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# EXTENDING UFDS TO PIDS WITHOUT ADDING UNITS

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ABSTRACT. If $U$ is a UFD, then there is a PID $P$ containing $U$ that has the same unit group as $U$. Moreover, $P$ can be taken so that its field of fractions is a pure transcendental extension of the field of fractions of $U$ with transcendence degree at most $|U|$.

**1. Introduction.** At a recent conference, A. Romanowska raised the question of whether there is a PID $P$ such that

(i) $P$ is a proper subring of the real numbers;
(ii) $P$ properly contains the ring of integers; and
(iii) $P$ has unit group $P^{\times} = \{\pm 1\}$.

She was presenting her joint paper with G. Czédli **[2]**, which studies convex sets in generalized affine spaces. A classical real affine space may be described algebraically as an $\mathbb{R}$-module equipped with the affine operations $ax + (1 - a)y$ for $a \in \mathbb{R}$. The convex subsets are those closed under the operations $ax + (1 - a)y$ where $a \in [0, 1]$. The Czédli-Romanowska generalization replaces $\mathbb{R}$ with a subring $P \leq \mathbb{R}$ that is a PID.

Affine spaces over $P$ are $P$-modules equipped with the operations $ax + (1 - a)y$, $a \in P$, and convex subsets of such faithful spaces are the subsets closed under those operations $ax + (1 - a)y$ where $a \in P \cap [0, 1]$. It turns out that any invertible element in $P \cap [0, 1]$ gives rise to a congruence (called an *aiming congruence*) on $C \times C$ for each convex subset $C$ of an affine space over $P$ **[2,** Section 5]. Such congruences play an essential role in the algebraic description of the topological closure of $C$.

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This explains the source of Romanowska's question: can the PID $P \subseteq \mathbb{R}$ be chosen so that its notion of convexity is nontrivial, i.e., $P \neq \mathbb{Z}$, and such that all aiming congruences are trivial, i.e., $P^\times = \{\pm 1\}$?

In this note, Romanowska's question is considered as a question of pure commutative ring theory, and the question is answered affirmatively. In fact, it will be shown that, if $U$ is any UFD, then there is a PID $P$ containing $U$ that has the same unit group as $U$. Moreover, $P$ can be taken so that its field of fractions is a pure transcendental extension of the field of fractions of $U$ with transcendence degree at most $|U|$.

This answers Romanowska's question as follows: $\mathbb{Z}[\pi]$ is a UFD that is a subring of $\mathbb{R}$ that properly contains $\mathbb{Z}$ and has only $\pm 1$ as units. Extend $\mathbb{Z}[\pi]$ to a PID $P$ without adding units using the theorem of this paper. The field of fractions of $P$ will be a pure transcendental extension of $\mathbb{Q}(\pi)$ of countable degree, and hence, will be embeddable in $\mathbb{R}$ since the field extension $\mathbb{R}/\mathbb{Q}(\pi)$ has uncountable transcendence degree. Thus, there is a PID $P$ contained properly between $\mathbb{Z}$ and $\mathbb{R}$ whose only units are $\pm 1$.

The main result is proven in Section 2. This note concludes with Section 3, where the following observations are explained:

(i) if $P$ is any PID answering Romanowska's question, then every number in the difference $P - \mathbb{Z}$ must be transcendental;
(ii) there are integral domains that cannot be extended to PID's without adding units; and
(iii) there are UFD's that can be extended to PID's without adding units but which cannot be extended further to Euclidean domains without adding units.

**2. The proof.** If $A$ is an integral domain, then $\widehat{A}$ denotes its field of fractions and $A^\times$ denotes its group of units. If $S \subseteq A$ is a multiplicatively closed subset, then the localization of $A$ at $S$ is denoted $S^{-1}A$, although, if $S = \{b^n\}_{n \geq 0}$ is generated by a single element $b$, then we typically write $A_b$ for $S^{-1}A$.

**Lemma 2.1.** *If $U$ is a* UFD, *$a, b \in U$ are coprime, and $X$ and $Y$ are indeterminates, then*

$$U[X, Y]/(aX + bY - 1)$$

*is a* UFD *that extends* $U$*. The field of fractions of* $U[X,Y]/(aX+bY-1)$ *is a pure transcendental extension of* $\widehat{U}$ *of transcendence degree* 1.

