y)) < \varepsilon$ for all $i \in I$. We say that $\{F_i(\cdot)\}_{i \in I}$ is equi-h*-u.s.c. if it is equi-h*-u.s.c. at every $y \in Y$.

Finally we would like to introduce a mode of set convergence, different from the Hausdorff convergence, which we are going to use in the sequel. So let $\{K_n\}_{n \geq 1}$ be a sequence of nonempty subsets of X and τ a topology on X. We say that K_n τ-converges to K in the Kuratowski sense if $\tau - \overline{\lim}_{n \to \infty} K_n \subseteq K \subseteq \tau - \underline{\lim}_{n \to \infty} K_n$, where

$$\tau - \overline{\lim_{n \to \infty}} K_n = \{x = \tau - \overline{\lim_{k \to \infty}} x_{n_k}, \quad x_{n_k} \in K_{n_k}, k \geq 1\}$$

and

$$\tau - \underline{\lim_{n \to \infty}} K_n = \{x = \tau - \overline{\lim_{n \to \infty}} x_n, \quad x_n \in K_n, n \geq 1\}.$$

Since we always have that $\tau - \underline{\lim}_{n \to \infty} K_n \subseteq \tau - \overline{\lim}_{n \to \infty} K_n$, we deduce that $K_n \tau$-converges to K in the Kuratowski sense if and only if $\tau - \overline{\lim}_{n \to \infty} K_n = K = \tau - \underline{\lim}_{n \to \infty} K_n$. When $w - \overline{\lim}_{n \to \infty} K_n = k = s - \underline{\lim}_{n \to \infty} K_n$, where w denotes the weak topology on X and s the strong (norm) topology, then we say that K_n converges to K in the Kuratowski-Mocso sense and we write that $K_n \stackrel{K-M}{\to} K$ as $n \to \infty$. For details we refer to Mosco [**12**] and Salinetti-Wests [**18**].

2. Existence theorems. Let $[0, T]$ be a compact interval in $\mathbf{R}$ with the Lebesgue measure dt and X a separable Banach space. We will study the Volterra integral inclusion

$$(*) \qquad x(t) \in p(t) + \int_0^t K(t, s) F(s, x(s)) ds,$$

where $F(\cdot, \cdot)$ is a multifunction (orientor field).

Before going into the first existence theorem, we need to state an auxiliary functional analytic result. This result is also very interesting in its own because it appears to be the most general theorem for weak compactness in the Lebesgue-Bochner space $L^1_X(\Omega)$. It was first obtained by the author in [**14**]. For completeness we include it here together with its proof.

Assume that $(\Omega, \sum, \mu)$ is a complete σ-finite measure space and X a separable Banach space.

THEOREM 2.1. *If $F : \Omega \to P_{wkc}(X)$ is integrably bounded, then S^1_F is a nonempty convex, w-compact subset of $L^1_X(\Omega)$.*

PROOF. Nonemptiness and convexity follow immediately from the fact that $F(\cdot)$ is integrably bounded and has convex values.

So it remains to show that S^1_F is a w-compact in $L^1_X(\Omega)$. For that purpose we are going to use James' theorem (see Floret [**6**, p. 59]). Thus we have to show that every element of $[L^1_X(\Omega)]^*$ achieves its supremum on S^1_F. From the Dinculeanu-Foias theorem (see Ionescu-Tulcea [**10**]) we know that $[L^1_X(\Omega)]^* = L^\infty_{X^*_{w^*}}(\Omega)$. Let $g(\cdot) \in L^\infty_{X^*_{w^*}}(\Omega)$. We have

$$\sup_{f \in S^1_F} (g, f) = \sup_{f \in S^1_F} \int_\Omega (g(\omega), f(\omega)) d\mu(\omega).$$

From Theorem 2.2 of Hiai-Umegaki [**8**] we know that

$$\sup_{f \in S^1_F} \int_\Omega (g(\omega), f(\omega)) d\mu(\omega) = \int_\Omega \sup_{x \in F(\omega)} (g(\omega), x) d\mu(\omega).$$

Let $M(\omega) = \{z \in F(\omega) : (g(\omega), z) = \sup_{x \in F(\omega)} (g(\omega), x)\}$. Since $F(\cdot)$ is w-compact valued for all $\omega \in \Omega, M(\omega) \neq \emptyset$. Also it is easy to see that $M(\cdot)$ is closed valued. Let $m(\omega) = \sup_{x \in F(\omega)} (g(\omega) x)$. If $\{f_n(\cdot)\}_{n \geq 1}$ is a Castaing representation of $F(\cdot)$ we can write that $m(\omega) = \sup_{n \geq 1} (g(\omega), f_n(\omega))$, which shows that $m(\cdot)$ is measurable. Let $r(\omega, z) = (g(\omega), z) - m(\omega)$. Clearly $r(\cdot, \cdot)$ is a Caratheodory function and so it is jointly measurable. Then note that

$$M(\omega) = \{z \in F(\omega) : r(\omega, z) = 0\}$$

$$\mathrm{Gr} M = \{(\omega, z) \in \Omega \times X : r(\omega, z) = 0\} \cap \mathrm{Gr} F.$$

