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EULER CLASS GROUP OF CERTAIN OVERRINGS OF A POLYNOMIAL RING

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ABSTRACT. Let A be a commutative Noetherian ring of dimension n and P a projective A-module of rank n with trivial determinant. In **[2]**, Bhatwadekar and Sridharan defined the nth Euler class group of A and studied the obstruction to the existence of unimodular element in P. For $R = A[T]$ and $R = A[T, T^{-1}]$, the nth Euler class groups of R are defined by Das and Keshari in **[8, 14]**, under certain assumption on A in the latter case. We define the nth Euler class group of the ring $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is a monic polynomial and the height of the Jacobson radical of A is ≥ 2. Also, we prove results similar to those in **[14]**.

1. Introduction. Let A be a commutative Noetherian ring of dimension n, and let P be a projective A-module. By a result of Serre **[21]**, if rank $P > n$, then P has a unimodular element (equivalently, P splits off a free summand of rank 1). It is well known that this result is not true in general if rank $P = n$. In **[19,** Theorem 3.8], Murthy proved that, if P is a projective module of rank n over the coordinate ring of a smooth n-dimensional affine variety X over an algebraically closed field, then a necessary and sufficient condition for P to split off a free summand of rank 1 is the vanishing of its top Chern class $C_n(P)$ in the Chow group $CH_0(X)$, see **[18, 19]**. However, this result of Murthy is not true for smooth affine varieties over non-algebraically closed fields, as we have the example of the tangent bundle of the real 2-sphere.

In order to tackle this question for smooth affine varieties over arbitrary base fields, Nori defined the notion of the 'Euler class group' of a smooth affine variety $X = \operatorname{Spec}(A)$. To any projective A-module P of rank $= \dim A$, he attached an element in this group, called the

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Euler class of P and asked whether the vanishing of the Euler class of P would ensure that P splits off a free summand of rank 1.

In **[2]**, Bhatwadekar and Sridharan settled this question of Nori in the affirmative. We ask the following:

Question 1.1. *Let R be a ring and P a projective R-module such that* $\operatorname{rank}(P) = \dim(R) - 1$. *What is the obstruction for P to split off a free summand of rank* 1*?*

Let A be a commutative Noetherian ring (containing $\mathbb{Q}$) of dimension n. In **[8]**, Das defined the notion of the nth Euler class group $E^n(A[T])$ of $A[T]$, studied the theory of the Euler class group of a polynomial algebra $A[T]$ and the relation between the Euler class groups of A and $A[T]$. In **[14]**, Keshari defined the nth Euler class group of a Laurent polynomial algebra $A[T, T^{-1}]$ and proved similar results as in **[2]**, under certain conditions on A. Note that the definitions of the Euler class groups $E^n(A[T])$ and $E^n(A[T, T^{-1}])$ are different from the definition of the Euler class group $E^n(A)$ (due to Bhatwadekar and Sridharan) and are not obtained by replacing A by $A[T]$ or $A[T, T^{-1}]$. In order to accommodate projective $A[T]$-modules with nontrivial determinant, in **[11]**, Das and Zinna defined $E^n(A[T], L)$, where L is a projective $A[T]$-module of rank 1.

In this paper, we study the nth Euler class group of the overrings of a polynomial ring. Let A be a commutative Noetherian ring of dimension $n \geq 3$ with $\operatorname{ht}\mathcal{J}(A) \geq 2$, where $\mathcal{J}(A)$ denotes the Jacobson radical of A. Let $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. We define $E^n(A[T, 1/f(T)])$ and extend the results proved in **[3]**. In Section 2, we provide definitions and state some basic results without proof. In Section 3, we prove addition and subtraction results for $R = A[T, 1/f(T)]$, which are the main ingredients for Euler class theory. In Section 4, we define the nth Euler class group of $R = A[T, 1/f(T)]$ and prove the results similar to those in **[14]**. In Section 5, we define the nth weak Euler class group $E_0^n(R)$ of $R = A[T, 1/f(T)]$.

2. Preliminaries. All rings are assumed to be commutative Noetherian, and all modules are finitely generated.

Let A be a ring, and let P be a projective A-module. Recall that $p \in P$ is called a *unimodular* element if there exists a $\phi \in P^* = \mathrm{Hom}_A(P, A)$, such that $\phi(p) = 1$. The set of all unimodular elements of P is denoted by $\mathrm{Um}(P)$. Let $p \in P$ and $\varphi \in P^*$ be such that $\varphi(p) = 0$. Let $\varphi_p \in \mathrm{End}(P)$ be defined as $\varphi_p(q) = \varphi(q)p$. Then $1 + \varphi_p$ is a unipotent automorphism of P. An automorphism of P of the form $1 + \varphi_p$ is called a *transvection* of P if either $p \in \mathrm{Um}(P)$ or $\varphi \in \mathrm{Um}(P^*)$. We denote by $E(P)$ the subgroup of $\mathrm{Aut}(P)$ generated by all the transvections of P. Throughout this paper, $\mathcal{J}(A)$ is the Jacobson radical of a ring A.

The following is a classical result due to Serre [**21**].

Theorem 2.1. *Let A be a ring of dimension d. Then any projective A-module P of rank $> d$ has a unimodular element. In particular, if dim $A = 1$, then any projective A-module of trivial determinant is free.*

Now, we state the following results due to Bhatwadekar and Roy, which are proved in [**6,** Lemma 4.1] and [**7,** Proposition 4.1], respectively.

Lemma 2.2. *Let $B \subset C$ be rings of dimension d and $x \in B$ such that $B_x = C_x$. Then*

(i) *$B/(1 + xb) = C/(1 + xb)$ for all $b \in B$.*
(ii) *If I is an ideal of C such that $\mathrm{ht}\,(I) \geq d$ and $I + xC = C$, then there exists an element $b \in B$ such that $1 + xb \in I$.*

Proposition 2.3. *Let A be a ring and $I \subset A$ an ideal. Let P be a projective A-module of rank n. Then any transvection θ of P/IP, i.e., $\theta \in E(P/IP)$, can be lifted to a (unipotent) automorphism Θ of P. In particular, if P/IP is free of rank n, then any element ψ of $E((A/I)^n)$ can be lifted to $\Psi \in \mathrm{Aut}(P)$. If, in addition, the natural map $\mathrm{Um}(P) \to \mathrm{Um}(P/IP)$ is surjective, then the natural map $E(P) \to E(P/IP)$ is surjective.*

The next result is proved in [**13,** Theorem 3.14].

Theorem 2.4. *Let A be a ring of dimension d, and let $R = A[X, f_1/g, \ldots, f_n/g]$, where $g, f_i \in A[X]$ with g a non-zerodivisor. Let P be a projective R-module of rank $r \geq \max\{2, d+1\}$. Then $E(R \oplus P)$ acts transitively on $\mathrm{Um}(R \oplus P)$.*

The following result is a consequence of a theorem of Eisenbud and Evans, as stated in [**20,** page 1420].