*Proof.* First, we show that $U[X,Y]/(aX+bY-1)$ is a domain that extends $U$. $U[X]$ is a UFD whose prime elements are the irreducibles of $U$ together with those polynomials in $U[X]$ having content 1 that are irreducible over $\widehat{U}$. One such is the linear polynomial $(aX-1)$ since the content is $\gcd(a,-1)=1$. Now, $U[X,Y]=(U[X])[Y]$ is a UFD whose prime elements are the irreducibles of $U[X]$, together with those polynomials in $U[X,Y]$ having content 1 that are irreducible over $\widehat{U}(X)$. One such is the linear polynomial $bY+(aX-1)$ due to the fact that the content is $\gcd(b,aX-1)=1$. Since $aX+bY-1$ is prime in $U[X,Y]$, $U[X,Y]/(aX+bY-1)$ is a domain. In order to see that it extends $U$, it suffices to note that the ideal $(aX+bY-1)$ restricts trivially to the subring $U\leq U[X,Y]$ consisting of constant polynomials. This follows from the fact that every nonzero element of $(aX+bY-1)$ has degree at least 1 with respect to $X$ or $Y$.

Next, we show that $U[X,Y]/(aX+bY-1)$ satisfies the ascending chain condition on principal ideals (ACCP). Suppose that $(d_1)\subseteq(d_2)\subseteq\cdots$ is an ascending chain of principal ideals in $U[X,Y]/(aX+bY-1)$. Choose elements $e_{k+1}\in U[X,Y]/(aX+bY-1)$ such that $d_k=d_{k+1}e_{k+1}$. Writing $U[X,Y]/(aX+bY-1)$ in the form $U[X,(1-aX)/b]$, consider it to be a subring of the localization

$$\left(U\left[X,\frac{1-aX}{b}\right]\right)_b=U_b[X]. \tag{2.1}$$

In the larger ring, $U_b[X]$, which is a UFD, the chain must stabilize. Assume that

$$(d_k)=(d_{k+1})=\cdots,$$

so, for sufficiently large $k$, there exist elements $f_k/b^{n_k}\in U_b[X]$ with $f_k\in U[X,Y]/(aX+bY-1)$ such that $d_k\cdot(f_k/b^{n_k})=d_{k+1}$. Since $U_b[X]$ is a domain in which $d_k=d_{k+1}e_{k+1}$ and $d_k\cdot(f_k/b^{n_k})=d_{k+1}$, it must be that

$$e_{k+1}\cdot(f_k/b^{n_k})=1\quad\text{in } U_b[X],$$

or

$$e_{k+1}f_k=b^{n_k}\quad\text{in } U[X,Y]/(aX+bY-1).$$

This shows that, for sufficiently large $k$, the element $e_{k+1}$ divides a power of $b$ in $U[X,Y]/(aX+bY-1)$. A similar argument shows that, for sufficiently large $k$, the element $e_{k+1}$ divides a power of $a$ in $U[X,Y]/(aX+bY-1)$. Since $aX+bY=1$ in $U[X,Y]/(aX+bY-1)$, $e_{k+1}$ is a unit for sufficiently large $k$. Since $d_k = d_{k+1}e_{k+1}$, it follows that $(d_k) = (d_{k+1}) = \cdots$ in $U[X,Y]/(aX+bY-1)$ for sufficiently large $k$.

Next, we claim that, if $q$ is a prime divisor of $b$ in $U$, then $q$ remains prime in $U[X,Y]/(aX+bY-1)$, i.e., $(q)$ is a prime ideal in $U[X,Y]/(aX+bY-1)$. For this, it suffices to establish the primeness of the ideal $(q, aX+bY-1) = (q, aX-1)$ in $U[X,Y]$. Now,

$$U[X,Y]/(q,aX-1) \cong U/(q)[X,Y]/(aX-1) \cong U/(q)[Y]_a,$$

where the last ring may be constructed in steps: form the quotient $U/(q)$; form the polynomial ring $U/(q)[Y]$; then, localize at the powers of $a$, $U/(q)[Y]_a$. Since $U$ itself is a domain, factoring by the prime ideal $(q)$ preserves and creates the domain property; forming the polynomial ring $U/(q)[Y]$ preserves the domain property; then, localizing at the nonzero element $a$, $U/(q)[Y]_a$, also preserves the domain property. (That $a$ is nonzero in $U/(q)[Y]$ follows from the fact that $q$ does not divide $a$ in $U$, since $q \mid b$ and $\gcd(a,b) = 1$.) This shows that $U[X,Y]/(q,aX-1)$ is a domain; thus, $(q,aX-1)$ is prime in $U[X,Y]$, and hence, $q$ is prime in $U[X,Y]/(aX+bY-1)$.