Recalling that Gr$F \in \sum \times B(X)$ we deduce that Gr$M \in \sum \times B(X)$ and so applying Aumann's selection theorem we can find $\hat{f} : \Omega \to X$ measurable such that $\hat{f}(\omega) \in M(\omega)$ for all $\omega \in \Omega$. Thus we have

$$\int_\Omega \sup_{x \in F(\omega)} (g(\omega), x) d\mu(\omega) = \int_\Omega (g(\omega), \hat{f}(\omega)) d\mu(\omega)$$

$$\Rightarrow \sup_{f \in S_F^1} (g, f) = (g, \hat{f}).$$

Since $g(\cdot) \in L^\infty_{X^*_{w^*}}(\Omega) = [L^1_X(\Omega)]^*$ was arbitrary, invoking James' theorem, we conclude that S_F^1 is w-compact in $L^1_X(\Omega)$. □0pt

REMARK. 1) If X is also weakly sequentially complete, then the converse of the above theorem is true. Namely given $F : \Omega \to P_f(X)$ integrably bounded if S_F^1 is a convex and w-compact subset of $L^1_X(\Omega)$, then, for all $\omega \in \Omega, F(\omega) \in P_{wkc}(X)$. This was proved by the author in [**15**].

2) An immediate important consequence of the theorem is that $\int_\Omega F(\omega) d\mu(\omega) \in P_{wkc}(X)$.

Now we are ready for the first existence result for the integral inclusion

$$(*) \qquad x(t) \in p(t) + \int_0^t K(t,s) F(s, x(s)) ds.$$

Here $F(\cdot, \cdot)$ is a multifunction (orientor field) and $K : \{(t,s) : 0 \le s \le t \le T\} \to \mathcal{L}(X) =$ Continuous linear operators from X into itself. By a solution to equation (*) we understand an $x(\cdot) \in C_X(T)$ satisfying (*) for all $t \in T$.

THEOREM 2.2. *If* 1) $F : T \times X \to P_{fc}(X)$ *is a multifunction such that:*

(a) $F(\cdot, \cdot)$ *is jointly measurable and for all* $x \in X, F(t,x) \subseteq G(t)$ *a.e. with* $G : T \to P_{wkc}(X)$ *integrably bounded,*

(b) *for all* $t \in T, F(t, \cdot) : X_w \to X_w$ *is u.s.c.;*

2) *for all* $t \in T, K(t, \cdot)$ *is essentially bounded on* [0,t],

3) $\lim_{t'-t \to 0^+} \left(\int_t^{t'} ||K(t', s)|| \, |G(s)| ds + \int_0^t ||K^t(t', s) - K(t, s)|| \, |G(s)| ds \right)$

$= 0$ *for fixed* t *or* t,

4) $p(t) \in C_X(T)$,

then $(*)$ *admits a solution.*

PROOF. Consider the set $W \subseteq C_X(T)$ defined by

$$W = \{x(\cdot) \in C_X(T) : x(t) = p(t) + \int_0^t K(t, x) g(s) ds, g(\cdot) \in S_G^1, t \in T\}.$$

Our claim is that W is a compact subset of $C_{X_w}(T)$. First note that, for all $x(\cdot) \in W$ and all $t \in T$, we have

$$x(t) \in p(t) + \int_0^t K(t, s) G(s) ds.$$

Since $K(t, \cdot) \in \mathcal{L}(X)$, it is also weakly continuous and so we have that $K(t, s)G(s) \in P_{wkc}(X)$ for all $s \in [0, t], t \in T$. Furthermore, for all $x^* \in X^*$ we have $\sigma_{K(t,s)G(s)}(x^*) = \sigma_{G(s)}(K^*(t, s)x^*)$, and using Theorem 3.8.1 of Hille-Phillips [**19**], we see that $s \to \sigma_{K(t,s)G(s)}(x^*)$ is measurable on $[0, t]$. Thus, invoking Theorem III-37 of Castaing-Valadier [**4**], we conclude that $s \to K(t, s)G(s)$ is a $P_{wkc}(X)$-valued integrably bounded multifunction on $[0, t]$. Then, using Theorem 2.1, we get that $\int_0^t K(t, s)G(s) ds \in P_{wkc}(X), t \in T$.

Thus, for every $t \in T$, we have that

$$\overline{\{x(t)\}_{x(\cdot) \in W}}^w \in P_{wk}(X).$$

Also, for $t, t \in T, t > t$, we can write that

$$
\begin{aligned}
||x(t) - x(t)|| &\leq ||p(t') - p(t)|| + \left|\left| \int_0^t K(t,s)g(s)ds \right.\right. \\
&\qquad \left.\left. - \int_0^t K(t,s)g(s)ds \right|\right| \\
&\leq ||p(t') - p(t)|| + \int_t^{t'} ||K(t',s)||\,||g(s)||ds \\
&\qquad + \int_0^t ||K(t',s) - K(t,s)|| \quad ||g(s)||ds \\
&\leq ||p(t') - p(t)|| + \int_t^{t'} ||K(t',s)||\,|G(s)|ds \\
&\qquad + \int_0^t ||K(t',s) - K(t,s)||\,|G(s)|ds.
\end{aligned}
$$

Passing to this limit as $t' \to t$, we get that

$$||x(t') - x(t)|| \to 0$$

uniformly in $x(\cdot) \in W$. So we deduce that W is equicontinuous, a fortiori, and then w-equicontinuous.

