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ON AN INTEGRAL EQUATION WITH A BESSEL FUNCTION KERNEL

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ABSTRACT. The integral equation

$$g(x) = \int_0^\infty J_x^2(y)f(y)dy, \quad x \geq 0,$$

which arises in certain problems in stereology, is solved for a wide class of input functions $g(x)$ using transform techniques. Practical sufficient conditions for the validity of the solution representation are given and illustrative examples are presented.

1. Introduction and formal analysis. Integral equations of the first kind are typically far more difficult to solve than those of the second kind. Exceptions occur in the case of difference kernels (see, for example, [**2**, pp. 301ff.], [**9**, pp. 364ff], [**10**]), product or quotient kernels [**3**, pp. 214ff], and when transform techniques are applicable. The problem-at-hand falls into this latter category.

Our interest is in finding the function $f(y)$ which satisfies the integral equation

$$g(x) = \int_0^\infty J_x^2(y)f(y)dy, \quad x \geq 0, \tag{1.1}$$

where $g(x)$ is a known function of the nonnegative real variable x and $J_x(y)$ designates the Bessel function of the first kind of order x and argument y. The importance of this equation in spatial statistics or stereology was first brought to the attention of the author by Eugene Church of DOA [**1**]. The squared Bessel function kernel and the appearance of the independent variable in (1.1) are a bit unusual. Nevertheless, an inversion formula for this integral equation does exist and can be derived using a procedure based upon analytic function theory and our knowledge of two familiar integral transforms.

We begin our formal analysis by noting that, as a function of x, the kernel $J_x^2(y)$ is very well behaved (see [**4**], for example). Indeed, it is an entire function of x, viewing x as a complex variable. Moreover,

$$J_x^2(y) \sim \begin{cases} \text{const.}\, y^{2x} \text{ for small } y, \\ \frac{\text{const.}}{y} \cos^2(y - x\pi/2 - \pi/4) \text{ as } y \to \infty, \\ \frac{\text{const.}}{x} (ey/2x)^{2x} \text{ as } x \to \infty, \mid \arg x \mid \leq \pi/2. \end{cases}$$

The implication of these observations is that, for reasonable $f(y)$, the right-hand side of (1.1) should be analytic in a domain including the right half of the complex plane $\Re x \geq 0$ and should tend to zero as $x \to \infty$, $\mid \arg x \mid \leq \pi/2 - \delta (\delta > 0)$. If we are to have any success in solving (1.1) then, the known function $g(x)$ should share this same behavior. In what follows we will also assume that $g(x)$ has a nontrivial imaginary part when x is purely imaginary, since $J_x^2(y)$ has this property. This is important since we will actually solve the given integral equation for purely imaginary values of x. The principle of the permanence of functional equations [**7**] then ensures that we have thereby in fact also solved (1.1) for real nonnegative x.

Our approach uses the easily verified Bessel function identity [**4**, p. 4]

$$J_x^2(y) - J_{-x}^2(y) = \sin \pi x [J_x(y) Y_{-x}(y) + J_{-x}(y) Y_x(y)],$$

and the integral representation

$$J_x(y) - Y_{-x}(y) + J_{-x}(y) Y_x(y) = -\frac{4}{\pi} \int_0^\infty J_0\,(2y \cosh t) \cosh 2xt\, dt,\ y > 0$$

(see [**4**, p. 97] or [**6**, p. 727]). As a consequence, if we consider (1.1) for purely imaginary values of the independent variable, it follows that (1.2)

$$\begin{aligned} \operatorname{Imag} g(ix) \equiv \frac{g(ix) - g(-ix)}{2i} &= \frac{1}{2i} \int_0^\infty \big(J_{ix}^2(y) - J_{-ix}^2(y)\big) f(y) dy \\ &= -\frac{2 \sinh \pi x}{\pi} \int_0^\infty \cos 2xt \Big(\int_0^\infty J_0(2y \cosh t) f(y) dy \Big) dt. \end{aligned}$$

To obtain this last expression we have formally interchanged the order of integration which resulted from the various substitutions in (1.1).

The right-hand side of (1.2) is in the form of a familiar Fourier cosine transform. In view of the nature of this classic transform ([**5**; Vol. I, Chapter I], [**11**]), if we replace x by $x/2$, it follows readily that

$$\int_0^\infty G(x)\cos xt\,dx = \int_0^\infty J_0(2y\cosh t)f(y)dy, \quad t \geq 0, \tag{1.3}$$

where

$$G(x) \equiv -\frac{\operatorname{Imag} g(ix/2)}{\sinh \pi x/2}. \tag{1.4}$$

Since $g(x)$ is given, the expression on the left-hand side of (1.3) is known, and this relation thus represents a new integral equation for the unknown function $f(y)$ equivalent to (1.1). The form of (1.3) with the Bessel function kernel occuring only to the first power, however, engenders a straightforward solution; the right-hand side of (1.3) is nothing more than (essentially) the Hankel transform of $f(y)/y$, ([**5**, Vol. II, Chapter VIII], [**11**]). A simple inversion therefore gives rise to the desired final result:

$$f(y) = y\int_0^\infty \tau J_0(y\tau)\Big(\int_0^\infty G(x)\cos xt\,dx\Big)d\tau \quad y \geq 0, \tag{1.5}$$

where $\tau \equiv 2\cosh t$.

2. Technical details. As agreed above, we restrict attention to functions $g(x)$ which are analytic in domains including the right half-plane $\Re x \geq 0$, are not purely real when $\Re x = 0$, and tend to zero as $x \to \infty, |\arg x| \leq \pi/2 - \delta(\delta > 0)$. There then are three areas of the formal derivation which need to be firmed up: the inversion of the Fourier cosine transform arising from (1.2), the inversion of the Hankel transform appearing in (1.3), and the necessary interchange of the order of integration which followed application of the integral representation for the Bessel function cross-product. We take up these matters in order.