Lemma 2.5. *Let A be a ring, and let P be a projective A-module of rank r. Let $(\alpha, a) \in (P^* \oplus A)$. Then, there exists an element $\beta \in P^*$ such that $\mathrm{ht} I_a \geq r$, where $I = (\alpha + a\beta)(P)$. In particular, if the ideal $(\alpha(P), a)$ has height $\geq r$, then $\mathrm{ht} I \geq r$. Further, if $(\alpha(P), a)$ is an ideal of height $\geq r$ and I is a proper ideal of A, then $\mathrm{ht} I = r$.*

Now we state some technical results due to Bhatwadekar and Keshari [**1,** Proposition 2.11, Lemma 3.3, Lemma 3.6, Lemma 4.4].

Lemma 2.6. *Let A be a ring, and let $I \subset A$ be an ideal of height n. Let $f \in A$ be such that it is not a zerodivisor $\mod I$. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Let $\alpha : P \to I$ be a linear map such that the induced map $\alpha_f : P_f \to I_f$ is a surjection. Then, there exists a $\Psi \in E(P_f)$ such that*

(1) *$\beta = \Psi(\alpha) \in P^*$, and*
(2) *$\beta(P)$ is an ideal of A of height n contained in I.*

Lemma 2.7. *Let A be a ring, and let $I = (c_1, c_2)$ be an ideal of A. Let $b \in A$ be such that $I + (b) = A$ and n is a positive even integer. Then $I = (e_1, e_2)$ with $c_1 - e_1 \in I^2$ and $b^n c_2 - e_2 \in I^2$.*

Lemma 2.8. *Let A be a ring, and let I_1 and I_2 be two comaximal ideals of A. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Let $\Phi : P \twoheadrightarrow I_1$ and $\Psi : P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Phi \otimes A/I_1 = \Psi \otimes A/I_1$. Assume that:*

(1) *$a = \Phi(0, 1)$ is a non zerodivisor mod the ideal $(\sqrt{(\Phi(P_1))})$.*
(2) *$n - 1 > \dim \overline{A}/\mathcal{J}(\overline{A})$, where $\overline{A} = A/(\Phi(P_1))$.*

Let $L \subset I_2^2$ be an ideal such that $\Phi(P_1) + L = A$. Then, the surjection $\Psi : P \twoheadrightarrow I_1 \cap I_2$ induces a surjection $\overline{\Psi} : P \twoheadrightarrow I_2/L$. Moreover, $\overline{\Psi}$ may be lifted to a surjection $\Lambda : P \twoheadrightarrow I_2$.

Lemma 2.9. *Let A be a ring with $\dim A/\mathcal{J}(A) = r$, and let P be a projective A-module of rank $\geq r+1$. Let I and L be ideals of A such that $L \subset I^2$. Let $\phi : P \twoheadrightarrow I/L$ be a surjection. Then ϕ can be lifted to a surjection $\psi : P \twoheadrightarrow I$.*

The next results are due to Bhatwadekar and Sridharan [**2,** 2.11, 4.2, 4.3, 4.4].

Lemma 2.10. *Let A be a ring, and let $I \subset A$ be an ideal. Let I_1 and I_2 be ideals of A contained in I such that $I_2 \subset I^2$ and $I_1 + I_2 = I$. Then $I = I_1 + (e)$ for some $e \in I_2$ and $I_1 = I \cap I'$, where $I_2 + I' = A$.*

Theorem 2.11. *Let A be a ring of dimension $n \geq 2$ containing $\mathbb{Q}$. Let I be an ideal of A of height n such that I/I^2 is generated by n elements. Let $w_I : (A/I)^n \twoheadrightarrow I/I^2$ be a surjection. Let P be a projective A-module of rank n with trivial determinant and an isomorphism $\chi : A \xrightarrow{\sim} \wedge^n(P)$. Then the following holds*:

(1) *If $(I, w_I) = 0$ in $E(A)$, then w_I can be lifted to a surjection from A^n to I.*
(2) *Suppose $e(P, \chi) = (I, w_I)$ in $E(A)$. Then, there exists a surjection $\alpha : P \twoheadrightarrow I$ such that (I, w_I) is obtained from (α, χ).*
(3) *$e(P, \chi) = 0$ in $E(A)$ if and only if P has a unimodular element.*

The following result is due to Mandal and Sridharan [**15,** Theorem 2.3].

Theorem 2.12. *Let A be a ring, and let I_1 and I_2 be two comaximal ideals of $A[T]$ such that I_1 contains a monic polynomial and $I_2 = I_2(0)A[T]$ is an extended ideal. Let $I = I_1 \cap I_2$. Suppose that P is a projective A-module of rank $n \geq \dim A[T]/I_1 + 2$. Let $\alpha : P \twoheadrightarrow I(0)$ and $\psi : P[T]/I_1P[T] \twoheadrightarrow I_1/{I_1}^2$ be two surjections such that $\phi(0) = \alpha \otimes A/I_1(0)$. Then there exists a surjective map $\Psi : P[T] \twoheadrightarrow I$ such that $\Psi(0) = \alpha$.*

The addition and subtraction principles, respectively, presented in Section 3 are due to Bhatwadekar and Keshari [**1,** Theorems 3.7, 5.6].

Proposition 2.13. *Let A be a ring of dimension d, and let I_1 and $I_2 \subset A$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Let $\Phi : P \twoheadrightarrow I_1$ and $\Psi : P \twoheadrightarrow I_2$ be two surjections. Then, there exists a surjection $\Delta : P \twoheadrightarrow I_1 \cap I_2$ with $\Delta \otimes A/I_1 = \Phi \otimes A/I_1$ and $\Delta \otimes A/I_2 = \Psi \otimes A/I_2$.*

Proposition 2.14. *Let A be a ring of dimension d and let I_1, $I_2 \subset A$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P_1 \oplus A$ be a projective A-module of rank n. Let $\Phi : P \twoheadrightarrow I_1$ and $\Psi : P \twoheadrightarrow I_1 \cap I_2$ be two surjections such that $\Phi \otimes A/I_1 = \Psi \otimes A/I_1$. Then, there exists a surjection $\Delta : P \twoheadrightarrow I_2$ such that $\Delta \otimes A/I_2 = \Psi \otimes A/I_2$.*

3. Addition and subtraction principles. In this section, we prove the addition and subtraction principles Theorems 3.8 and 3.9, respectively, as per our requirement. We begin by giving the next definition.

Definition 3.1. Let $f(T)$ be a monic polynomial in $A[T]$. We call $g(T) \in A[T]$ a special polynomial relative to $f(T)$ if $g(T) = 1 + f(T)h(T)$ for some monic polynomial $h(T) \in A[T]$.

Let S be the set of all special polynomials relative to $f(T)$ in $A[T]$. Clearly, S is a multiplicatively closed subset of $A[T]$.