Finally we will show that W is closed in $C_{X_W}(T)$. Then, invoking the Arzela-Ascoli theorem, we will have our claim. So let $\{x_a(\cdot)\}_{a\in A}$ be a net in W such that $x_a(\cdot) \xrightarrow{C_{X_w}(T)} x(\cdot)$. Then, for all $t \in T$, we have

$$x_a(t) = p(t) + \int_0^t K(t,s)g_a(s)ds$$

with $g_a(\cdot) \in S_G^1$, $a \in A$. But recall that S_G^1 is a w-compact subset of $L_X^1(T)$ (Theorem 2.1). So, by passing to a subnet, if necessary, we may assume that $g_a(\cdot) \xrightarrow{w-L_X^1(T)} g(\cdot) \in S_G^1$. Then we can write

$$\int_0^t K(t,s)g_a(s)ds \xrightarrow{w} \int_0^t K(t,s)g(s)ds \qquad (\text{since } ||K(t,\cdot)|| \in L^\infty)$$

$$\Rightarrow x_a(t) \xrightarrow{w} p(t) + \int_0^t K(t,s)g(s)ds.$$

On the other hand we already have by hypothesis that

$$x_a(t) \xrightarrow{w} x(t)$$

for all $t \in T$. Since weak limits are unique (the weak topology being Hausdorff) we deduce that, for all $t \in T$,

$$x(t) = p(t) + \int_0^t K(t,s)g(s)ds$$

with $g(\cdot) \in S_G^1$. Hence $x(\cdot) \in W$.

Therefore W is closed in $C_{X_w}(T)$. An application of the Arzela-Ascoli theorem tells us that W is a compact subset of $C_{X_w}(T)$.

Now consider the multifunction

$$\Phi : W \to 2^{C_X(T)},$$

defined by

$$\Phi(x) = \Big\{ z(\cdot) \in C_X(T) : z(t) = p(t) + \int_0^t K(t,s)f(s)ds, f(\cdot) \in S^1_{F(\cdot,x(\cdot))}, t \in T \Big\}.$$

Since $F(\cdot,\cdot)$ is jointly measurable for every $x : T \to X$ measurable, $s \to F(s, x(s))$ is measurable (just note that $s \to d_{F(s,x(s))}(z)$ is measurable for all $z \in X$). So $S^1_{F(\cdot,x(\cdot))} \neq \emptyset$. Hence we see that $\Phi(\cdot)$ has nonempty, convex closed values in W. Also note that W is metrizable, since the topology of W is equal to the topology of pointwise convergence on a countable dense subset of T.

Now we will show that $\mathrm{Gr}\Phi \subseteq W \times W$ is closed and, since W is a compact subset of $C_{X_w}(T)$, this will imply that $\Phi(\cdot)$ is u.s.c. (see Theorem 7.1.16 of Klein-Thompson [**11**]). So consider a sequence

$$\{x_n(\cdot), y_n(\cdot)\}_{n\geq 1} \subseteq \mathrm{Gr}\Phi \text{s.t.} x_n(\cdot) \xrightarrow{C_{X_w}(T)} x(\cdot) \text{ and } y_n(\cdot) \xrightarrow{C_{X_w}(T)} y(\cdot).$$

Then, for every $n \geq 1$, we have

$$y_n(t) = p(t) + \int_0^t K(t,s)f_n(s)ds,$$

for all $t \in T$, with $f_n(\cdot) \in S^1_{F(\cdot,x_n(\cdot))}$. Since $\{f_n(\cdot)\}_{n\geq 1} \subseteq S^1_G$ and the latter is w-compact in $L^1_X(T)$, by passing to a subsequence, if necessary, we may assume that

$$f_n(\cdot) \xrightarrow{w-L^1_X(T)} f(\cdot) \in S^1_G.$$

Applying Mazur's lemma we can find

$$h_m(\cdot) \in \operatorname{conv} \cup_{n\geq m} f_n(\cdot)$$

such that $h_m(\cdot) \xrightarrow{s-L^1_X(T)} f(\cdot)$, and by passing to a further subsequence we may assume that

$$h_m(s) \xrightarrow{s} f(s) \quad \text{a.e.}$$

For every $x^* \in X^*$ we have

$$\begin{aligned}(x^*, h_m(s)) &\leq \sigma\text{conv} \cup_{n\geq m} F(s, x_n(x))^{(x^*)} \\ &= \sigma_{\cup_{n\geq m} F(s,x_n(x))}(x^*) = \sup_{n\geq m} \sigma_{F(s,x_n(s))}(x^*) \text{ a.e.}\end{aligned}$$

$$\Rightarrow \lim_{m\to\infty} (x^*, h_m(s)) = (x^*, f(s)) \leq \overline{\lim_{n\to\infty}}\, \sigma_{F(s,x_n(s))}(x^*) \text{ a.e.}$$

But, by hypothesis for all $t \in T, F(t,\cdot)$ is u.s.c. from X_w into X_w. So Proposition 2 of Aubin-Ekeland [**2**, p. 122] tells us that $x \to \sigma_{F(t,x)}(x^*)$ is u.s.c. from X_w into $\mathbf{R}$. So, for all $x^* \in X^*$, we have

$$\begin{aligned}&\overline{\lim_{n\to\infty}}\, \sigma_{F(s,x_n(x))}(x^*) \leq \sigma_{F(s,x(s))}(x^*) \text{ a.e.} \\ &\Rightarrow (x^*, f(s)) \leq \sigma_{F(s,x(s))}(x^*) \text{ a.e.} \\ &\Rightarrow f(s) \in F(s, x(s)) \text{ a.e.} \\ &\Rightarrow f(\cdot) \in S^1_{F(\cdot,x(\cdot))}.\end{aligned}$$

