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NUMERICAL SOLUTIONS OF INTEGRAL EQUATIONS ON THE HALF LINE II. THE WIENER-HOPF CASE.

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ABSTRACT. Numerical approximation schemes of quadrature type are investigated for integral equations of the form

$$x(s) - \int_0^\infty \kappa(s-t)x(t)dt = y(s), \quad 0 \le s < \infty.$$

The principal hypotheses are that κ is integrable, bounded, and uniformly continuous on R, and that x and y are bounded and continuous or, alternatively, bounded and uniformly continuous, on R^+. The convergence of numerical integration approximations is established, along with error bounds in some cases. The analysis involves the collectively compact operator approximation theory and a variant of that theory in which the role of compact sets is played by bounded uniformly equicontinuous sets of functions on R^+.

1. Introduction. Consider a Wiener-Hopf integral equation

$$x(s) - \int_0^\infty \kappa(s-t)x(t)dt = y(s), \qquad 0 \le s < \infty, \tag{1.1}$$

where x and y are bounded and continuous on $[0,\infty)$, and κ is bounded, uniformly continuous, and integrable on $(-\infty,\infty)$. For example, $\kappa(u) = e^{-|u|}$ or $\kappa(u) = 1/(1+u^2)$.

Finite-section approximations for (1.1) are given by

$$x_\beta(s) - \int_0^\beta \kappa(s-t)x_\beta(t)dt = y(s), \qquad 0 \le s < \infty, \tag{1.2}$$

for $\beta \ge 0$. Numerical integration yields discrete approximations $x_{\beta n}$ for x_β and hence for x. As an illustration, the rectangular quadrature rule gives

$$x_{\beta n}(s) - \frac{1}{n}\sum_{i=1}^{\beta n} \kappa(s - \frac{i}{n})x_{\beta n}(\frac{i}{n}) = y(s), \qquad 0 \le s < \infty, \tag{1.3}$$

for $\beta, n = 1, 2, \dots$, which reduces to a finite linear system for $x_{\beta n}(\frac{i}{n}), i = 1, \dots, \beta n$. This is an example of the Nyström method. The rectangular quadrature rule usually would not be recommended. More general quadrature formulas are introduced in §2. They include the familiar repeated rules.

This paper continues an investigation begun in [**3**] and carried forward in [**4**]. In [**3**], we compared solutions of integral equations such as (1.1) and (1.2) with a more general class of kernels $k(s,t)$. In [**4**], we studied equations analogous to (1.1) - (1.3) with $\kappa(s-t)$ replaced by the kernel $k(s,t)$ of a compact operator.

The following notation is adopted. Let $Z^+ = \{1, 2, \dots\}, R = (-\infty, \infty)$ and $R^+ = [0, \infty)$. Let X^+ be the Banach space of bounded, continuous, real or complex functions f on R^+ with $||f|| = \sup|f(t)|$. Thus, convergence in norm is uniform convergence. Let $\mathcal{B}(X^+)$ denote the space of bounded linear operators on X^+.

Equations (1.1) - (1.3) are expressed in operator forms on X^+ by

$$(I - K)x = y, \tag{1.1}$$

$$(I - K_\beta)x_\beta = y, \tag{1.2}$$

$$(I - K_{\beta n})x_{\beta n} = y. \tag{1.3}$$

The operators $K, K_\beta, K_{\beta n}$ are defined for $f \in X^+$ by

$$Kf(s) = \int_0^\infty \kappa(s-t)f(t)dt, \tag{1.4}$$

$$K_\beta f(s) = \int_0^\beta \kappa(s-t)f(t)dt, \tag{1.5}$$

$$K_{\beta n}f(s) = \frac{1}{n}\sum_{i=1}^{\beta n} \kappa\Big(s - \frac{i}{n}\Big) f\Big(\frac{i}{n}\Big), \tag{1.6}$$

where (1.6) is a special case. The general formula for $K_{\beta n}$ is given in §3. The hypotheses on κ and the quadrature formula ensure that $K, K_\beta, K_{\beta n} \in \mathcal{B}(X^+)$.

In [**3**] we showed that solutions of $(I-K)x = y$ and $(I-K_\beta)x_\beta = y$ satisfy

$$(1.7) \qquad x_\beta(s) \to x(s) \text{ as } \beta \to \infty, \text{ uniformly on finite intervals.}$$

This extended earlier work of Atkinson [**5**]. The literature on error bounds associated with (1.7) is meager. Estimates for $|x_\beta(s) - x(s)|$ in particular cases have been obtained by Atkinson [**5**], by de Hoog and Sloan [**7**], and by Anselone and Baker [**2**]. Silbermann [**8**] obtained related results by Banach algebra methods.

Here, our main purpose is to compare solutions of $(I-K_\beta)x_\beta = y$ and $(I-K_{\beta n})x_{\beta n} = y$. We first consider β to be fixed. It will be shown that

$$(1.8) \qquad ||x_{\beta n} - x_\beta|| \to 0 \text{ as } n \to \infty \qquad \forall \beta \in R^+,$$

together with error bounds. These results are derived by means of the collectively compact operator approximation theory in [**1**]. In view of (1.7) and (1.8), $x(s)$ is the iterated limit of $x_{\beta n}(s)$ as $n \to \infty$ and $\beta \to \infty$ in that order, with error bounds in some cases.

We shall obtain stronger results in the closed subspace X_u^+ of X^+ which consists of the bounded, *uniformly* continuous functions on R^+. In this setting,

$$(1.9) \qquad ||x_{\beta n} - x_\beta|| \to 0 \text{ as } n \to \infty, \text{ uniformly for } \beta \in R^+,$$

with error bounds that are uniform in β. The proof is based on a variant of the collectively compact theory in which the role of relatively compact sets is played by bounded, uniformly equicontinuous sets in X_u^+. Such sets are not relatively compact in general. By (1.7) and (1.9), $x(s)$ is the double limit of $x_{\beta n}(s)$ as β and n increase independently. The convergence is uniform on finite intervals. Several numerical examples in the paper by Atkinson [**5**] are covered by our analysis. They illustrate the uniform convergence in (1.9).