Lemma 3.2. *Let A be a ring of dimension d, and let $R = A[T, 1/f(T)]$, where $f(T)$ is monic. Let S be the multiplicative set of all special polynomials relative to $f(T)$. Then* $\dim S^{-1}R = \dim A$.

Proof. We show that any maximal ideal R of height $d+1$ contains a special polynomial relative to $f(T)$. Let $\mathfrak{M}$ be a maximal ideal of R such that $\mathrm{ht}\mathfrak{M} = d+1$. It is easy to see that $\mathfrak{M}$ contains a $f(T)g(T)$ for some monic polynomial $g(T) \in A[T]$. Since $\mathfrak{M} + f(T)R = R$, applying 2.2 to $A[T]$ and R with $x = f(T)$, we get $1 + f(T)h(T) \in \mathfrak{M}$ for some $h(T) \in A[T]$. A suitable combination of $f(T)g(T)$ and $1 + f(T)h(T)$ will give the required element. Therefore, dim $S^{-1}R =$ dim A. $\square$

Notation 3.3. Let R and S be as above. We denote the localized ring $S^{-1}R$ by $\mathcal{R}$. From Lemma 3.2, we have $\dim \mathcal{R} = \dim A$.

The next result is proved in [**3,** Lemma 3.6] in the case where A is an affine algebra over a field. A more general version of this result is due to Das and Keshari [**10,** Lemma 3.1].

Lemma 3.4. *Let A be a ring of dimension d and $R = A[T, 1/f(T)]$, where $f(T)$ is monic. Let P be a projective R-module of rank n, where $2n \geq d+3$. Let $I \subset R$ be an ideal of height n. Let $J \subset I \cap A$ be any ideal of height $\geq d-n+2$, and let $g \in R$ be any element. Assume that we are given a surjection $\phi : P \twoheadrightarrow I/(I^2 g)$. Then, ϕ has a lift $\widetilde{\phi} : P \to I$ such that $\widetilde{\phi}(P) = I_2$ satisfies the following properties*:

(1) $I_2 + (J^2 g) = I$,
(2) $I_2 = I \cap I_1$, *where* $\mathrm{ht} I_1 \geq n$, *and*
(3) $I_1 + (J^2 g) = R$.

Proof. Let $\phi' : P \to I$ be any lift of ϕ. Since $\phi'(P) + I^2 g = I$, by Lemma 2.10, we can choose $b \in I^2 g$ such that $(\phi'(P), b) = I$. Let $C = R/(J^2 g)$ and the bar denote reduction modulo the ideal $(J^2 g)$. Now, applying 2.5 to the element $(\overline{\phi}', \overline{b})$ of $\overline{P}^* \oplus C$, there exists a $\beta \in P^*$ such that, if $N = (\phi' + b\beta)(P)$, then $\mathrm{ht}(\overline{N}_{\overline{b}}) \geq n$.

Since $b \in (I^2 g)$, the element $\phi' + b\beta$ is also a lift of ϕ. Therefore, replacing ϕ' by $\phi' + b\beta$, we may assume that $N = \phi'(P)$. Now, as $(N, b) = I$ and $b \in (I^2 g)$, it follows that $N = I \cap K$, $(K, b) = R$.

Since $b \in I$, $N_b = K_b$. Therefore, we have:

(1) $\overline{N} = \overline{I} \cap \overline{K}$ with $\mathrm{ht}(\overline{K}) = \mathrm{ht}(\overline{K}_{\overline{b}}) = \mathrm{ht}(\overline{N}_{\overline{b}}) \geq n$.
(2) $\overline{b} + \overline{K} = C$.

Now we show that $\overline{K} = C$. Assume, to the contrary, that $\overline{K}$ is a proper ideal of C. Since $b \in I^2 g$, in view of (1) and (2), we have

$$\begin{aligned} n &\leq \mathrm{ht}(\overline{K}) = \mathrm{ht}(\overline{K}_{\overline{g}}) \\ &\leq \dim C_{\overline{g}} = \dim(A/J^2)\left[T, \frac{1}{fg}\right] \\ &= \dim A/J + 1 \leq d - (d-n+2) + 1 = n-1. \end{aligned}$$

This is a contradiction. Thus, $\overline{K} = C$ and, from (1), we have $\phi'(P) + (J^2 g) = I$. By Lemma 2.10, there is an element $c \in (J^2 g)$ such that $(\phi'(P), c) = I$. It follows that $\phi'(P) = I \cap L$ and $(L, c) = R$. Take $I_2 = \phi'(P)$, $I_1 = L$ and $\phi' = \widetilde{\phi}$. Then (1), (2) and (3) follow. □

The next result is an analogue of [**1,** Lemma 4.5] for $A[T, 1/f(T)]$. When $f(T) = T$, it is proved in [**14,** Lemma 3.3].

Lemma 3.5. *Let A be a ring with $\dim A/\mathcal{J}(A) = d$ and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is a monic polynomial. Let I and L be ideals of R such that $L \subset I^2$ and L contains a special polynomial relative to $f(T)$. Let Q be a projective R-module of rank $n \geq d+1$. Let $\phi : Q \oplus R \twoheadrightarrow I/L$ be a surjection. Then, we can lift ϕ to a surjection $\Phi : Q \oplus R \twoheadrightarrow I$ with $\Phi(0,1)$, a special polynomial relative to $f(T)$.*

Proof. Let $1 + g(T)f(T) \in L$ be a special polynomial relative to $f(T) \in A[T]$. Let $\Phi'(= (\Theta, h)) : Q \oplus R \to I$ be a lift of ϕ. By adding some suitable multiple of $1 + g(T)f(T)$ to $h(T)$, we can assume that $h(T)$ is a special polynomial relative to $f(T)$. (If (Θ, h) is a lift of ϕ, then, for any $b \in L$, $(\Theta, h + b)$ is also a lift of ϕ. Now take $b = -h(1 + gf) + (1 + gf)^r$ for some large integer $r > 0$).