Nagata's Criterion states that, if $A$ is an integral domain with ACCP, $S$ is a multiplicatively closed subset of $A$ that is generated by prime elements, and the localization $S^{-1}A$ is a UFD, then $A$ itself is a UFD. Apply this to the ring $A = U[X,Y]/(aX+bY-1)$ with $S$ equal to the multiplicatively closed subset of $U[X,Y]/(aX+bY-1)$ that is generated by the set of all prime divisors of $b$ in $U$. Here, it helps to write $A = U[X,Y]/(aX+bY-1)$ in the form

$$U\left[X, \frac{1-aX}{b}\right].$$

It has been shown that $A$ has ACCP. In the localization $S^{-1}A$, the element $b$ is a unit; hence,

$$S^{-1}A = S^{-1}\left(U\left[X, \frac{1-aX}{b}\right]\right) = S^{-1}U[X, 1-aX] = S^{-1}U[X],$$

which is a UFD since it is a localization of a polynomial ring over a UFD. By Nagata's Criterion, $A = U[X,Y]/(aX + bY - 1)$ is itself a UFD.

For the final statement of the theorem, the element $b$ becomes a unit in the field of fractions of $U[X,Y]/(aX + bY - 1)$. Hence, the field of fractions of $U[X,Y]/(aX+bY-1)$ is the same as the field of fractions of the ring $(U[X,Y]/(aX+bY-1))_b = U_b[X]$. This field of fractions is easily seen to be $\widehat{U}(X)$, which has transcendence degree 1 over $\widehat{U}$. □

**Lemma 2.2.** *Assume that $U$ is a* UFD *and $a, b \in U$ are coprime. If*

$$f, g \in U[X,Y]/(aX + bY - 1),$$

*$g$ divides $f$, and $f \in U \setminus \{0\}$, then $g \in U$. In particular (when $f = 1$) any unit of $U[X,Y]/(aX + bY - 1)$ lies in $U$. Moreover, if $f_1, f_2 \in U$ are coprime in $U$, then they remain coprime in the extension $U[X,Y]/(aX + bY - 1)$.*

*Proof.* Every element of

$$U[X,Y]/(aX + bY - 1) = U\left[X, \frac{1 - aX}{b}\right] \quad (\le U_b[X])$$

is a polynomial in $X$ over the localization $U_b$. If $f \in U \setminus \{0\}$, then $f$ has degree zero with respect to $X$; hence, any divisor of $f$ must have degree zero with respect to $X$. This forces $g \in U_b$. A similar argument using the representation

$$U[X,Y]/(aX + bY - 1) = U\left[\frac{1 - bY}{a}, Y\right] \le U_a[Y]$$

shows that $g \in U_a$. Therefore, $g \in U_a \cap U_b = U$, where the last equality follows from the facts that $U$ is a UFD and $a$ and $b$ are coprime.

The last two assertions of the lemma follow from the first. □

**Theorem 2.3.** *If $U$ is a* UFD*, then $U$ has an extension $P$ that is a* PID *such that $U$ and $P$ have exactly the same set of units. Moreover, $P$ can be chosen so that the field of fractions $\widehat{P}$ is a pure transcendental extension of $\widehat{U}$ of degree at most $|U|$.*

*Proof.* In this first paragraph, we describe the strategy of the proof. If $U$ is already a PID, then there is nothing to do. Otherwise, $U$ is infinite and contains elements $a$ and $b$ such that the ideal $(a, b)$ is not principal. If $c = \gcd(a, b)$, then $a = a'c$ and $b = b'c$ for some coprime elements $a'$ and $b'$ such that the ideal $(a', b')$ is not principal. The proof consists of a construction designed to kill off all such "bad pairs" of coprime elements, i.e., pairs of coprime elements that generate nonprincipal ideals.

The proof begins now. Assume that $U$ is an infinite UFD. Let $\kappa = |U|$, and enumerate with $\kappa$ a set of coprime pairs of elements of $U$ which includes all bad pairs of $U$, that is, all pairs of coprime elements generating nonprincipal ideals. Here, $(1, 1)$ is a coprime pair, and pairs are allowed to be reused in the enumeration, so this kind of enumeration is possible.

If the enumeration function is $\beta\colon \kappa \to U^2$, then define rings $V_i$, $i < \kappa$, as follows.