The stronger results in X_u^+ apply to most of the cases that are likely to arise in practice. The restriction to X_u^+ merely excludes non-uniformly continuous functions such as $y(s) = \sin s^2$. The analysis in X_u^+ requires somewhat stricter conditions on κ and the quadrature formula which, however, are satisfied in typical examples.

The extension of the convergence results in X^+ and X_u^+ to compact perturbations of Winer-Hopf operators will be pursued in a separate investigation. Recently, Chandler and Graham [**6**] obtained convergence results for the numerical solution of Wiener-Hopf equations, as well as for compact perturbations, in a quite different setting. They assume that the functions x and y in (1.1) decay exponentially and that κ is infinitely differentiable.

2. The quadrature formula. The quadrature formula is defined first on R^+ and then restricted to finite intervals. On R^+ the quadrature rule has the general form

$$\sum_{i=1}^{\infty} \omega_{ni} f(t_{ni}) \approx \int_0^{\infty} f(t)dt, \qquad n \in Z^+, \tag{2.1}$$

$$0 \leq t_{nl} < t_{n2} < \cdots, \qquad \omega_{ni} > 0. \tag{2.2}$$

We have in mind primarily the standard repeated rules obtained by translating a convergent rule from $[0,1]$ to the successive unit intervals. Examples include the trapezoidal rule and Simpson's rule with step length $1/n$, and the n-point Gauss rule. More general composite rules provide other examples.

Restrictions of the quadrature formula to finite intervals $[0, \beta]$ with $\beta > 0$ are expressed by

$$\sum_{0}^{\beta} {}^*\omega_{ni} f(t_{ni}) \approx \int_0^{\beta} f(t)dt, \qquad n \in Z^+, \tag{2.3}$$

where the sum is over the terms with $0 \leq t_{ni} \leq \beta$. The star in (2.3) means that, if $t_{ni} = \beta$ for some t_{ni} and β, then the corresponding weight ω_{ni} may have to be multiplied by an appropriate factor to recover the correct repeated or composite rule on $[0, \beta]$. The factor is $1/2$ for the trapezoidal rule. For convenience, the sum in (2.3) is defined to be zero for $\beta = 0$.

For the analysis in X^+ we shall assume that the quadrature formula has the basic convergence property

$$\mathbf{Q}\ \sum_0^\beta *\omega_{ni} f(t_{ni}) \to \int_0^\beta f(t)dt \text{ as } n \to \infty \quad \forall f \in C[0,\beta], \qquad \forall \beta \in R^+.$$

The convergence is uniform for f in any bounded, equicontinuous set. This follows from the general proposition that pointwise convergence of bounded linear operators is uniform on compact sets.

A consequence of $\omega_{ni} > 0$ is that, if the convergence in Q holds for all $\beta \in Z^+$, then it holds for all $\beta \in R^+$. The main ideas of a proof are as follows. Let $f \in C[0,\beta]$ and $\beta \in R^+$. It suffices to consider $f \geq 0$ with $||f|| = 1$. Let $\gamma \in Z^+$ and $\gamma > \beta$. Extend f to $[0,\gamma]$ by defining $f(t) = 0$ for $\beta < t \leq \gamma$. Approximate f by functions $f_\varepsilon, f^\varepsilon \in C[0,\gamma]$ such that

$$f_\varepsilon \leq f \leq f^\varepsilon, \qquad \int_0^\gamma [f^\varepsilon(t) - f_\varepsilon(t)]dt < \varepsilon.$$

To be more specific, let f_ε and f^ε equal f except in one-sided neighborhoods of $t = \beta$, where f_ε and f^ε are linear with slope $-1/\varepsilon$. Then

$$\int_0^\gamma f_\varepsilon(t)dt \leq \int_0^\beta f(t)dt \leq \int_0^\gamma f^\varepsilon(t)dt,$$

$$\sum_0^\gamma *\omega_{ni} f_\varepsilon(t_{ni}) \leq \sum_0^\beta *\omega_{ni} f(t_{ni}) \leq \sum_0^\gamma *\omega_{ni} f^\varepsilon(t_{ni}).$$

Apply Q to f_ε and f^ε on $[0,\gamma]$ to complete the proof. A similar argument shows that Q extends to all bounded Riemann integrable functions (see [**1**], Ch. 2).

In view of the foregoing discussion, the standard repeated rules satisfy Q. Although not recommended for our purposes, the n-point Gauss-Laguerre formula also satisfies Q.

For $0 \leq \alpha \leq \beta < \infty$ define

$$\text{(2.4)} \qquad \sum_\alpha^\beta *\omega_{ni} f(t_{ni}) = \sum_0^\beta *\omega_{ni} f(t_{ni}) - \sum_0^\alpha *\omega_{ni} f(t_{ni}).$$

The quadrature formula is additive in the sense that

$$\sum_{\alpha}^{\beta} {}^*\omega_{ni} f(t_{ni}) + \sum_{\beta}^{\gamma} {}^*\omega_{ni} f(t_{ni}) = \sum_{\alpha}^{\gamma} {}^*\omega_{ni} f(t_{ni}).$$

It follows from (2.4) and Q that

$$\sum_{\alpha}^{\beta} {}^*\omega_{ni} f(t_{ni}) \to \int_{\alpha}^{\beta} f(t)dt \text{ as } n \to \infty \qquad \forall f \in C[\alpha,\beta]. \tag{2.5}$$

The convergence is uniform for f in any bounded, equicontinuous set. Let $f \equiv 1$ in (2.5) to obtain

$$\sum_{\alpha}^{\beta} {}^*\omega_{ni} \to \beta - \alpha \text{ as } n \to \infty, \tag{2.6}$$

$$m_{\alpha\beta} = \sup_{n \in Z^+} \sum_{\alpha}^{\beta} {}^*\omega_{ni} < \infty. \tag{2.7}$$