Let $B = R/(h)$. Since $h = 1 + g_1 f$, we have $A \hookrightarrow B$ is an integral extension, and hence, $\mathcal{J}(A) = \mathcal{J}(B) \cap A$, where $g_1 \in A[T]$ is monic. Since $A \hookrightarrow B$ is an integral extension, $A/\mathcal{J}(A) \hookrightarrow B/\mathcal{J}(B)$ is also an integral extension. Let "bar" denote reduction modulo the ideal (h). Let $\alpha : \overline{Q} \twoheadrightarrow \overline{I}/\overline{L}$ be the surjection induced by Θ. As $\dim(B/\mathcal{J}(B)) = d$ and $n \geq d + 1$, by Lemma 2.9, α can be lifted to a surjection from $\overline{Q}$ to $\overline{I}$. Therefore, a map $\Gamma : Q \to I$ exists such that $\Gamma(Q) + (h) = I$ and $(\Theta - \Gamma)(Q) = K \subset L + (h)$. Hence,

$$\Theta - \Gamma \in KQ^* \subseteq (L + h)Q^*.$$

This shows that $\Theta - \Gamma = \Theta_1 + h\Gamma_1$, where $\Theta_1 \in LQ^*$ and $\Gamma_1 \in Q^*$. Let $\Phi_1 = \Gamma + h\Gamma_1$, and let $\Phi = (\Phi_1, h)$. Then

$$\Phi(Q \oplus R) = (\Gamma + h\Gamma_1)(Q) + (h) = \Gamma(Q) + (h) = I.$$

Therefore, $\Phi : Q \oplus R \twoheadrightarrow I$ is a surjection and moreover, $\Phi(0,1) = h(T)$ is a special polynomial relative to $f(T)$. Since $\Phi - \Phi' = (\Phi_1 - \Theta, 0)$, we have $\Phi_1 - \Theta \in LQ^*$, and Φ' is a lift of ϕ. Hence, Φ is a surjective lift of ϕ. $\square$

The next result is crucial for the proof of addition and subtraction principles. For the polynomial ring, the following result is proved in [**1,** Lemma 4.6]. Our proof closely follows that proof. Let $R = A[T, 1/f(T)]$ and S be the set of all special polynomials relative to $f(T)$ in $A[T]$. Recall that we denote the localized ring $S^{-1}R$ by $\mathcal{R}$.

Lemma 3.6. *Let A be a ring of dimension d, and let $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let n be an integer such that $2n \geq d+3$. Let I be an ideal of R of height n such that $I + \mathcal{J}(A)R = R$. Assume that $\text{ht}\mathcal{J}(A) \geq d - n + 2$. Let $P = Q \oplus R^2$ be a projective R-module of rank n, and let $\phi : P \twoheadrightarrow I/I^2$ be a surjection. If the surjection $\phi \otimes \mathcal{R} : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}/I^2\mathcal{R}$ can be lifted to a surjection from $P \otimes \mathcal{R}$ to $I\mathcal{R}$, then ϕ can be lifted to a surjection $\Phi : P \twoheadrightarrow I$.*

Proof. By choosing the common denominator $h \in S$, see Lemma 3.3, there is a surjective map $\Phi' : P_h \twoheadrightarrow I_h$ which is a lift of $\phi_h : P_h \twoheadrightarrow I_h/{I_h}^2$. Since $I + \mathcal{J}(A)R = R$, I is not contained in any maximal ideal of R which contains a special polynomial relative to $f(T)$. Therefore, h is a unit modulo I. Since $\Phi' \in \text{Hom}_{R_h}(P_h, I_h)$, choose a positive even integer N such that $\Phi'' = h^N \Phi' \in \text{Hom}_R(P, I)$. Clearly $\Phi''_h : P_h \twoheadrightarrow I_h$ is a surjection.

Since h is a unit modulo I, the canonical map $R/I \to R_h/I_h$ is an isomorphism, and therefore, $I/I^2 = I_h/I_h^2$. Clearly, $\phi'' = \Phi'' \otimes R/I : P \twoheadrightarrow I/I^2$ is surjective and $\phi'' = h^N \phi$. $\square$

Now, we prove the following claim.

Claim 3.7. *The map $\phi'' : P \twoheadrightarrow I/I^2$ can be lifted to a surjection form P to I.*

Proof of Claim 3.7. We know that, if Δ is an automorphism of P and if the surjection $\phi''\Delta : P \twoheadrightarrow I/I^2$ has a surjective lift form P to I, then ϕ'' also has a surjective lift from P to I. We know that any element of $E(P/IP)$ can be lifted to an automorphism of P. By Lemma 2.6, there exists a $\Delta_1 \in E(P_h)$ such that:

(1) $\Psi = \Delta_1^*(\Phi'') \in \mathrm{Hom}_R(P, I)$, where Δ_1^* is an element of $E(P_h^*)$ induced from Δ_1, and
(2) $\Psi(P)$ is an ideal of R of height n.

Since $\Psi_h(P_h) = I_h$ and h is unit modulo I, we have $I = \Psi(P) + I^2$. By Lemma 2.10, $\Psi(P) = I_1 = I \cap I'$, where $I' + I = R$. Then, since $(I_1)_h = I_h$, we have $I'_h = R_h$. Since $I'_h = R_h$, I' contains h^r, a special polynomial relative to $f(T)$ for some integer r. Since $\Delta_1 \in E(P_h)$,

$$\overline{\Delta} = \Delta_1 \otimes R_h/I_h \in E(P_h/IP_h).$$

Due to $P/IP = P_h/IP_h$, we can regard $\overline{\Delta}$ as an element of $E(P/IP)$. By (2.9), $\overline{\Delta}$ can be lifted to an automorphism Δ of P. The map $\Psi : P \twoheadrightarrow I \cap I'$ induces a surjection $\psi : P \twoheadrightarrow I/I^2$, and we see that $\psi = \phi''\Delta$. Therefore, to prove the claim, it is enough to show that ψ can be lifted to a surjection from P to I.

If $I' = R$, then Ψ is a required surjective lift of ψ. Hence, we assume that I' is an ideal of height n. The map $\Psi : P \twoheadrightarrow I \cap I'$ induces a surjection $\psi' : P \twoheadrightarrow I'/I'^2$. Since $P = Q \oplus R^2$ and I' contains h^r, a special polynomial relative to $f(T)$ for some r, by Lemma 3.5, ψ' can be lifted to a surjection $\Psi'(= (\Gamma, a_1, a_2)) : P \twoheadrightarrow I'$, where $\Gamma \in Q^*$ and $a_1, a_2 \in R$, with a_1 a special polynomial relative to $f(T)$. If necessary, by Lemma 2.5, we can replace Γ by $\Gamma + a_2^2\Gamma_1$ for suitable $\Gamma_1 \in Q^*$ and assume that $\mathrm{ht}K = n-1$, where $K = \Gamma(Q) + Ra_1$. Let $\overline{R} = R/K$ and $\overline{A} = A/K \cap A$. Then, $\overline{A} \hookrightarrow \overline{R}$ is an integral extension, and hence,

$$\begin{aligned}\dim(\overline{R}/\mathcal{J}(\overline{R})) &= \dim(\overline{A}/\mathcal{J}(\overline{A}))\\ &\le \dim(A/\mathcal{J}(A)) \le n-2 < n-1.\end{aligned}$$

Let $P_1 = Q \oplus R$. Then $P = P_1 \oplus R$. Since K contains a_1, a special polynomial relative to $f(T)$, $K + I^2 = R$. Moreover, surjections $\Psi : P \twoheadrightarrow I \cap I'$ and $\Psi' : P \twoheadrightarrow I'$ are such that $\Psi \otimes R/I' = \Psi' \otimes R/I'$. Therefore, since $\overline{R} = R/K$ and $\dim \overline{R}/\mathcal{J}(R) < n-1$, by Lemma 2.8, there exists a surjection $\Lambda_1 : P \twoheadrightarrow I$ such that

$$\Lambda_1 \otimes R/I = \Psi \otimes R/I = \psi.$$

Therefore, $\Lambda = \Lambda_1\Delta^{-1} : P \twoheadrightarrow I$ is a lift of ϕ''. This completes the proof of claim.