(i) $V_0 = U$.
(ii) $V_{i+1} = V_i[X_i, Y_i]/(a_iX_i + b_iY_i - 1)$ if $\beta(i) = (a_i, b_i)$.
(iii) If $\lambda \leq \kappa$ is limit, then $V_\lambda = \bigcup_{i<\lambda} V_i$.

The statement "$V_\mu$ is a UFD and the pairs enumerated by $\beta$ remain coprime in $V_\mu$" can be established for all $\mu \leq \kappa$ by transfinite induction using Lemmas 2.1 and 2.2. When $\mu = 0$, the statement holds by our initial hypothesis that $U$ is a UFD and by the definition of $\beta$. When $\mu = i + 1$ is a successor ordinal, Lemma 2.1 proves that $V_\mu$ is a UFD, while Lemma 2.2 proves that the pairs enumerated by $\beta$ remain coprime in $V_\mu$. If $\mu$ is a limit ordinal, any element

$$f \in V_\mu = \bigcup_{i<\mu} V_i$$

occurs first at some successor stage $V_{i+1}$ or else in $V_0$, and when it first occurs, all divisors of $f$ that lie in $V_\mu$ already exist in $V_i$ or $V_0$, respectively. Thus, unique factorization of elements and coprimeness of $\beta$-enumerated pairs in $V_\mu$ is inherited from $V_{i+1}$ or $V_0$.

Thus, $U_1 := V_\kappa$ is a UFD containing $U_0 := U$ as a subring. Since new divisors of elements are not introduced during the construction, no new units are introduced. Hence, the UFD $U_1$ is an extension of $U_0$

that has the same unit group; however, all bad pairs in $U_0$ have been "killed" in $U_1$.

In order to keep track of the transcendence degree of the field of fractions as the construction progresses, we note the following:

**Claim 2.4.** *Let $\kappa$ be a cardinal, and let $F_i$, $i < \kappa$, be a sequence of fields such that*

(i) *$F_{i+1}/F_i$ is a pure transcendental extension with transcendence base $T_i$ for all $i < \kappa$; and*

(ii)
$$F_\lambda := \bigcup_{i<\lambda} F_i$$
*when $\lambda \leq \kappa$ is limit.*

*Then, $F_\kappa/F_0$ is a pure transcendental extension with transcendence base $\bigcup_{i<\kappa} T_i$.*

*Sketch.* To prove the claim, it may be argued by transfinite induction on $\lambda$ that $\bigcup_{i<\lambda} T_i$ is algebraically independent and, together with $F_0$, generates $F_\lambda$ as a field. □

Applying Claim 2.4 to the situation where $F_i = \widehat{V}_i$, $i < \kappa$, we obtain that $\widehat{U}_1/\widehat{U}_0$ is a pure transcendental extension of transcendence degree $\kappa$. (In particular, $|U_1| = \kappa = |U|$.)

We may iterate the construction from above to produce a chain $U = U_0 \leq U_1 \leq \cdots$, where each $U_i$ is a UFD with the same group of units as $U$, in each $U_{i+1}$, all bad pairs from $U_i$ have been killed, any divisor of an element that first appears at the $i$th stage also exists at the $i$th stage, and each $\widehat{U_{i+1}}$ is a pure transcendental extension of $\widehat{U_i}$ of degree $\kappa = |U|$. The union

$$P = \bigcup_{i<\omega} U_i$$

is therefore a UFD with no bad pairs. Such a ring is necessarily a PID, as the following argument shows. No pair of coprime elements in $P$ can generate a nonprincipal ideal; thus, $P$ is a Bezout domain. In order to show that $P$ is a PID, it suffices to show that it is Noetherian. If this is not the case, then there is a strictly increasing chain of ideals

$$I_0 \subsetneq I_1 \subsetneq \cdots \quad \text{in } P.$$

This can be adjusted to a strictly increasing chain of principal ideals, as follows. Choose $d_{i+1} \in I_{i+1} \setminus I_i$ for all $i$. Now, choose $f_i$ so that $(f_i) = (d_1, \ldots, d_i)$ in $P$ for all $i$. This is possible since $P$ is Bezout. The chain

$$(f_1) \subsetneq (f_2) \subsetneq \cdots$$

of principal ideals in $P$ has been constructed so that it is strictly increasing. This is impossible, since $P$ is a UFD.