The analysis in X_u^+ will require a stronger convergence condition than Q on the quadrature formula which, however, is satisfied by the standard repeated rules. This convergence property will involve bounded, *uniformly* equicontinuous sets in X_u^+. Although such a set S is not relatively compact, its restrictions $S_{[\alpha,\beta]}$ to closed intervals $[\alpha,\beta]$ are relatively compact in a certain uniform sense. To explain this, fix $\gamma \in R^+$ and vary $\alpha, \beta \in R^+$ with $\beta - \alpha = \gamma$. Translate all the restrictions $S_{[\alpha,\beta]}$ to $[0,\gamma]$. This yields a subset of $C[0,\gamma]$ which is bounded and equicontinuous, hence relatively compact.

We shall assume that the quadrature formula on X_u^+ has the "translationally invariant" convergence property

$$\sum_{\alpha}^{\beta} {}^*\omega_{ni} f(t_{ni}) \to \int_{\alpha}^{\beta} f(t)dt \;\text{ as } n \to \infty \qquad \forall f \in X_u^+,$$

$\mathbf{Q_u}$ uniformly for $0 \le \beta - \alpha \le \gamma$ with any fixed $\gamma \in R^+$,
and uniformly for f in any bounded,
uniformly equicontinuous set in X_u^+.

An adaptation of the argument for Q shows that if the convergence in Q_u holds for $\alpha, \beta \in Z^+$ then it holds for all $\alpha, \beta \in R^+$ (see also [**1**, Ch. 2)]. It follows that Q_u has the simpler equivalent form:

$$\sum_{\beta-1}^{\beta} {}^*\omega_{ni} f(t_{ni}) \to \int_{\beta-1}^{\beta} f(t)dt \text{ as } n \to \infty \qquad \forall f \in X_u^+,$$

uniformly for $\beta \in Z^+$, and uniformly for f in any bounded, uniformly equicontinuous set in X_u^+.

It follows easily that Q_u is satisfied by the standard repeated rules.

Let $f \equiv 1$ in Q_u to obtain

$$\sum_{\alpha}^{\beta} {}^*\omega_{ni} \to \beta - \alpha \text{ as } n \to \infty, \text{ uniformly for } 0 \le \beta - \alpha \le \gamma \text{ with any fixed } \gamma \in R^+. \tag{2.8}$$

There exists $n_0 \in Z^+$ such that the sums in (2.8) are bounded uniformly for $0 \le \beta-\alpha \le \gamma$ and $n \ge n_0$. To avoid making unimportant exceptions for small values of n, modify the quadrature formula for $n < n_0$ if necessary so that

$$m_\gamma = \sup_{\substack{0 \le \beta-\alpha \le \gamma \\ n \in Z^+}} \sum_{\alpha}^{\beta} {}^*\omega_{ni} < \infty \qquad \forall \gamma \in R^+. \tag{2.9}$$

This is satisfied by the standard repeated rules. For reference purposes, we subsume (2.9) in Q_u.

For repeated rules that are exact for constant functions on the successive unit intervals,

$$m_1 = \sum_{\beta-1}^{\beta} {}^*\omega_{ni} = 1 \qquad \forall \beta \in R^+. \tag{2.10}$$

Whenever the infinite sum in (2.1) exists, define

$$\sum_{\alpha}^{\infty} {}^*\omega_{ni} f(t_{ni}) = \sum_{0}^{\infty} \omega_{ni} f(t_{ni}) - \sum_{0}^{\alpha} {}^*\omega_{ni} f(t_{ni}). \tag{2.11}$$

The following lemma gives estimates for quadrature sums that will be used later.

LEMMA 2.1. *Assume Q_u. Let $\alpha, \beta \in Z^+$ and $\alpha < \beta$. Then*

a) $\sum_\alpha^\beta *\omega_{ni} f(t_{ni}) \leq m_1 \int_{\alpha+1}^{\beta+1} f(t)dt$ *for $f \geq 0$, f nondecreasing on $[\alpha, \beta+1]$,*

b) $\sum_\alpha^\beta *\omega_{ni} f(t_{ni}) \leq m_1 \int_{\alpha-1}^{\beta-1} f(t)dt$ *for $f \geq 0, f$ nonincreasing on $[\alpha-1, \beta]$,*

c) $\sum_\alpha^\infty *\omega_{ni} f(t_{ni}) \leq m_1 \int_{\alpha-1}^{\infty} f(t)dt$ *for $f \geq 0, f$ nonincreasing, f integrable on $[\alpha-1, \infty)$.*

PROOF. For $f \geq 0$ and f nondecreasing,

$$\sum_{j-1}^{j} *\omega_{ni} f(t_{ni}) \leq m_1 f(j) \leq m_1 \int_j^{j+1} f(t)dt.$$

Sum on j to obtain (a). For $f \geq 0$ and f nonincreasing,

$$\sum_{j}^{j+1} *\omega_{ni} f(t_{ni}) \leq m_1 f(j) \leq m_1 \int_{j-1}^{j} f(t)dt.$$

Sum on j to obtain (b), which implies (c). □

3. Convergence results in X^+. The operators $K, K_\beta, K_{\beta n}$ are defined on X^+ by

$$Kf(s) = \int_0^\infty \kappa(s-t) f(t)dt, \tag{3.1}$$

$$K_\beta f(s) = \int_0^\beta \kappa(s-t) f(t)dt, \tag{3.2}$$

$$K_{\beta n} f(s) = \sum_0^\beta *\omega_{ni} \kappa(s - t_{ni}) f(t_{ni}), \tag{3.3}$$

for $\beta \in R^+$ and $n \in Z^+$. We assume that the quadrature formula in (3.3) has the basic convergence property Q. The conditions on κ are

WH1 $\kappa \in L^1(R)$,

WH2 κ bounded, uniformly continuous on R,

WH3 $\kappa(u) \to 0$ as $u \to \underline{+}\infty$.