Let L denote the ideal of R generated by $\mathcal{J}(A)h(T)$, and let $D = R/L$. Since $L + I = R$ and $\Delta(P) = I$, $\Delta \otimes D$ is a unimodular element of $P^* \otimes D$. Let $\Delta = (\lambda, b_1, b_2)$, where $\lambda \in \mathrm{Hom}_R(Q, R)$ and $b_1, b_2 \in R$. Since $h(T)$ is a special polynomial relative to $f(T)$, $D/\mathcal{J}(D) = A/\mathcal{J}(A)[T, 1/f(T)]$ and $\dim(A/\mathcal{J}(A)) \leq n-2$. By Lemma 2.4, the unimodular element $(\lambda, b_1, b_2) \otimes D$ can be taken to $(0,0,1)$ by an element of $E(P^* \otimes D)$. By Lemma 2.3, every element of $E(P^* \otimes D)$ can be lifted to an elementary automorphism of P^*. Moreover, since $I + (h) = R$, a lift can be chosen so that it is identity modulo I. Therefore, there exists an elementary automorphism Ω of P such that Ω is identity modulo I and $\Omega^*(\Lambda) = \Lambda\Omega = (0,0,1)$ modulo L. Therefore, replacing Λ by $\Lambda\Omega$, we can assume that $\Lambda = (\lambda, b_1, b_2)$ with $1 - b_2 \in L$.

Choose $h_1 \in R$ such that $hh_1 = 1$ modulo (b_2), and hence, modulo I. Let $\mathcal{I} = (h_1^N b_1, b_2)$ be an ideal. By Lemma 2.7, $\mathcal{I} = (e_1, e_2)$ with $e_1 - h_1^N b_1 \in \mathcal{I}^2$ and $e_2 - h_1^N b_2 \in \mathcal{I}^2$. Since $\Lambda = (\lambda, b_1, b_2), \Lambda(P) = I$ and $Rh_1 + Rb_2 = R$, we see that

$$\begin{aligned} I = \lambda(Q) + (b_1, b_2) &= h_1^N \lambda(Q) + (h_1^N b_1, b_2) \\ &= h_1^N \lambda(Q) + (e_1, e_2). \end{aligned}$$

Let $\Phi = (h_1^N \lambda, e_1, e_2) \in \mathrm{Hom}_R(P, I)$. We can see that $\Phi : P \twoheadrightarrow I$ is surjective. Moreover, since $1 - hh_1 \in I$, $\Phi \otimes R/I = h_1^N \Lambda \otimes R/I$ and $\Lambda \otimes R/I = h^N \phi \otimes R/I$. Therefore, Φ is a lift of ϕ. $\square$

If $2n \geq d+4$, the following addition and subtraction principles (Lemmas 3.8 and 3.9, respectively) are due to Bhatwadekar and Keshari for any $f(T) \in A[T]$ and without any condition on the Jacobson radical of A, see [**1**]. The only case remaining is when $2n = d+3$. Since the proof of the following results equally work in the case $2n \geq d+3$, we give the proof for the general case. In the case $f(T) = T$, this is proved in [**14,** Proposition 3.5].

Theorem 3.8. *Let A be a ring of dimension d and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let $I_1, I_2 \subset R$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P' \oplus R^2$ be a projective R-module of rank n. Assume that* $\mathrm{ht}\mathcal{J}(A) \geq d-n+2$. *Let $\Phi : P \twoheadrightarrow I_1$ and $\Psi : P \twoheadrightarrow I_2$ be two surjections. Then, there exists a surjection $\Delta : P \twoheadrightarrow I_1 \cap I_2$ with $\Delta \otimes R/I_1 = \Phi \otimes R/I_1$ and $\Delta \otimes R/I_2 = \Psi \otimes R/I_2$.*

Proof.

Step 1. Write $I = I_1 \cap I_2$ and $J = (I \cap A) \cap \mathcal{J}(A)$. Let $\Gamma : P \twoheadrightarrow I/I^2$ be a surjection induced by the surjections Φ and Ψ such that $\Gamma \otimes R/I_1 = \Phi \otimes R/I_1$ and $\Gamma \otimes R/I_2 = \Psi \otimes R/I_2$. Therefore, it is enough to show that Γ has a surjective lift from P to I. Clearly, we have $\text{ht}J \geq d-n+2$, as $\text{ht}(I \cap A) \geq n-1 \geq d-n+2$. Now, applying Lemma 3.4 to $\Gamma : P \twoheadrightarrow I/I^2$ with $g = 1$, we get a lift $\Gamma_1 : P \to I$ of Γ such that the ideal $\Gamma_1(P) = I''$ satisfies the following properties:

(1) $I = I'' + J^2$;
(2) $I'' = I \cap K$, where $\text{ht}K \geq n$;
(3) $K + J = R$.

Clearly, $\dim \mathcal{R} = d$ and $I\mathcal{R} = I_1\mathcal{R} \cap I_2\mathcal{R}$. Applying Lemma 2.13 in the ring $\mathcal{R}$ for the surjections $\Phi \otimes \mathcal{R} : P \otimes \mathcal{R} \twoheadrightarrow I_1\mathcal{R}$ and $\Psi \otimes \mathcal{R} : P \otimes \mathcal{R} \twoheadrightarrow I_2\mathcal{R}$, we obtain a surjection $\Delta : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ such that

$$\Delta \otimes \mathcal{R}/I_1\mathcal{R} = \Phi \otimes \mathcal{R}/I_1\mathcal{R}$$

and

$$\Delta \otimes \mathcal{R}/I_2\mathcal{R} = \Psi \otimes \mathcal{R}/I_2\mathcal{R}.$$

From the construction of Γ, it follows that Δ is a lift of $\Gamma \otimes \mathcal{R}$. We have two surjections $\Gamma_1 : P \twoheadrightarrow I \cap K$ and $\Delta : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Since Γ_1 is a lift of Γ, we have $\Gamma_1 \otimes \mathcal{R}/I\mathcal{R} = \Delta \otimes \mathcal{R}/I\mathcal{R}$.