Applying Claim 2.4 to the chain

$$\widehat{U} = \widehat{U_0} \leq \widehat{U_1} \leq \cdots \leq \bigcup_{i<\omega} \widehat{U_i} = \widehat{P},$$

we obtain that $\widehat{P}$ is a pure transcendental extension of $\widehat{U}$ of transcendence degree $\omega \cdot \kappa = \kappa = |U|$. □

**3. Problems and discussion.** If $\mathcal{D}$ is a subcategory of a category $\mathcal{C}$, it may be asked whether each $\mathcal{C}$-object has a morphism to some $\mathcal{D}$-object. If the inclusion functor $\mathcal{D} \to \mathcal{C}$ has a left adjoint, then, indeed each $\mathcal{C}$-object has a *universal* morphism to a $\mathcal{D}$-object given by the unit of the adjunction. This is the case, for example, when $\mathcal{C}$ is the category of integral domains equipped with embeddings and $\mathcal{D}$ is the full subcategory of fields. A universal embedding of an integral domain into a field is its embedding into its field of fractions.

It may happen that each $\mathcal{C}$-object has a morphism to a $\mathcal{D}$-object, but not a universal such morphism, such as when $\mathcal{C}$ is the category of fields and $\mathcal{D}$ is the full subcategory of algebraically closed fields. The author does not know a conventional term for this situation, so (borrowing a term from order theory) we call $\mathcal{D}$ a *cofinal* subcategory if each $\mathcal{C}$-object has a morphism to a $\mathcal{D}$-object.

Every ring homomorphism

$$\varphi \colon R \longrightarrow S$$

preserves units in the sense that $u \in R^\times$ implies $\varphi(u) \in S^\times$. Say that $\varphi$ reflects units if $v \in S^\times$ implies that $\varphi^{-1}(v)$ is a nonempty subset of $R^\times$. Thus, a unit-reflecting embedding $\varphi : R \to S$ restricts to an isomorphism between unit groups.

The theorem of this paper may be expressed as follows: if $\mathcal{C}$ is the category of UFD's equipped with unit-reflecting embeddings, then the full subcategory of PID's is cofinal.

This paper does not resolve the more general question:

**Question 3.1.** *Does the inclusion functor from the category of* PID's (*with unit-reflecting embeddings*)*into the category of* UFD's (*with unit-reflecting embeddings*) *have a left adjoint? Is there a* universal *unit-reflecting embedding of a* UFD *into a* PID*?*

Now, we turn to another observation and question. Recall that Romanowska's original question was whether there is a subring $P \leq \mathbb{R}$ of the field of real numbers such that

(i) $P$ properly contains $\mathbb{Z}$;
(ii) $P$ is a PID; and
(iii) the only units of $P$ are $+1$ and $-1$.

**Claim 3.2.** *If $P$ is such a ring, then any algebraic number in $P$ must be a rational integer.*

To see this, choose any algebraic number $\alpha \in P$. The rings $K := \mathbb{Q}[\alpha]$ (a field) and $P$ (a PID) are integrally closed; thus, the intersection

$$I := K \cap P$$

is integrally closed and lies between $\mathbb{Z}[\alpha]$ and $P$. It follows that $I$ contains the integral closure of $\mathbb{Z}$ in $K$, which is the ring $\mathcal{O}_K$ of algebraic integers in $K$. By Dirichlet's unit theorem, the group of units in $\mathcal{O}_K$ is the product of a finite group of roots of unity and a free abelian group of rank $r+s-1$, where $r$ is the number of real embeddings of $K$ and $s$ is the number of pairs of conjugate complex embeddings. Since $\mathcal{O}_K \leq I \leq P$ has $+1$ and $-1$ as its only units, it follows that $r + s - 1 = 0$. Since $K$ is real, it follows that $r \geq 1$, while, of course, $s \geq 0$; hence, $r = 1$, $s = 0$, and $K$ has inclusion as its unique embedding into $\mathbb{C}$. If $d$ is the degree of the minimal polynomial of $\alpha$ over $\mathbb{Q}$, then there are at least $d$ embeddings of $K$ into $\mathbb{C}$, so, $d = 1$ and $\alpha$ is rational. If $\alpha = p/q$ where $\gcd(p, q) = 1$, then, choose $u, v \in \mathbb{Z}$ such that $pu + qv = 1$, or

$\alpha u + v = 1/q$. Since $\alpha, u, v, q \in S$, we obtain $q, 1/q \in P$; hence, $q = \pm 1$. Therefore, $\alpha \in \mathbb{Z}$.