The following functions κ satisfy WH1-3.

EXAMPLE 3.1. $\kappa(u) = e^{-|u|}$.

EXAMPLE 3.2. $\kappa(u) = \frac{1}{1+u^2}$.

EXAMPLE 3.3. $\kappa(u) = \sin u/(1+u^2)$.

The conditions in WH1-3 are not independent. Thus,

$$\text{WH1, WH2} \Rightarrow \text{WH3}, \qquad \text{WH1, WH3}, \kappa\text{continuous} \Rightarrow \text{WH2}.$$

For ease of reference, we have included in the hypotheses on κ all of the basic properties that will be needed.

It follows from WH1 that

A
$$\sup_{s\in R^+} \int_0^\infty |\kappa(s-t)|dt = ||k||_1 < \infty,$$

B $\displaystyle\int_0^\infty |\kappa(s'-t) - \kappa(s-t)|dt \to 0$ as $s' \to s$, uniformly for $s \in R^+$,

which imply that $K, K_\beta \in \mathcal{B}(X^+)$. From WH2, $K_{\beta n} \in B(X^+)$.

The operator norms are given by

$$||K|| = \sup_{s\in R^+} \int_0^\infty |\kappa(s-t)|dt = ||\kappa||_1, \tag{3.4}$$

$$||K_\beta|| = \sup_{s\in R^+} \int_0^\beta |\kappa(s-t)|dt \le ||\kappa||_1, \tag{3.5}$$

$$(3.6) \qquad \|K_{\beta n}\| = \sup_{s\in R^+} \sum_0^\beta {}^*\omega_{ni}|\kappa(s-t_{ni})| \le m_{0\beta}\|\kappa\|_\infty,$$

where $m_{0\beta}$ is defined in (2.7) with $\alpha = 0$. The operators K_β are bounded uniformly. For each fixed β, the operators $K_{\beta n}, n \in Z^+$, are bounded uniformly.

For the time being, we focus our attention on K and K_β. Assume that κ satisfies WH1. By (3.4), $K = O$ if and only if $\kappa = 0$ a.e. The following discussion is adapted from [**3**], where more general operators are considered and further details are available. See also [**5**].

It is not true in general that $\|K_\beta f - Kf\| \to 0$ as $\beta \to \infty$. For example, let $\kappa \ge 0$ and $f \equiv 1$. Then, for all $\beta \in R^+, \|K_\beta f - Kf\| = \|\kappa\|_1 \ne 0$ if $K \ne 0$. In the study of K and K_β, the role ordinarily played by norm convergence in X^+ will be taken by uniform convergence on finite intervals. Let

$$\|f\|_{[0,\alpha]} = \max_{t\in[0,\alpha]} |f(t)|, \qquad f \in X^+, \alpha \in R^+.$$

Then

$$\begin{aligned}&f_\beta(t) \to f(t) \text{ as } \beta \to \infty, \text{ uniformly on finite intervals,}\\ &\Leftrightarrow \|f_\beta - f\|_{[0,\alpha]} \to 0 \text{ as } \beta \to \infty \qquad \forall \alpha \in R^+.\end{aligned}$$

From (3.1) and (3.2),

$$Kf(s) - K_\beta f(s) = \int_\beta^\infty \kappa(s-t)f(t)dt,$$

$$|K_\beta f(s) - Kf(s)| \le \|f\| \int_\beta^\infty |\kappa(s-t)|dt = \|f\| \int_{-\infty}^{s-\beta} |\kappa(u)|du,$$

$$\|K_\beta f - Kf\|_{[0,\alpha]} \le \|f\| \int_{-\infty}^{\alpha-\beta} |\kappa(u)|du.$$

Therefore,

$$(3.7) \qquad \begin{aligned}&K_\beta f \to Kf \text{ as } \beta \to \infty, \text{ uniformly on finite intervals },\\ &\forall f \in X^+.\end{aligned}$$

Next, consider $Kf_\beta - Kf$. We find that

$$Kf_\beta(s) - Kf(s) = \Big(\int_0^\alpha + \int_\alpha^\infty\Big)\kappa(s-t)[f_\beta(t) - f(t)]dt,$$
$$||Kf_\beta - Kf||_{[0,\gamma]} \le ||\kappa||_1 ||f_\beta - f||_{[0,\alpha]} + (||f_\beta|| + ||f||)\int_{-\infty}^{\gamma-\alpha} |\kappa(u)|du,$$

and hence

(3.8) $\{f_\beta\}$ bounded , $f_\beta \to f$ uniformly on finite intervals
$\Rightarrow Kf_\beta \to Kf$ uniformly on finite intervals .

Similarly, for $K_\beta f_\beta - Kf$,

$$K_\beta f_\beta(s) - Kf(s) = \int_0^\alpha \kappa(s-t)[f_\beta(t) - f(t)]dt + \int_\alpha^\beta \kappa(s-t)f_\beta(t)dt - \int_\alpha^\infty \kappa(s-t)f(t)dt,$$
$$||K_\beta f_\beta - Kf||_{[0,\gamma]} \le ||\kappa||_1 ||f_\beta - f||_{[0,\alpha]} + (||f_\beta|| + ||f||)\int_{-\infty}^{\gamma-\alpha} |\kappa(u)|du.$$

This yields

(3.9) $\{f_\beta\}$ bounded , $f_\beta \to f$ uniformly on finite intervals
$\Rightarrow K_\beta f_\beta \to Kf$ uniformly on finite intervals.