Applying Lemma 2.14 in the ring $\mathcal{R}$ for the surjections $\Gamma_1 \otimes \mathcal{R}$ and Δ, we get a surjection $\Delta_1 : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ with $\Delta_1 \otimes \mathcal{R}/K\mathcal{R} = \Gamma_1 \otimes \mathcal{R}/K\mathcal{R}$. Since K is comaximal with J, we have $K\mathcal{R} + \mathcal{J}(A)\mathcal{R} = \mathcal{R}$. Applying Lemma 3.6 to the surjection $\Gamma_1 \otimes R/K$, we obtain a surjection $\Delta_2 : P \twoheadrightarrow K$ which is a lift of $\Gamma_1 \otimes R/K : P \twoheadrightarrow K/K^2$.

Step 2. Recall that $P = P' \oplus R^2$, $J = (I \cap A) \cap \mathcal{J}(A)$ and $J + K = R$. Write $P_1 = P' \oplus R$ and $P = P_1 \oplus R$. We have two surjections $\Gamma_1 : P \twoheadrightarrow I \cap K$ and $\Delta_2 : P \twoheadrightarrow K$ with $\Gamma_1 \otimes R/K = \Delta_2 \otimes R/K$.

Since $\text{ht}J \geq d-n+2$, $\dim A/J \leq d-(d-n+2) = n-2$. Let "bar" denote reduction modulo J^2. Then, $\overline{R} = A/J^2[T, 1/f(T)]$. By Lemma 2.4, after applying an automorphism of $P_1 \oplus R$, we can assume that $\Delta_2(P_1) = R$ modulo J^2 and $\Delta_2(0,1) \in J^2$. Assume that $\Delta_2(0,1) = \lambda \in J^2$. By Lemma 2.5, replacing Δ_2 by $\Delta_2 + \lambda\Delta_3$ for some $\Delta_3 \in P_1^*$, we can assume that $\text{ht}(\Delta_2(P_1)) = n-1$. Let $p_1 \in P_1$ such that $\Delta_2(p_1) = 1$ modulo J^2. Further, replacing λ by $\lambda + \Delta_2(p_1)$, we can assume that $\lambda = 1 \mod J^2$.

Let K_1 and K_2 be two ideals of $R[Y]$ defined by $K_1 = (\Delta_2(P_1), Y+\lambda)$ and $K_2 = IR[Y]$. Since $\Delta_2(P_1) + J = R$ and $J \subset I$, K_1 and K_2 are comaximal. Write $K_3 = K_1 \cap K_2$; hence, $K_3(0) = I \cap K$. Then, we have two surjections $\Gamma_1 : P \twoheadrightarrow K_3(0)$ and $\Lambda_1 : P[Y] \twoheadrightarrow K_1$ defined by $\Lambda_1 = \Delta_2$ on P_1 and $\Delta_1(0,1) = Y + \lambda$. Then,

$$\Lambda_1(0) = \Gamma_1 \mod K_1(0)^2 \quad \text{and} \quad \Delta_2 \otimes R/K = \Gamma_1 \otimes R/K.$$

Since $\text{ht}(\Delta_2(P_1)) = n-1$ and $\Delta_2(P_1) + \mathcal{J}(A) = R$, $\dim R[Y]/K_1 = \dim R/\Delta_2(P_1) \leq d-n+1 \leq n-2$. Hence, applying Lemma 2.12, we obtain a surjection $\Lambda_2 : P[Y] \twoheadrightarrow K_3$ with $\Lambda_2(0) = \Gamma_1$. Putting $Y = 1-\lambda$, we get a surjection $\widetilde{\Delta} = \Lambda_2(1-\lambda) : P \twoheadrightarrow I$ with $\widetilde{\Delta} \otimes R/I = \Gamma_1 \otimes R/I$. Since Γ_1 is a lift of $\Gamma : P \twoheadrightarrow I/I^2$, we have $\widetilde{\Delta} \otimes R/I = \Gamma \otimes R/I$. This completes the proof. $\square$

Theorem 3.9. *Let A be a ring of dimension d and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let $I_1, I_2 \subset R$ be two comaximal ideals of height n, where $2n \geq d+3$. Let $P = P' \oplus R^2$ be a projective R-module of rank n. Assume that* $\text{ht}\mathcal{J}(A) \geq d-n+2$. *Let $\Phi : P \twoheadrightarrow I_1 \cap I_2$ and $\Psi : P \twoheadrightarrow I_1$ be two surjections with $\Phi \otimes R/I_1 = \Psi \otimes R/I_1$. Then, there exists a surjection $\Delta : P \twoheadrightarrow I_2$ such that $\Phi \otimes R/I_2 = \Delta \otimes R/I_2$.*

Proof. Let $\phi : P \twoheadrightarrow I_2/I_2^2$ be a surjection induced by Φ. Let $J = (I_2 \cap A) \cap \mathcal{J}(A)$. Then, $\text{ht}J \geq d-n+2$, since $\text{ht}(I_2 \cap A) \geq n-1$ and $n-1 \geq d-n+2$. Applying Lemma 3.4, to surjection $\phi : P \twoheadrightarrow I_2/{I_2}^2$ with $g=1$, we get a lift $\widetilde{\phi} : P \to I$ of ϕ such that $\widetilde{\phi}(P) = I''$ satisfies the following properties:

(1) $I_2 = I'' + J^2$;
(2) $I'' = I_2 \cap K$, where $\text{ht}K \geq n$, and
(3) $K + J^2 = R$.

Note that we have surjections Φ and Ψ such that $\Phi \otimes R/I_1 = \Psi \otimes R/I_1$. Therefore, we have two surjections $\Phi \otimes \mathcal{R}$ and $\Psi \otimes \mathcal{R}$ such that

$$\Phi \otimes \mathcal{R}/I_1\mathcal{R} = \Psi \otimes \mathcal{R}/I_1\mathcal{R}.$$

Since $\dim \mathcal{R} = d$, applying Lemma 2.14 in the ring $\mathcal{R}$ for the surjections $\Phi \otimes \mathcal{R}$ and $\Psi \otimes \mathcal{R}$, there exists a surjection $\Gamma : P \otimes \mathcal{R} \twoheadrightarrow I_2\mathcal{R}$ such that

$$\Gamma \otimes \mathcal{R}/I_2\mathcal{R} = \Phi \otimes \mathcal{R}/I_2\mathcal{R} = \widetilde{\phi} \otimes \mathcal{R}/I_2\mathcal{R}.$$

Applying Lemma 2.14 for the surjections Γ and $\widetilde{\phi} \otimes \mathcal{R}$, there exists a surjection $\Gamma_1 : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ such that $\Gamma_1 \otimes \mathcal{R}/K\mathcal{R} = \widetilde{\phi} \otimes \mathcal{R}/K\mathcal{R}$. Since K is comaximal with $\mathcal{J}(A)$, applying Lemma 3.6, we obtain a surjection $\Gamma_2 : P \twoheadrightarrow K$ with $\Gamma_2 \otimes R/K = \widetilde{\phi} \otimes R/K$.