The well-known Fourier theory itself suggests a reasonable sufficient condition for the inversion with the cosine kernel (see [**11**, pp. 13ff.], for example):

[A] $G(x)$ given by (1.4) belongs to $L(0,\infty)$ and, say, is continuously differentiable.

Titchmarsh [**11**, pp. 232ff.] has also extended the classical theory and provided an analogue of the Fourier single-integral formula for a wide class of kernels whose Mellin transforms do not differ greatly from that of $\cos x$. As a special case he has established

THEOREM (TITCHMARSH). *If $F(\tau)$ belongs to $L(0,\infty)$ and is of bounded variation near the point τ, then, for $\nu \geq -1/2$,*

$$\frac{1}{2}\{F(\tau+0)+F(\tau-0)\} = \int_0^\infty J_\nu(\tau y)\sqrt{\tau y}dy \int_0^\infty J_\nu(y\tau')\sqrt{y\tau'}F(\tau')d\tau'.$$

The application of this result to our investigations leads to the following sufficient condition, in the spirit of [A], for the solution of the integral equation (1.3):

[B] $\sqrt{\tau}\int_0^\infty G(x)\cos xt\,dx$, where $G(x)$ is given by (1.4) and $\tau \equiv 2\cosh t$, belongs to $L(0,\infty)$ as a function of τ and, say, is continuously differentiable.

Fubini's theorem governs the interchange of order of integration. Owing to the many repeated integrals which occur when the representation (1.5) is substituted in (1.2), however, it is easier to state the needed condition(s) in terms of the behavior of $f(y)$ rather than of $G(x)$. Accordingly, using the Tonelli-Hobson extension of the Fubini result (see [**8**, p. 630] for the finite case), we assume

[C]
$$\text{Either} \int_0^\infty dy \int_0^\infty H(x,y,t)dt < \infty$$
$$\text{or} \int_0^\infty dt \int_0^\infty H(x,y,t)dy < \infty,$$

where $H(x,y,t) \equiv | J_0(2y\cosh t)\cos 2xtf(y) \ |$ with $f(y)$ given by the formula (1.5).

It should be noted that the conditions [A], [B], [C] are unnecessarily stringent, and the solution of (1.1) can be effected by the representation (1.5) in many cases when one or more of these three criteria does not prevail. Nevertheless, the conditions as given constitute a good practical guide to when the various steps taken to effect the solution of the original integral equation are completely justified.

3. Applications. The following two examples have been chosen to indicate the nature of the calculations implicit in utilization of the formula (1.5).

I. Let

$$g(x) = \frac{1}{4a\sqrt{a^2+1}}(\sqrt{a^2+1}-a)^{2x}$$

with $a > 0$. This input function is entire in x. A simple calculation shows

$$G(x) = \frac{-1}{4a\sqrt{a^2+1}} \frac{\sin(x\ln(\sqrt{a^2+1}-a))}{\sinh \pi x/2}$$

and

$$\int_0^\infty G(x)\cos xt\, dx = \frac{1}{4a^2+\tau^2},$$

where $\tau \equiv 2\cosh t$ [**6**, p. 503]. Thence

$$\begin{aligned} f(y) &= y\int_0^\infty \frac{\tau J_0(y\tau)}{4a^2+\tau^2} d\tau \\ &= yK_0(2ay), \quad y \geq 0, \end{aligned}$$

by virtue of [**6**, p. 678], where K_0 designates the zero-order modified Bessel function of the third kind.

As a check of our calculations we note that

$$\int_0^\infty J_x^2(y) yK_0(2ay)dy = \frac{1}{4a\sqrt{a^2+1}}(\sqrt{a^2+1}-a)^{2x}, \quad x \geq 0,$$

[**6**, p. 672]. For completeness, we also observe that all three conditions [A], [B], and [C] are valid in this example. The verification of [C] is a consequence of the fact that

$$| J_0(2y \cosh t) | \leq \text{const.} (y \cosh t)^{-1/2}$$

and

$$K_0(2ay) \leq \text{const.} e^{-2ay}$$

for large values of the respective arguments.

II. Let

$$g(x) = \sqrt{\pi/2}\frac{\Gamma(x+1/4)}{\Gamma(x+3/4)}.$$

Owing to the nature of the gamma function, $g(x)$ is analytic for $\Re x > -1/4$ and behaves like $x^{-1/2}$ for large x. In this case we have

$$G(x) = \frac{1}{2}\sqrt{\pi} \mid \Gamma(ix/2 + 1/4) \mid^2$$

and

$$\int_0^\infty G(x) \cos xt \, dx = \pi\tau^{-1/2},$$

where $\tau \equiv 2\cosh t$ [**6**, p. 657]. From this intermediate result we are then led to

$$\begin{aligned} f(y) &= y \int_0^\infty \tau J_0(y\tau)\pi\tau^{-1/2} d\tau \\ &= \Gamma^2(3/4) y^{-1/2}, \quad y > 0, \end{aligned}$$

in view of [**6**, p. 684].

A check of our calculations in this example verifies that

$$\int_0^\infty J_x^2(y)\Gamma^2(3/4)y^{-1/2} dy = \sqrt{\pi/2}\frac{\Gamma(x+1/4)}{\Gamma(x+3/4)}$$

owing to [**6**, p. 692]. Only condition [A], however, is satisfied. This case, therefore, typifies a wide class of examples in which the alternating sign character of the cosine and Bessel function kernels is able to overcome a (marginal) lack of integrability on the infinite interval.

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