The Wiener-Hopf operator K is not compact unless $K = O$. However, as we shall see, the operators K_β are compact, i.e., for each $\beta \in R^+, \{K_\beta f : ||f|| \le 1\}$ is relatively compact. Although K and K_β differ in this respect, they share the following related property which serves some of the same purposes. From A and B,

(3.10) $\{Kf : ||f|| \le 1\}$ is bounded, uniformly equicontinuous,

$$\{K_\beta f : ||f|| \le 1, \beta \in R^+\} \text{ is bounded, uniformly equicontinuous.} \tag{3.11}$$

Repeated use of the Arzelà-Ascoli lemma on the successive intervals $[0, n], n \in Z^+$, followed by a diagonal argument, yields

$$\begin{aligned} &\{f_\beta\} \text{ bounded } \Rightarrow \exists\{\beta_i\} \text{ and } \exists g \in X^+ \text{ such that} \\ &K f_{\beta_i} \to g \text{ as } \beta_i \to \infty, \text{ uniformly on finite intervals,} \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &\{f_\beta\} \text{ bounded } \Rightarrow \exists\{\beta_i\} \text{ and } \exists g \in X^+ \text{ such that} \\ &K_{\beta_i} f_{\beta_i} \to g \text{ as } \beta_i \to \infty, \text{ uniformly on finite intervals,} \end{aligned} \tag{3.13}$$

Now consider the equations

$$(I - K)x = y, \qquad (I - K_\beta)x_\beta = y. \tag{3.14}$$

THEOREM 3.1. *Assume* WH1 *and* $(I - K)^{-1} \in \mathcal{B}(X^+)$. *Then there exists* $\beta_0 \in R^+$ *such that*

$$(I - K_\beta)^{-1} \in \mathcal{B}(X^+), \textit{ bounded uniformly for } \beta \ge \beta_0, \tag{3.15}$$

and

$$x_\beta(s) \to x(s) \textit{ as } \beta \to \infty, \textit{ uniformly on finite intervals.} \tag{3.16}$$

PROOF. See [**3**, Theorem 10.2], or [**7**, Theorem 5.2]. □

In [**3**], (3.15) is proved by contradiction, and (3.16) comes from (3.13) and (3.9). The arguments do not yield error bounds. The analysis in [**7**], based on Fourier transforms, yields theoretical bounds for $|x_\beta(s) - x(s)|$ which show how the error varies with s. For the special case with $||K|| < 1$, computable bounds for $|x_\beta(s) - x(s)|$ are derived in [**2**] and in [**5**]. See also Silbermann [**8**].

Next, we relate the operators K_β and $K_{\beta n}$.

THEOREM 3.2. *Assume* WH1-3 *and* Q. *Then, for any fixed* $\beta \in R^+$,

$$K_\beta \text{ is compact,} \tag{3.17}$$

$$\{K_{\beta n} : n \in Z^+\} \text{ is collectively compact,} \tag{3.18}$$

$$\|K_{\beta n} f - K_\beta f\| \to 0 \quad \text{as } n \to \infty \qquad \forall f \in X^+. \tag{3.19}$$

PROOF. Fix $\beta \in R^+$. By (3.2),

$$|K_\beta f(s)| \leq \|f\| \int_0^\beta |\kappa(s-t)|dt = \|f\| \int_{s-\beta}^s |\kappa(u)|du.$$

Hence WH1 yields

$$K_\beta f(s) \to 0 \text{ as } s \to \infty, \text{ uniformly for } \|f\| \leq 1. \tag{3.20}$$

By (3.11) and (3.20),

$$\begin{array}{l}\{K_\beta f : \|f\| \leq 1\} \text{ is bounded, uniformly equicontinuous,} \\ \text{and equiconvergent to zero at infinity.}\end{array} \tag{3.21}$$

Any such set is relatively compact in X^+. Therefore,

$$\{K_\beta f : \|f\| \leq 1\} \text{ is relatively compact,}$$

which means that K_β is a compact operator. This is a consequence of WH1 alone. For further details, see Atkinson [**5**].

From (3.3) and (2.7),

$$|K_{\beta n} f(s') - K_{\beta n} f(s)| \leq m_{0\beta} \|f\| \sup_{t \in R^+} |\kappa(s'-t) - \kappa(s-t)|.$$

Hence, by WH2,

$$\begin{array}{c}\{K_{\beta n} f : \|f\| \leq 1, n \in Z^+\} \\ \text{is bounded, uniformly equicontinuous.}\end{array} \tag{3.22}$$

Also from (3.3) and (2.7),

$$|K_{\beta n}f(s)| \leq m_{0\beta}||f|| \max_{s-\beta\leq u\leq s} |\kappa(u)|.$$

Hence, by WH3,

$$\text{(3.23)} \qquad \begin{array}{l}\{K_{\beta n}f : ||f|| \leq 1, n \in Z^+\} \\ \text{is equiconvergent to zero at infinity.}\end{array}$$

From (3.22) and (3.23),

$$\{K_{\beta n}f : ||f|| \leq 1, n \in Z^+\} \text{ is relatively compact,}$$

which means that $\{K_{\beta n} : n \in Z^+\}$ is collectively compact.

It remains to prove (3.19). Fix $f \in X^+$. Let

$$g_s(t) = \kappa(s-t)f(t).$$

Then

$$K_\beta f(s) = \int_0^\beta g_s(t)dt,$$

$$K_{\beta n}f(s) = \sum_0^\beta {}^*\omega_{ni}g_s(t_{ni}).$$

By WH2, $\{g_s : s \in R^+\}$ is a bounded, equicontinuous set in X^+. Hence, the basic convergence property Q implies that

$$K_{\beta n}f(s) \to K_\beta f(s) \text{ as } n \to \infty, \text{ uniformly for } s \in R^+,$$

which is equivalent to (3.19). □

A generalization of Theorem 3.2, for operators with kernels $k(s,t)$, is proved by different means in [**4**], Theorems 2.3 and 4.4.