Thus, if $P$ is any PID, answering Romanowska's question, then any number in $P - \mathbb{Z}$ is transcendental. This suggests:

**Question 3.3.** *Given a* UFD $U$*, what is the minimum transcendence degree of the extension* $\widehat{P}/\widehat{U}$*, where* $P$ *is a* PID *that contains* $U$ *and satisfies* $P^\times = U^\times$*? Is it always possible to find a* PID $P$ *such that the transcendence degree of* $\widehat{P}/\widehat{U}$ *is finite? Is it always possible to find a* PID $P$ *such that the transcendence degree of* $\widehat{P}/\widehat{U}$ *is* $1$*?*

The strategy used in this paper to construct a PID satisfying $\mathbb{Z} \subsetneq P \subsetneq \mathbb{R}$ and $P^\times = \{\pm 1\}$ is to first adjoin a transcendental number to $\mathbb{Z}$ (forming, say, $\mathbb{Z}[\pi]$), and then to eliminate all occurrences of nonprincipal ideals via a sequence of extensions. But observe that this must be done carefully. If, for example, at some point of the construction, we have a ring containing the transcendentals

$$\pi, \pi^{1/2}, \pi^{1/4}, \pi^{1/8}, \dots,$$

then the ring cannot be further extended to a PID without adding units. More generally, if at some point of the construction, we have a domain containing any strictly increasing sequence of principal ideals $(d_1) \subsetneq (d_2) \subsetneq \cdots$, then, in any larger domain with no additional units, this chain remains a properly increasing chain. This cannot occur in a PID. A stronger statement is true: if $P \leq \mathbb{R}$ answers Romanowska's question, then, in any subring of $P$ any principal ideal is contained in only finitely many other principal ideals. This observation shows that there exist integral domains, such as $\mathbb{Z}[\pi, \pi^{1/2}, \pi^{1/4}, \dots]$, which cannot be extended to PID's without adding units.

We change the question. Rather than extending a UFD to a PID without adding units, can we extend a UFD to a Euclidean domain without adding units? Interestingly, this is not always possible: there exist UFD's that cannot be extended to Euclidean domains without adding units.

For example, it is well known that $P = \mathbb{Z}[(1+\sqrt{-19})/2]$ is a PID that is not a Euclidean domain [**3,** pages 277, 282]. This ring $P$ cannot even be extended to a Euclidean domain without adding units. This can be established by a slight modification, described below, of the argument used in [**3,** page 277] to show that $P$ is not a Euclidean domain.

Suppose that $\varphi\colon P \to E$ is a unit-reflecting embedding of $P$ into a Euclidean domain $E$. $E$ has nonunits, so $E$ is not a field. Therefore, $E$ has an element $u$ of least Euclidean norm among elements in $E - (E^\times \cup \{0\})$. Such an element $u \in E$ is a *universal side divisor* for $E$, which means that every nonzero coset of the ideal $(u)$ contains a unit. In particular, $E/(u)$ has cardinality at most

$$|E^\times \cup \{0\}| = |E^\times| + 1.$$

$E/(u)$ has cardinality at least 2, since $u$ is not a unit, so,

$$2 \le [E : (u)] \le |E^\times| + 1.$$

The units of $P = \mathbb{Z}[(1+\sqrt{-19})/2]$ are only $\pm 1$, as may be shown with a norm argument. If $\varphi\colon P \to E$ is a unit-reflecting embedding, then $E^\times = \{\pm 1\}$. The previous displayed equation becomes

$$2 \le [E : (u)] \le 3,$$

so $E$ must have an ideal $(u)$ of index 2 or 3. Restricting $(u)$ to $P$, we obtain an ideal $(u)|_P = \varphi^{-1}((u))$ of index 2 or 3 in $P$. However, there is no such ideal in $Z[(1+\sqrt{-19})/2]$, as can be shown by a norm argument [**3,** page 277]. Hence, there is no unit-reflecting embedding of the PID $\mathbb{Z}[(1+\sqrt{-19})/2]$ into a Euclidean domain.

**Question 3.4.** *What conditions on a* PID $P$ *are necessary for there to exist a unit-reflecting embedding from* $P$ *into a Euclidean domain?*

**Problem 3.5.** Let $\mathcal{ID}$ be the category of integral domains equipped with unit-reflecting embeddings. Discover interesting instances $(\mathcal{C}, \mathcal{D})$ of pairs of full subcategories where $\mathcal{C} \supsetneq \mathcal{D}$, and $\mathcal{D}$ is cofinal in $\mathcal{C}$.

For example, this paper shows that $(\mathcal{C}, \mathcal{D}) =$ (UFD's, PID's) is an instance, while (PIDs, Euclidean domains) is not an instance.

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