Then, for all $t \in T, y(t) = p(t) + \int_0^t K(t,s)f(s)ds$ with $f(\cdot) \in S^1_{F(\cdot,x(\cdot))}$. Hence $(x(\cdot), y(\cdot)) \in \text{Gr}\Phi$, which proves that $\text{Gr}\Phi$ is a closed subset of $W \times W$. Thus we deduce that $\Phi(\cdot)$ is u.s.c. Applying the infinite dimensional version of Kakutani's fixed point theorem (see [**2**], p. 344) we get that there exists $\hat{x}(\cdot) \in W$ such that $\hat{x}(\cdot) \in \Phi(\hat{x}(\cdot))$. Clearly, then, $\hat{x}(\cdot)$ is the desired solution of $(*)$. □0pt

Next we will state an existence theorem for integral inclusions in which the multifunction is l.s.c. The remarkable feature of that result is that the orientor field need not be convex valued.

Assume that $T = [0, T]$ is as before and that X is a separable Banach space.

THEOREM 2.3. *If* 1) $F : T \times X \to P_f(X)$ *is a multifunction such that* :

(a) $F(\cdot,\cdot)$ *is jointly measurable and, for all* $x \in X, |F(t,x)| \leq \psi(t) a.e. with\ \psi(\cdot) \in L^1$,

(b) *for all* $t \in T, F(t,\cdot)$ *is* l.s.c.;

2) $K(t,s)$ *is compact, linear and* $||K(t,s)|| \leq M$;

3) $\lim_{t'-t\to 0^+} \Big(\int_t^{t'} ||K(t',s)||\psi(s)ds + \int_0^{t'} ||K(t,s) - K(t',s)||\psi(s)ds \Big)$
$= 0$ *for fixed* t' *or* t; *and*

4) $p : T \to X$ *is continuous;*
then (*) *admits a solution.*

PROOF. Again consider the set $W \subseteq C_X(T)$ defined by

$$W = \{x(\cdot) \in C_X(T) : x(t) = p(t) + \int_0^t K(t,s)g(s)ds,$$
$$g(\cdot) \text{ measurable, } ||g(s)|| \leq \psi(s) \text{ a.e.}, t \in T\}.$$

As in the proof of Theorem 2.2, using hypotheses 2) and 3), we ge that W is compact in $C_X(T)$ (Arzela-Ascoli theorem).

Let $L : X \to 2^{L^1_X} - \{\emptyset\}$ be defined by

$$L(x) = S^1_{F(\cdot,x(\cdot))}$$

We claim $L(\cdot)$ is l.s.c. So let $x_n \to x$ in W, and $f(\cdot) \in L(x)$. A straightforward application of Aumann's selection theorems (see [**21**]) gives us $f_n(\cdot) \in S^1_{F(\cdot,x_n(\cdot))}$ s.t. $||f_n(s) - f(s)|| = d_{F(s,x_n(s))}(f(s))$. But since $F(s,\cdot)$ is l.s.c. we have

$$\begin{aligned} F(s,x(s)) &\subseteq \underline{\lim}_{n\to\infty} F(s,x_n(s)) \\ &\Rightarrow f(s) \in \underline{\lim}_{n\to\infty} F(s,x_n(s)) \\ &\Rightarrow \underline{\lim}_{F(s,x_n(s))} d(f(s)) = 0 \text{ a.e.} \\ &\Rightarrow ||f_n(s) - f(s)|| \to 0 \text{ a.e. and } f_n \in S^1_{F(\cdot,x_n(\cdot))} \end{aligned}$$

Hence $L(x) \subseteq \underline{\lim}_{L(x_n)}$ and so we get that $L(\cdot)$ is l.s.c. Apply Theorem 3.1 of Fryszkowski [**7**] to get $v : W \to L^1_X(T)$ continuous s.t. $v(x) \in L(x)$. Let $k(x)(\cdot) \in C_X(T)$ be defined by

$$k(x)(t) = p(t) + \int_0^t K(t,s)v(x)(s)ds.$$

Clearly $k : W \to W$ and is continuous. Apply the Schauder fixed point theorem to get $x(\cdot) \in W$ s.t. $x = k(x)$. It is easy to see that $x(\cdot)$ is the desired solution. □0

3. A convergence result. Before stating and proving the convergence result that we have for Banach space valued differential inclusions, we need to develop some auxiliary material, which is also interesting in its own as general results about measurable multifunctions.

For the first lemma assume that X is any Banach space.