COROLLARY 3.3. *Assume* WH1-3 *and* Q. *Then, for any fixed* $\beta \in R^+$,

$$\text{(3.24)} \qquad ||(K_{\beta n} - K_\beta)K_\beta|| \to 0 \quad \text{as } n \to \infty,$$

$$(3.25) \qquad ||(K_{\beta n} - K_\beta)K_{\beta_n}|| \to 0 \text{ as } n \to \infty,$$

PROOF. Since pointwise convergence of bounded linear operators, as in (3.19), is always uniform on compact sets, (3.17)-(3.19) imply (3.24) and (3.25). □

With this preparation, we are ready to compare solutions of the equations

$$(3.26) \qquad (I - K_\beta)x_\beta = y, \qquad (I - K_{\beta n})x_{\beta n} = y.$$

THEOREM 3.4. *Assume* WH1-3 *and* Q. *For some fixed* $\beta \in R^+$, *assume* $(I - K_{\beta n})^{-1} \in \mathcal{B}(X^+)$. *Then there exists* $n_0(\beta)$ *such that*

$$(3.27) \qquad (I - K_\beta n)^{-1} \in \mathcal{B}(X^+), \text{ bounded uniformly for } n \geq n_0(\beta),$$

and

$$(3.28) \qquad ||x_{\beta n} - x_\beta|| \to 0 \text{ as } n \to \infty.$$

More specifically,

$$(3.29) \qquad \Delta_{\beta n} = ||(I - K_\beta)^{-1}|| \; ||(K_{\beta n} - K_\beta)K_{\beta n}|| < 1 \qquad \forall n \geq n_0(\beta),$$

which implies

$$(3.30) \qquad ||(I - K_{\beta n})^{-1}|| \leq \frac{1 + ||(I - K_\beta)^{-1}|| \; ||K_{\beta n}||}{1 - \Delta_{\beta n}} \qquad \forall n \geq n_0(\beta)$$

and

$$(3.31) \quad ||x_{\beta n} - x_\beta|| \leq ||(I - K_{\beta n})^{-1}|| \; ||K_{\beta n}x_\beta - K_\beta x_\beta|| \to 0 \text{ as } n \to \infty.$$

PROOF. These results are consequences of the collectively compact operator approximation theory in [**1**], §1.8. □

Other bounds for $||x_{\beta n} - x_\beta||$ are available from [**1**]. A companion theorem reverses the roles of K_β and $K_{\beta n}$.

Theorems 3.1 and 3.4 enable us to relate solutions of

$$(3.32)\qquad (I-K)x = y, \qquad (I-K_{\beta n})x_{\beta n} = y$$

By (3.16) and (3.28), $x(s)$ is the iterated limit of $x_{\beta n}(s)$ as first $n \to \infty$ and then $\beta \to \infty$. By the triangle inequality,

$$(3.33)\qquad |x_{\beta n}(s) - x(s)| \le ||x_{\beta n} - x_\beta|| + |x_\beta(s) - x(s)|.$$

In some cases, there are error bounds for $|x_\beta(s) - x(s)|$ and hence for $|x_{\beta n}(s) - x(s)|$.

4. The restriction from $\mathbf{x}^+$ to $\mathbf{x}_u^+$. We shall compare solutions of the equations (1.1)-(1.3) in X_u^+. But first, we make some general observations on the effect of the restriction of the setting from X^+ to the closed subspace X_u^+.

It follows from WH1-3, with the aid of A and B, that

$$(4.1)\qquad K, K_\beta, K_{\beta n} \in \mathcal{B}(X^+), \qquad K, K_\beta, K_{\beta n} : X^+ \to X_u^+.$$

Consider the operator K. Similar conclusions will hold for K_β and $K_{\beta n}$. Suppose that

$$(4.2)\qquad (I-K)x = y, \qquad x, y \in X^+.$$

From (4.1) and (4.2),

$$(4.3)\qquad x \in X_u^+ \Leftrightarrow y \in X_u^+.$$

Temporarily, let I_u and K_u denote the restrictions of I and K to X_u^+. From (4.1), or from (4.2) and (4.3) with $y = 0$,

$$x \in X^+, \ (I-K)x = 0 \ \Leftrightarrow \ x \in X_u, \ (I_u - K_u)x = 0.$$

It follows that

$$(4.4)\qquad I-K \text{ is one-to-one on } X^+ \Leftrightarrow I_u - K_u \text{ is one-to-one on } X_u^+.$$

THEOREM 4.1. *Assume* WH1. *Then*

$$(I - K)^{-1} \in \mathcal{B}(X^+) \Leftrightarrow (I_u - K_u)^{-1} \in \mathcal{B}(X_u^+), \tag{4.5}$$

in which case

$$(I_u - K_u)^{-1}y = (I - K)^{-1}y \quad \forall y \in X_u^+. \tag{4.6}$$

PROOF. First, assume $(I - K)^{-1} \in \mathcal{B}(X^+)$. By (4.4), $I_u - K_u$ is one-on-one on X_u^+. To show that $(I_u - K_u)X_u^+ = X_u^+$, choose any $y \in X_u^+$. Let $x = (I - K)^{-1}y$. By (4.3), $x \in X_u^+$. Hence,

$$y = (I - K)x = (I_u - K_u)x \in (I_u - K_u)X_u^+, \qquad (I_u - K_u)X_u^+ = X_u^+.$$

Both (4.6) and $(I_u - K_u)^{-1} \in \mathcal{B}(X_u^+)$ follow.