We have two surjections $\widetilde{\phi} : P \twoheadrightarrow I_2 \cap K$ and $\Gamma_2 : P \twoheadrightarrow K$ such that $\Gamma_2 \otimes R/K = \widetilde{\phi} \otimes R/K$. Recall that $K + \mathcal{J}(A) = R$. We get a surjection $\Delta : P \twoheadrightarrow I_2$ such that $\Delta \otimes R/I_2 = \widetilde{\phi} \otimes R/I_2 = \Phi \otimes R/I_2$, by following Step (2) of the proof of Theorem 3.8. □

In the case of $f(T) = T$, the following result is [**14,** Theorem 3.8].

Theorem 3.10. *Let A be a ring of dimension d and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let n be an integer such that $2n \geq d+3$. Let I be an ideal of R of height n. Assume that $\mathrm{ht}\mathcal{J}(A) \geq d-n+2$. Let $P = P' \oplus R^2$ be a projective R-module of rank n, and let $\phi : P \twoheadrightarrow I/I^2$ be a surjection. Assume that $\phi \otimes \mathcal{R} : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}/I^2\mathcal{R}$ can be lifted to a surjection $\Phi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Then, ϕ can be lifted to a surjection $\Delta : P \twoheadrightarrow I$.*

Proof. Let $J = (I \cap A) \cap \mathcal{J}(A)$. We have $\mathrm{ht}J \geq d - n + 2$, as $\mathrm{ht}(I_2 \cap A) \geq n-1$ and $n - 1 \geq d - n + 2$. Applying Lemma 3.4 to the surjection $\phi : P \twoheadrightarrow I/I^2$ with $g = 1$, we obtain a lift $\Phi_1 : P \to I$ of ϕ such that the ideal $\Phi_1(P) = I''$ satisfies the following properties:

(1) $I = I'' + J^2$;
(2) $I'' = I \cap K$, where $\mathrm{ht}K \geq n$;
(3) $K + J^2 = R$.

If $\mathrm{ht}K > n$, then $K = R$, and hence, $I'' = I$. Therefore, we can take Φ_1 as a required lift of the surjection ϕ. Hence, we assume that $\mathrm{ht}K = n$. We have two surjections $\Phi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ and $\Phi_1 : P \twoheadrightarrow I \cap K$ such that$\Phi \otimes \mathcal{R}/I\mathcal{R} = \Phi_1 \otimes \mathcal{R}/I\mathcal{R}$. Applying Lemma 2.14 in the ring $\mathcal{R}$ for the surjections Φ and $\Phi_1 \otimes \mathcal{R}$, we obtain a surjection $\Psi : P \otimes \mathcal{R} \twoheadrightarrow K\mathcal{R}$ such that $\Psi \otimes \mathcal{R}/K\mathcal{R} = \Phi_1 \otimes \mathcal{R}/K\mathcal{R}$.

Since $K + \mathcal{J}(A) = R$, applying Lemma 3.6, we get a surjection $\Delta_1 : P \twoheadrightarrow K$, which is a lift of $\Phi_1 \otimes R/K$. We have two surjections Φ_1 and Δ_1 with $\Phi_1 \otimes R/K = \Delta_1 \otimes R/K$. Applying Lemma 3.9, we obtain a surjection $\Delta : P \twoheadrightarrow I$ such that $\Delta \otimes R/I = \Phi_1 \otimes R/I = \phi$. This completes the proof. $\square$

As a consequence of Theorem 3.10, we have the following:

Corollary 3.11. *Let A be a Noetherian ring of dimension $n \geq 3$ with $\mathrm{ht}\mathcal{J}(A) \geq 2$, and let $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is a monic. Let $I \subset R$ be an ideal of height n and $\phi : (R/I)^n \twoheadrightarrow I/I^2$ be a surjection. Assume that $\phi \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}$ to $I\mathcal{R}$. Then, ϕ can be lifted to a surjection $\Phi : R^n \twoheadrightarrow I$.*

4. Euler class group of $A[T, 1/f(T)]$.

Assumption 4.1. *Throughout this section, let A be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension $n \geq 3$ with $\mathrm{ht}\mathcal{J}(A) \geq 2$ and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is a monic.*

The results of this section are similar to [**14,** Section 4], where it is proved for $f(T) = T$. We proceed to define the nth Euler class group of the ring $R = A[T, 1/f(T)]$, where $f(T)$ is monic.

Clearly, $\dim R = n + 1$. Let I be an ideal R of height n such that I/I^2 is generated by n elements. We define a relation on the set of all surjections from $(R/I)^n$ to I/I^2. Let α and β be two surjections from $(R/I)^n$ to I/I^2. We say α and β are *related* if there exists $\sigma \in SL_n(R/I)$ such that $\alpha\sigma = \beta$. It is easy to see that this is an equivalence relation. Let $[\alpha]$ denote the equivalence class of α. We call such an equivalence class $[\alpha]$, a *local orientation* of I.

Let $\alpha : (R/I)^n \twoheadrightarrow I/I^2$ be a surjection, which can be lifted to a surjection $\Theta : R^n \twoheadrightarrow I$. Then, any β, related to α can also be lifted to a surjection $R^n \twoheadrightarrow I$. For, let $\beta = \alpha\sigma$ for some $\sigma \in SL_n(R/I)$. If $I\mathcal{R} = \mathcal{R}$, then $\beta \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n$ to $I\mathcal{R}$ and hence, by Lemma 3.11, β can be lifted to surjection. Therefore, we assume that $I\mathcal{R}$ is a proper ideal of $\mathcal{R}$. Since $\dim \mathcal{R} = n$, we have $\dim \mathcal{R}/I\mathcal{R} = 0$, and hence, $SL_n(\mathcal{R}/I\mathcal{R}) = E_n(\mathcal{R}/I\mathcal{R})$. Therefore, by Lemma 2.3, $\sigma \otimes \mathcal{R}$ can be lifted to an element of $SL_n(\mathcal{R})$. Thus, $\beta \otimes \mathcal{R}$

can be lifted to a surjection from $\mathcal{R}^n$ to $I\mathcal{R}$. Again, by Lemma 3.11, β can be lifted to a surjection from R^n to I. Therefore, from now on, we shall identify a surjection α with the equivalence class $[\alpha]$ to which it belongs.

We call a local orientation $[\alpha]$ of I, a *global orientation* of I, if the surjection $\alpha : (R/I)^n \twoheadrightarrow I/I^2$ can be lifted to a surjection $\Theta : R^n \twoheadrightarrow I$.

Let S be the set of pairs (I, w_I), where $I \subset R$ is an ideal of height n such that I/I^2 is generated by n elements, having the property that $\operatorname{Spec}(R/I)$ is connected, and $w_I : (R/I)^n \twoheadrightarrow I/I^2$ is a local orientation of I. Let G be a free abelian group on S.