LEMMA 3.1. *If $\{K_n\}_{n\geq 1} \subseteq P_f(X)$ and for all $n \geq 1, K_n \subseteq G, G \in P_{wk}(X)$, then, for all $x^* \in X^*, \overline{\lim}_{n\to\infty} \sigma_{K_n}(x^*) \leq \sigma_{w-\overline{\lim}_{K_n}(x^*)}$.*

PROOF. Fix $x^* \in X^*$. Because the sets are w-compact, for every $n \geq 1$ we can find $x_n \in A_n$ such that $x^*, x_n) = \sigma_{K_n}(x^*)$. Then we can find a subsequence $\{x_{n_k} = x_k\}_{k\geq 1}$ such that $(x^*, x_k) \to \overline{\lim}_{n\to\infty} \sigma_{K_n}(x^*)$. But $\{x_k\}_{k\geq 1} \subseteq G$ and G is w-compact (and, by the Eberlein-Smulian theorem, w-sequentially compact). So by passing to a subsequence, if necessary, we may assume that $x_k \xrightarrow{w} x \in G$. So $x \in w - \overline{\lim}_{n\to\infty} K_n$. Thus $(x^*, x) \leq \sigma_{w-\overline{\lim}_{K_n}(x^*)} \Rightarrow \overline{\lim}_{n\to\infty} \sigma_{K_n}(x^*) \leq \sigma_{w-\overline{\lim}_{n\to\infty} K_n}(x^*)$. Since $x^* \in X^*$ was arbitrary, the result follows. □0pt

We will use that lemma to establish an important generalization of the Schmeidler-Artstein theorem (see [**1**, **19**]) on weak sequential convergence. Assume that $(\Omega, \sum, \mu)$ is a σ-finite measure space and X a Banach space.

THEOREM 3.1. *If* $\{f_n(w), f(w)\}_{n\geq 1} \subseteq G(w) \in D_{wk}(x)$ *and* $f_n(\cdot) \xrightarrow{w-L^1_X} f(\cdot)$, *then* $f(\omega) \in \overline{\text{conv}}\ w - \overline{\lim}_{n\to\infty}\{f_n(\omega)\}_{n\geq 1}\mu - a.e.$

PROOF. Using Mazur's lemma we have that, for all $k \geq 1$,

$$f(\omega) \in \overline{\text{conv}} \cup_{n\geq k} \{f_n(\omega)\}\mu - \text{a.e.}$$

Let $x^* \in X^*$. Then we have, for all $k \geq 1$,

$$\begin{aligned}
(x^*, f(\omega)) &\leq \sigma_{\overline{\text{conv}} \cup_{n\geq k}\{f_n(\omega)\}}(x^*) \\
&= \sigma_{\cup_{n\geq k}\{f_n(\omega\}}(x^*) = \sup_{n\geq k}(x^*, f_n(\omega))\mu - \text{a.e.} \\
\Rightarrow (x^*, f(\omega)) &\leq \overline{\lim_{n\to\infty}}\,(x^*, f_n(\omega)) = \overline{\lim_{n\to\infty}}\,\sigma_{\{f_n(\omega)\}_{n\geq 1}}(x^*).
\end{aligned}$$

Using Lemma 3.1 we can write that

$$(x^*, f(\omega)) \leq \sigma_{w-\overline{\lim}}\{f_n(\omega)\}(x^*)\mu - \text{a.e.}$$

Since $x^* \in X^*$ was arbitrary we conclude that

$$f(\omega) \in \overline{\text{conv}}\ w - \overline{\lim}\{f_n(\omega)\}_{n\geq 1}. \ \square 0pt$$

The importance of the above theorem lies in the fact that, starting from a weakly-$L^1_X(\Omega)$ convergent sequence, we get a pointwise result.

We believe that this result will be very useful in several areas of pure and applied mathematics. Its first consequence is the following remarkable set valued version of Fatou's lemma.

Here $(\Omega, \sum, \mu)$ is a nonatomic, complete, σ-finite measure space and X a separable Banach space.

THEOREM 3.2. *If, for all* $n \geq 1$, $F_n : \Omega \to P_f(X)$ *is measurable,* $F_n(\omega) \subseteq G(\omega)$ *where* $G : \Omega \to P_{wkc}(X)$ *is integrably bounded and* $w - \overline{\lim}_{n\to\infty} F_n(\omega) \in P_f(X)$, *then* $w - \overline{\lim}_{n\to\infty} \int_\Omega F_n(\omega)d\mu(\omega) \subseteq cl \int_\Omega w - \overline{\lim}_{n\to\infty} F_n(\omega)d\mu(\omega)$.

PROOF. Let $x \in w - \overline{\lim}_{n\to\infty} \int_\Omega F_n(\omega)d\mu(\omega)$. Then there exists $x_k \in \int_\Omega F_{n_k}(\omega)d\mu(\omega)$ such that $x_k \overset{w}{\to} x$. From the definition of the Aumann integral we know that

$$x_k = \int_\Omega f_k(\omega)d\mu(\omega)$$

with $f_k(\cdot) \in S^1_{F_{n_k}}$. Since $S^1_{F_{n_k}} \subset S^1_G$ and the latter is w-compact in $L^1_X(\Omega)$, we can assume without any loss of generality that $f_k \overset{w-L^1_X}{\longrightarrow} f$. Hence $x = \int_\Omega f(\omega)d\mu(\omega)$. Also, from Theorem 3.2, we know that

$$\begin{aligned} & f(\omega) \in \overline{\text{conv}}\ w - \overline{\lim_{\{f_n(\omega)\}_{n\geq 1}}} \quad \mu - \text{a.e.} \\ \Rightarrow\ & f(\omega) \in \overline{\text{conv}}\ w - \overline{\lim_{n\to\infty}} F_n(\omega)\mu - \text{a.e.} \\ \Rightarrow\ & x \in \int_\Omega \overline{\text{conv}}\ w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega). \end{aligned}$$