It remains to prove the reverse implication in (4.5). Assume $(I_u - K_u)^{-1} \in \mathcal{B}(X_u^+)$. By (4.4), $I - K$ is one-to-one on X^+. To show that $(I-K)X^+ = X^+$, choose any $y \in X^+$. For $\beta \in R^+$, define $y_\beta(t) = y(t)$ on $[0, \beta]$ and $y_\beta(t) = y(\beta)$ on $[\beta, \infty)$. Then $y_\beta \in X_u^+$,

$$y_\beta(t) \to y(t) \text{ as } \beta \to \infty, \text{ uniformly on finite intervals,} \tag{4.7}$$

and $\{y_\beta\}$ is bounded. Let $x_\beta = (I_u - K_u)^{-1}y_\beta$. Then $x_\beta \in X_u^+, \{x_\beta\}$ is bounded, and $(I - K)x_\beta = (I_u - K_u)x_\beta = y_\beta$. Now, $x_\beta = Kx_\beta + y_\beta$. From (3.12) and (4.7), there exist $\{\beta_i\}$ and $x \in X^+$ such that

$$x_{\beta_i} \to x \text{ as } \beta_i \to \infty, \text{ uniformly on finite intervals.}$$

By (3.8),

$$y_{\beta_i} = x_{\beta_i} - Kx_{\beta_i} \to x - Kx, \text{ uniformly on finite intervals.}$$

In view of (4.7), $x - Kx = y$. Therefore, $y = (I - K)x \in (I - K)X^+$ and $(I - K)X^+ = X^+$. Since X^+ is complete, $(I - K)^{-1} \in \mathcal{B}(X^+)$. □

The analogues of Theorem 4.1 with K replaced by K_β and by $K_{\beta n}$ are easier to prove because K_β and $K_{\beta n}$ are compact and, hence, satisfy the Fredholm alternative:

$$I - K_\beta \text{ one-to-one on } X^+ \Leftrightarrow (I - K_\beta)X^+ = X^+, \tag{4.8}$$

$$(4.9)\qquad I - K_{\beta n} \text{ one-to-one on } X^+ \Leftrightarrow (I - K_{\beta n})X^+ = X^+.$$

We conclude from (4.1)-(4.6) and their counterparts for K_β and $K_{\beta n}$ that the effect of the restriction from X^+ to X_u^+ is merely to exclude nonuniformly continuous functions $y(s)$, such as $y(s) = \sin s^2$. Thus, most cases of interest should still be covered.

5. Convergence results in X_u^+. Restrict the setting to the space X_u^+ of bounded, uniformly continuous functions on R^+. As in §3, the operators $K, K_\beta, K_{\beta n}$ are expressed by

$$(5.1)\qquad Kf(s) = \int_0^\infty \kappa(s-t)f(t)dt,$$

$$(5.2)\qquad K_\beta f(s) = \int_0^\beta \kappa(s-t)f(t)dt,$$

$$(5.3)\qquad K_{\beta n} f(s) = \sum_0^\beta {}^*\omega_{ni}\kappa(s - t_{ni})f(t_{ni}).$$

Assume that the quadrature formula has the translationally invariant convergence property Q_u. The hypotheses on κ are

WH1 $\kappa \in L^1(R)$,

WH2 κ bounded, uniformly continuous on R,

WH3 $\kappa(u) \to 0$ as $u \to \pm\infty$,

WH4 $\kappa \geq 0, \kappa$ nonincreasing on $R^-; \kappa$ nondecreasing on R^-; or $|\kappa(u)| \leq \lambda(u)$ for some $\lambda \in L^1(R)$ with λ nonincreasing on R^+, λ nondecreasing on R^-.

These conditions on κ are not independent. The new condition WH4 is satisfied for typical functions κ, such as those in Examples 3.1-3.3. Although WH1-3 do not imply WH4, only rather unusual functions κ satisfy WH1-3 but not WH4.

The present circumstances are narrower in three respects. The setting has been restricted from X^+ to X_u^+. The function κ satisfies the additional condition WH4. The quadrature formula has the stronger convergence property Q_u. These limitations are not severe. Most of the examples likely to be met in practice should satisfy them.

By (4.1), $K, K_\beta, K_{\beta n} \in \mathcal{B}(X_u^+)$. The operator norms are still given by (3.4)-(3.6).

It follows from WH1 that

$$\int_{|s-t|\geq r} |\kappa(s-t)|dt = \int_{|u|\geq r} |\kappa(u)|du \to 0 \text{ as } r \to \infty. \tag{5.4}$$

A discrete analogue of (5.4) is given by

LEMMA 5.1. *Assume* WH1-4 *and* Q_u. *Then*

$$\sum_{|s-t_{ni}|\geq r} {}^*\omega_{ni}|\kappa(s-t_{ni})| \to 0 \text{ as } r \to \infty, \tag{5.5}$$

$$\text{uniformly for } s \in R^+ \text{ and } n \in Z^+.$$

PROOF. We give the proof for the second form of WH4. Let $r \in Z^+$ and $r \geq 2$. Let $s \in R^+$ and $p \leq s < p+1$, with p an integer. In (5.5),

$$\sum_{|s-t_{ni}|\geq r} {}^*\omega_{ni}|\kappa(s-t_{ni})| = \sum_{s+r}^{\infty} {}^*\omega_{ni}|\kappa(s-t_{ni})| + \sum_{0}^{s-r} {}^*\omega_{ni}|\kappa(s-t_{ni})|,$$

where the last sum is zero if $s \leq r$. By Lemma 2.1,

$$\begin{aligned}\sum_{s+r}^{\infty} {}^*\omega_{ni}|\kappa(s-t_{ni})| &\leq \sum_{p+r}^{\infty} {}^*\omega_{ni}\lambda(s-t_{ni}) \\ &\leq m_1 \int_{p+r-1}^{\infty} \lambda(s-t)dt \leq m_1 \int_{-\infty}^{2-r} \lambda(u)du.\end{aligned}$$

Similarly, for $s \geq r$,

$$\begin{aligned}\sum_{0}^{s-r} {}^*\omega_{ni}|\kappa(s-t_{ni})| &\leq \sum_{0}^{p-r+1} {}^*\omega_{ni}\lambda(s-t_{ni}) \\ &\leq m_1 \int_{1}^{p-r+2} \lambda(s-t)dt \leq m_1 \int_{r-2}^{\infty} \lambda(u)du.\end{aligned}$$

Therefore,

$$\sum_{|s-t_{ni}|\geq r} {}^*\omega_{ni}|\kappa(s-t_{ni})| \leq m_1 \int_{|u|\geq r-2} \lambda(u)du \to 0 \text{ as } r \to \infty,$$

and (5.5) follows. □

All of the ensuing results remain valid if WH4 is replaced by (5.5).

The next lemma gives discrete analogues of the properties A and B in §3.

LEMMA 5.2. *Assume* WH1-4 *and* Q_u. *Then*

A′

$$\sup_{\substack{s\in R^+\\ n\in Z^+}} \sum_0^\infty \omega_{ni}|\kappa(s-t_{ni})| < \infty,$$

B′

$$\sum_0^\infty \omega_{ni}|\kappa(s'-t_{ni})-\kappa(s-t_{ni})| \to 0 \text{ as } s' \to s,$$

uniformly for $s \in R^+$ and $n \in Z^+$.

PROOF. The proofs are elementary, but tedious. Merely break the summations into two parts, with $|s-t_{ni}| \leq r$ and $|s-t_{ni}| \geq r$. Apply Q_u, (2.9) and (5.5). For the proof of B' it is convenient to restrict s' to $|s'-s| < 1$.

By (3.5), the operators K_β are bounded uniformly for $\beta \in R^+$. By (3.6) and A', the operators $K_{\beta n}$ are bounded uniformly for $\beta \in R^+$ and $n \in Z^+$.

Now we come to the principal results. The following theorem relates the operators K_β and $K_{\beta n}$, uniformly with respect to $\beta \in R^+$.

THEOREM 5.3. *Assume* WH1-4 *and* Q_u. *Then*

(5.6)
$$\{K_\beta f : ||f|| \leq 1, \beta \in R^+\}$$
is bounded, uniformly equicontinuous,

$$\{K_{\beta n} f : ||f|| \leq 1, \beta \in R^+, n \in Z^+\} \tag{5.7}$$
$$\text{is bounded, uniformly equicontinuous,}$$

$$||K_{\beta n} f - K_\beta f|| \to 0 \text{ as } n \to \infty \quad \forall f \in X_u^+, \tag{5.8}$$
$$\text{uniformly for } \beta \in R^+, \text{ and uniformly for}$$
$$f \text{ in any bounded, uniformly equicontinuous set.}$$

PROOF. Since the operators K_β and $K_{\beta n}$ are bounded uniformly, the sets in (5.6) and (5.7) are bounded. For $||f|| \leq 1$,

$$|K_\beta f(s') - K_\beta f(s)| \leq \int_0^\infty |\kappa(s'-t) - \kappa(s-t)|dt,$$

$$|K_{\beta n} f(s') - K_{\beta n} f(s)| \leq \sum_0^\infty \omega_{ni} |\kappa(s' - t_{ni}) - K(s - t_{ni}|.$$

Therefore, by B and B', the sets in (5.6) and (5.7) are uniformly equicontinuous. It remains to prove (5.8). Without loss of generality, $||f|| \leq 1$. Let

$$J = J(s, r, \beta) = [s - r, s + r] \cap [0, \beta].$$

Thus, J is an interval of length $2r$ or less. Now

$$|K_{\beta n} f(s) - K_\beta f(s)| \leq \left| \sum_J {}^*\omega_{ni} \kappa(s - t_{ni}) f(t_{ni}) - \int_J \kappa(s-t) f(t) dt \right|$$
$$+ \sum_{|s - t_{ni}| \geq r} {}^*\omega_{ni} |\kappa(s - t_{ni})| + \int_{|s-t| \geq r} |\kappa(s-t)| dt.$$

Therefore, (5.4), (5.5) and Q_u imply (5.8). □

COROLLARY 5.4. *Assume* WH1-4 *and* Q_u. *Then*

$$||(K_{\beta n} - K_\beta) K_\beta|| \to 0 \textit{ as } n \to \infty, \textit{ uniformly for } \beta \in R^+, \tag{5.9}$$

$$||(K_{\beta n} - K_\beta) K_{\beta n}|| \to 0 \textit{ as } n \to \infty, \textit{ uniformly for } \beta \in R^+, \tag{5.10}$$

With this preparation, we relate the equations in X_u^+:

$$(5.11) \qquad (I-K)x = y, \;\; (I-K_\beta)x_\beta = y, \;\; (I-K_{\beta n})x_{\beta n} = y.$$

THEOREM 5.5. *Assume* WH1-4 *and* Q_u. *Assume also that* $(I - K)^{-1} \in \mathcal{B}(X_u^+)$. *Then there exists* $\beta_0 \in R^+$ *such that*

$$(5.12) \qquad (I-K_\beta)^{-1} \in \mathcal{B}(X_u^+), \text{ bounded uniformly for } \beta \geq \beta_0,$$

and

$$(5.13) \qquad x_\beta(s) \to x(s) \text{ as } \beta \to \infty, \text{ uniformly on finite intervals.}$$

Furthermore, there exists $n_0 \in Z^+$, *independent of* β, *such that*

$$(5.14) \;\; (I-K_{\beta n})^{-1} \in \mathcal{B}(X_u^+), \text{ bounded uniformly for } \beta \geq \beta_0, n \geq n_0,$$

and

$$(5.15) \qquad ||x_{\beta n} - x_\beta|| \to 0 \text{ as } n \to \infty, \text{ uniformly for } \beta \geq \beta_0.$$

(The bounds in Theorem 3.4 carry over without change.)

PROOF. Theorem 3.1 gives (5.12) and (5.13). Corollary 5.4 and [**1**], Theorem 1.10, give (5.14) and (5.15). □

It follows from (5.13) and (5.15) that

$$(5.16) \qquad x_{\beta n}(s) \to x(s) \text{ as } \beta, n \to \infty, \text{ uniformly on finite intervals.}$$

This double limit contrasts with the iterated limit obtained in X^+. There are error bounds for $|x_{\beta n}(s) - x(s)|$ in some cases.

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