Since $\mu(\cdot)$ is nonatomic, Corollary 4.3 of [8] tells us that

$$\begin{aligned} & \text{cl} \int_\Omega \overline{\text{conv}} w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega) \\ & = \int_\Omega \overline{\text{conv}} w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega) = \text{cl} \int_\Omega w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega). \end{aligned}$$

So we have that

$$x \in \text{cl} \int_\Omega w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega)$$

and since $x \in w - \overline{\lim}_{n\to\infty} \int_\Omega F_n(\omega)d\mu(\omega)$ was arbitrary we conclude that

$$w - \overline{\lim_{n\to\infty}} \int_\Omega F_n(\omega)d\mu(\omega) \subset \mathrm{cl} \int_\Omega w - \overline{\lim_{n\to\infty}} F_n(\omega)d\mu(\omega). \square 0pt$$

The next result provides a new, useful necessary condition for the Kuratowski-Mosco convergence of a sequence of convex sets.

Assume that X is any reflexive Banach space.

THEOREM 3.3. *If* $\{K_n\}_{n\geq 1} \subseteq P_{fc}(X)$, $sup|K_n| < \infty$ *and* $K_n \xrightarrow{K-M} K$ *as* $n \to \infty$, *then* $K \neq \emptyset$ *and* $\sigma_{K_n}(\cdot) \to \sigma_k(\cdot)$ *as* $n \to \infty$.

PROOF. That $K \neq \emptyset$ follows immediately from the reflexivity of X and the definition of the Kuratowski-Mosco limit of sets.

Also from Lemma 3.1 we know that, for all $x^* \in X^*$,

$$\overline{\lim_{n\to\infty}} \sigma_{K_n}(x^*) \leq \sigma_{w-\overline{\lim}_{K_n}}(x^*) = \sigma_K(x^*). \tag{1}$$

Furthermore, from Theorem 3.1 of Mosco [**13**], we know that

$$\sigma_{K_n}(\cdot) \xrightarrow{\tau} \sigma_K(\cdot)$$

where convergence in the τ-sense means that epi $\sigma_{K_n}(\cdot) \xrightarrow{K-M}$ epi $\sigma_K(\cdot)$ as $n \to \infty$ (see [**13**]). But from Lemma 1.10 of [**13**] (see also Lemma 1.1 of [**17**]) we know that

$$\sigma_K(x^*) \leq \underline{\lim}_{n\to\infty} \sigma_{K_n}(x^*). \tag{2}$$

From (1) and (2) we conclude that $\sigma_{K_n}(\cdot) \to \sigma_K(\cdot)$ as $n \to \infty$. □0pt

REMARKS. 1) In finite dimensional Banach spaces we can have the following converse of the above theorem: "*If* $\{K_n, K\} \subseteq P_{fc}(X), K$ *is bounded and* $\sigma_{K_n}(\cdot) \to \sigma_K(\cdot)$, *then* $K_n \xrightarrow{h} K$." This follows immediately if we combine Corollary 2C of Salinetti-Wets [**17**], Theorem 3.1 of Mosco [**13**] and Theorem 3 of Salinetti-Wets [**18**].

2) Although Corollary 2E of Salinetti-Wets [**17**] appears to provide a converse to our Theorem 3.3 for infinite dimensional Banach spaces, that corollary is not valid in an infinite dimensional setting as the following counterexample indicates: Let X be an infinite dimensional reflexive Banach space and take $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ but $x_n \xrightarrow{s} x$. Note that, for all $x^* \in X^*, \sigma_{\{x_n\}}(x^*) = (x^*, x_n) \to (x^*, x) = \sigma_{\{x\}}(x^*)$. If Corollary 2E of [**17**] was true for infinite dimensional spaces we would have had that $\sigma_{\{x_n\}}(\cdot) \xrightarrow{\tau} \sigma_{\{x\}}(\cdot)$ which, by Theorem 3.1 of [**13**], implies that $\{x_n\} \xrightarrow{K-M} \{x\}$ as $n \to \infty$, a contradiction since $x_n \xrightarrow{s} x$.

The next lemma can be viewed as a converse of Lemma 3.1. Again X is any Banach space.

LEMMA 3.2. *If* $\{K_n, K\}_{n\geq 1} \subseteq 2^X \backslash \{\emptyset\}$ *and, for all* $x^* \in X^*, \overline{\lim}_{n\to\infty} \sigma_{K_n}(x^*) \leq \sigma_K(x^*)$, *then* $w - \overline{\lim}_{n\to\infty} K_n \subseteq \overline{\mathrm{conv}} K$.

PROOF. Let $x \in w - \overline{\lim}_{n\to\infty} K_n$. Then we can find $x_k \in K_{n_k}$ such that

$$x_k \xrightarrow{w} x \Rightarrow (x^*, x_k) \to (x^*, x) \Rightarrow (x^*, x) \leq \overline{\lim}_{n\to\infty} \sigma_{K_n}(x^*) \leq \sigma_K(x^*)$$
$$\Rightarrow x \in \overline{\mathrm{conv}} K.$$

Thus finally we conclude that $w - \overline{\lim}_{n\to\infty} K_n \subseteq \overline{\mathrm{conv}} K$. □0pt

This allows us to prove the following interesting superpositional convergence result. Assume that X is a reflexive Banach space.

THEOREM 3.4. *If* $F_n : X \to P_{fc}(X)$ *is a sequence of equi-*h^**-u.s.c. multifunctions such that, for all* $x \in X, \sup_{n\geq 1} |F_n(x)| < +\infty$ *and* $F_n(x) \xrightarrow{K-M} F(x)$, *then, for any* $x_n \xrightarrow{s} x$, *we have* $w - \overline{\lim}_{n\to\infty} F_n(x_n) \subseteq F(x)$.

PROOF. For every $x^* \in B_1^* = \{z^* \in X^* : ||z^*|| \leq 1\}$, we have

$$\sigma_{F_n(x_n)}(x^*) - \sigma_{F(x)}(x^*) = \sigma_{F_n(x_n)}(x^*) - \sigma_{F_n(x)}(x^*)$$
$$+ \sigma_{F_n(x)}(x^*) - \sigma_{F(x)}(x^*).$$

From Hörmander's formula we know that

$$\sigma_{F_n(x_n)}(x^*) - \sigma_{F_n(x)}(x^*) \le h^*(F_n(x_n), F_n(x)).$$

Because of the equi-h*-u.s.c. hypothesis we have that

$$h^*(F_n(x_n), F_n(x)) \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow \sigma_{F_n(x_n)}(x^*) - \sigma_{F_n(x)}(x^*) \to 0 \quad \text{as } n \to \infty.$$

Also, since $F_n(x) \overset{K-M}{\to} F(x)$ as $n \to \infty$, Theorem 3.3 tells us that

$$\sigma_{F_n(x)}(x^*) \to \sigma_{F(x)}(x^*).$$

So we get that

$$\overline{\lim_{n\to\infty}} \left(\sigma_{F_n(x_n)}(x^*) - \sigma_{F(x)}(x^*) \right) \le 0$$

$$\Rightarrow \overline{\lim_{n\to\infty}} \sigma_{F_n(x_n)}(x^*) \le \sigma_{F(x)}(x^*), \qquad x^* \in B_1^*.$$

Exploiting the fact that the support function is positively homogeneous we get that

$$\overline{\lim_{n\to\infty}} \sigma_{F_n}(x_n)(x^*) \le \sigma_{F(x)}(x^*)$$

for all $x^* \in X^*$. Hence Lemma 3.2 tells us that

$$w - \overline{\lim_{n\to\infty}} F_n(x_n) \subseteq F(x).$$

The final auxiliary result that we will need, in order to prove the convergence theorem, is the following. Let X be any Banach space. By $\overset{uo}{\to}$ we will denote the convergence in the uniform operator topology on $\mathcal{L}(X)$.

THEOREM 3.5. *If* $\{K_n, K\}_{n\ge1} \subseteq P_f(X), K_n \subseteq G$ *for all* $n \ge 1$ *where* $G \in P_{wkc}(X), w - \overline{\lim}_{n\to\infty} K_n \subseteq K$ *and* $\{A_n, A\}_{n\ge1} \subseteq \mathcal{L}(X)$ *with*

$A_n \overset{uo}{\rightarrow} A$, *then* $w - \overline{\lim}_{n\to\infty} A_n(K_n) \subseteq A(K)$.

PROOF. For any $x^* \in X^*$ and $n \geq 1$ we have that

$$\sigma_{A_n(K_n)}(x^*) = \sigma_{K_n}(A_n^* x^*) \quad \text{and} \quad \sigma_{A(K)}(x^*) = \sigma_K(A^* x^*).$$

Then

$$\begin{aligned} \sigma_{A_n(K_n)}(x^*) - \sigma_{A(K)}(x^*) &= \sigma_{K_n}(A_n^* x^*) - \sigma_K(A^* x^*) \\ &= \sigma_{K_n}(A_n^* x^*) - \sigma_{K_n}(A^* x^*) \\ &\quad + \sigma_{K_n}(A^* x^*) - \sigma_K(A^* x^*). \end{aligned}$$

But

$$\begin{aligned} \sigma_{K_n}(A_n^* x^*) - \sigma_{K_n}(A^* x^*) &\leq \sigma_{K_n}(A_n^* x^* - A^* x^*) \\ &\leq |K_n| \; ||A_n^* x^* - A^* x^*|| \\ &\leq |G| \; ||A_n^* x^* - A^* x^*||, \end{aligned}$$

because

$$A_n \overset{uo}{\rightarrow} A \Rightarrow A_n^* \overset{uo}{\rightarrow} A^* \Rightarrow ||A_n^* x^* - A^* x^*|| \to 0 \text{ as } n \to \infty.$$

Also from Lemma 3.1 and the hypothesis that $w - \overline{\lim}_{n\to\infty} K_n \subseteq K$ we get that

$$\overline{\lim_{n\to\infty}} \sigma_{K_n}(A^* x^*) \leq \sigma_{w-\overline{\lim} K_n}(A^* x^*) \leq \sigma_K(A^* x^*).$$

So we have that

$$\overline{\lim_{n\to\infty}} \sigma_{A_n(K_n)}(x^*) \leq \sigma_{A(K)}(x^*)$$

which, by Lemma 3.2, implies that

$$w - \overline{\lim_{n\to\infty}} A_n(K_n) \subseteq A(K). \ \square 0pt$$

Now we are ready for the promised convergence result.

Again, $T = [0,T]$ with the Lebesgue measure dt, while X is a separable, reflexive Banach space. First we will prove a closure theorem.

We consider the following sequence of integral inclusions

$$(*_n) \qquad \begin{cases} \dot{x}_n(t) \in p_n(t) + \int_0^t K_n(t,s)F_n(s,x(s))\,ds \\ x(\cdot) \in C_X(T). \end{cases}$$

and

$$(*) \qquad \begin{cases} \dot{x}(t) \in p(t) + \int_0^t K(t,s)F(s,x(s))\,ds \\ x(\cdot) \in C_X(t). \end{cases}$$

THEOREM 3.6. *If* 1) $F_n : T \times X \to P_{fc}(X)$ *are multifunctions such that*:

(a) *for all* $n \geq 1, F_n(\cdot,\cdot)$ *is jointly measurable and, for all* $x \in X, |F_n(t,x)| \leq \psi(t)$ *a.e. with* $\psi(\cdot) \in L^1$,

(b) *for all* $t \in T, \{F_n(t,\cdot)\}_{n\geq 1}$ *is equi-*h^**-u.s.c.*,

(c) *For all* $(t,x) \in T \times X, F_n(t,x) \overset{K-M}{\to} F(t,x)$ *and* $F(t,x) \neq \emptyset$;

(2) $\{p_n(\cdot), p(\cdot)\}_{n\geq 1} \subseteq C_X(T)$ *and, for all* $t \in T, p_n(t) \xrightarrow{w} p(t)$;

(3) *for all* $t \in T, K_n(t,s) \xrightarrow{s} K(t,s)$ *a.e. on* $[0,t]$ and $\sup_{n\geq 1} ||K_n(t,\cdot)|| \in L^\infty([0,t])$;

and if $\{x_n(\cdot)\}_{n\geq 1}$ *is a sequence of solutions of* $(*_n), n \geq 1$ *such that, for all* $t \in T, x_n(t) \xrightarrow{s} x(t), x(\cdot) \in C_X(T)$; *then* $x(\cdot)$ *solves* $(*)$.

PROOF. By hypothesis, for all $n \geq 1$ and all $t \in T$, we have

$$\begin{aligned} x_n(t) &\in p_n(t) + \int_0^t K_n(t,s)F_n(s,x_n(s))ds \\ \Rightarrow x(t) &\in w - \overline{\lim_{n\to\infty}} \left(p_n(t) + \int_0^t K_n(t,s)F_n(s,x_n(s))ds\right) \\ &\subseteq w - \overline{\lim_{n\to\infty}}\, p_n(t) + w - \overline{\lim_{n\to\infty}} \int_0^t K_n(t,s)F_n(s,x_n(s))ds. \end{aligned}$$

Using our multivalued version of Fatou's lemma (Theorem 3.2), we can write that

$$w-\overline{\lim_{n\to\infty}}\int_0^t K_n(t,s)F_n(s,x_n(s))ds$$
$$\subseteq \mathrm{cl}\int_0^t w-\overline{\lim}_{n\to\infty}K_n(t,s)F_n(s,x_n(s))ds.$$

An application of Theorems 3.4 and 3.5 gives us

$$\int_0^t w-\overline{\lim_{n\to\infty}}\,K_n(t,s)F_n(s,x_n(s))ds\subseteq\int_0^t K(t,s)F(s,x(s))ds$$

$$\Rightarrow x(t)\in p(t)+\int_0^t K(t,s)F(s,x(s))ds,\quad t\in T.$$

Thus $x(\cdot)\in C_X(T)$ solves (*). □0pt

If we impose additional assumptions on $\{F_n(\cdot,\cdot)\}_{n\geq 1}$ and $\{K_n(\cdot,\cdot)\}_{n\geq 1}$, we can have a convergence result analogous to Theorem 1 of Strauss-Yorke [**20**].

THEOREM 3.7. *If the orientor fields $\{F_n(\cdot,\cdot)\}_{n\geq 1}$ take values in $P_{fc}(X)$, for all $x\in X, n\geq 1, |F_n(t,x)|\leq\psi(t)$ a.e. with $\psi(\cdot)\in L^\infty$, for all $t\in T, \sup_{n\geq 1}||K_n(t,s)||\leq M,\quad p_n(\cdot)\xrightarrow{C_X}p(\cdot)$ as $n\to\infty$ and the rest of the hypotheses of Theorem 3.6 hold, then if $\{x_n(\cdot)\}_{n\geq 1}$ are solutions of the $\{(*_n)\}_{n\geq 1}$, we can find a subsequence converging to $x(\cdot)\in C_X(T)$, a solution of $(_*)$.*

PROOF. Let $D=\{p_n(\cdot),p(\cdot)\}_{n\geq 1}\subseteq C_X(T)$. Note that

$$\{x_n(\cdot)\}_{n\geq 1}\subseteq W=\Big\{z(\cdot)\in C_X(T): z(t)=q(t)+\int_0^t f(s)ds,$$
$$||f(s)||\leq M||\psi||_\infty,\quad q\in D, t\in T\Big\}.$$

As in the proof of Theorem 2.2 we can show that W is a compact subset of $C_{X_W}(T)$. So, by passing to a subsequence if necessary, we may assume that $x_n(\cdot) \xrightarrow{C_{X_w}} x(\cdot) \in W \Rightarrow x_n(t) \xrightarrow{w} x(t), t \in T$. An application of the closure result (Theorem 3.6) proves this theorem. □0pt

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