Assume that $I \subset R$ is an ideal of height n such that I/I^2 is generated by n elements. Let $I = I_1 \cap \cdots \cap I_r$ be a decomposition of I, where the I_ks are pairwise comaximal ideals of height n and $\operatorname{Spec}(R/I_k)$ is connected. By [**8,** Lemma 4.4], it follows that such a decomposition is unique. We say that the I_ks are connected components of I. Let $w_I : (R/I)^n \twoheadrightarrow I/I^2$ be a surjection. Then, w_I induces surjections $w_{I_k} : (R/I_k)^n \twoheadrightarrow I_k/I_k^2$. By (I, w_I), we denote the element $\sum (I_k, w_{I_k})$ of G.

Let $S' = \{(I, w_I) \in G \mid w_I : (R/I)^n \twoheadrightarrow I/I^2 \text{ is a global orientation}\}$. Let H be the free subgroup of G generated by S'. We define the nth *Euler class group* of R, denoted by $E^n(R)$, to be G/H. By abuse of notation, we will write $E(R)$ for $E^n(R)$ throughout this paper.

Let P be a projective R-module of rank n having trivial determinant. Let $\chi : R \xrightarrow{\sim} \wedge^n P$ be an isomorphism. To the pair (P, χ), we associate an element $e(P, \chi)$ of $E(R)$ as follows:

Let $\lambda : P \twoheadrightarrow I_1$ be a surjection, where $I_1 \subset R$ is an ideal of height n (by Lemma 2.5, such a surjection always exists). Let $\overline{\lambda} : P/I_1P \twoheadrightarrow I_1/I_1^2$ be the induced surjection, where "bar" denotes reduction modulo I_1. By Lemma 2.1, P/I_1P is a free R/I_1-module of rank n, as $\dim R/I_1 \leq 1$ and P has a trivial determinant. We choose an isomorphism $\overline{\gamma} : (R/I_1)^n \xrightarrow{\sim} P/I_1P$ such that $\wedge^n(\overline{\gamma}) = \chi$. Let w_{I_1} be the surjection $\overline{\lambda}\overline{\gamma} : (R/I_1)^n \twoheadrightarrow I_1/{I_1}^2$. Let $e(P, \chi)$ be the image of (I_1, w_{I_1}) in $E(R)$. We say that (I_1, w_{I_1}) is *obtained* from the pair (λ, χ).

Lemma 4.2. *The assignment, sending (P, χ) to the element $e(P, \chi)$, is well defined.*

Proof. Recall that $w_{I_1} : (R/I_1)^n \twoheadrightarrow I_1/I_1^2$ is a surjection. Let $\mu : P \twoheadrightarrow I_2$ be another surjection, where I_2 is an ideal R of height n. Let (I_2, w_{I_2}) be obtained from the pair (μ, χ). Let $J = (I_1 \cap I_2) \cap A$. By Lemma 3.4, w_{I_1} can be lifted to $\Phi : R^n \twoheadrightarrow I_1 \cap K$, where $\mathrm{ht} K = n$ and $K + J = R$.

Since K and I_1 are comaximal, Φ induces a local orientation w_K of K. Clearly, $(I_1, w_{I_1}) + (K, w_K) = 0$ in $E(R)$. Let $L = K \cap I_2$. Since $K + I_2 = R$, w_K and w_{I_2} together induce a local orientation w_L of L, it is enough to show that $(L, w_L) = 0$ in $E(R)$ (since $(L, w_L) = (K, w_K) + (I_2, w_{I_2})$ in $E(R)$ and $(L, w_L) = 0$ implies $(I_1, w_{I_1}) = (I_2, w_{I_2})$ in $E(R)$).

Due to the fact that $\dim \mathcal{R} = n = \mathrm{rank} P$, $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R})$ is well defined in $E(\mathcal{R})$ [**2,** Section 4]. Hence, it follows that $w_L \otimes \mathcal{R}$ is a global orientation of $L\mathcal{R}$. Therefore, by Lemma 3.11, w_L is a global orientation of L, i.e., $(L, w_L) = 0$ in $E(R)$. This proves Lemma 4.2. □

Notation 4.3. We define the Euler class of (P, χ) to be $e(P, \chi)$.

Remark 4.4. From [**12,** Remark 2.16], since the ring extension $R \to \mathcal{R}$ is flat, there is a group homomorphism $\Gamma : E(R) \to E(\mathcal{R})$. For more details, we refer to [**16,** Section 3]. Further, it is easy to see that Γ is an injective group homomorphism.

Theorem 4.5. *Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements, and let $w_I : R^n \to I/I^2$ be a local orientation of I. If the image of (I, w_I) is zero in $E(R)$, then w_I is a global orientation of I.*

Proof. Let $(I, w_I) = 0$ in $E(R)$. By Remark 4.4, we have $(I\mathcal{R}, w_I \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. Therefore, by Lemma 2.11, $w_I \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n \twoheadrightarrow I\mathcal{R}$ (as $\dim \mathcal{R} = n$). By Lemma 3.11, w_I can be lifted to a surjection R^n to I, and hence, w_I is a global orientation of I. □

Theorem 4.6. *Let P be a projective R-module of rank n with trivial determinant, and let I be an ideal R of height n. Let $\psi : P \twoheadrightarrow I/I^2$ be a surjection such that $\psi \otimes \mathcal{R}$ can be lifted to a surjection $\Psi : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$. Then, there exists a surjection $\widetilde{\Psi} : P \twoheadrightarrow I$, which is a lift of ψ.*

Proof. Recall that $\mathrm{ht}\mathcal{J}(A) \geq 2$. Let $J = I \cap \mathcal{J}(A)$. Then, $\mathrm{ht}J \geq 2$. By Lemma 3.4, ψ can be lifted to a surjection $\Phi : P \twoheadrightarrow I \cap I'$, where $\mathrm{ht}I' = n$ and $I' + J = R$.

Fix a trivialization $\chi : R \xrightarrow{\sim} \wedge^n P$. Let $\lambda : (R/(I \cap I'))^n \xrightarrow{\sim} P/(I \cap I')P$ be an isomorphism such that $\wedge^n(\lambda) = \chi \otimes R/(I \cap I')$. Then, $e(P, \chi) = (I \cap I', w_{I \cap I'})$ in $E(R)$, where $w_{I\cap I'} = (\Phi \otimes R/(I \cap I'))\lambda$. Therefore, $e(P, \chi) = (I, w_I) + (I', w_{I'})$, where w_I and $w_{I'}$ are local orientations of I and I' respectively, induced from $w_{I\cap I'}$.

Since $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = (I\mathcal{R}, w_I \otimes \mathcal{R})$ (using Ψ), $(I'\mathcal{R}, w_{I'} \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$, i.e., $w_{I'} \otimes \mathcal{R}$ can be lifted to a surjection from $\mathcal{R}^n$ to $I'\mathcal{R}$. By Lemma 3.11, $w_{I'}$ can be lifted to an n set of generators of I', say $I' = (f_1, \ldots, f_n)$. Since $I' + \mathcal{J}(A)R = R$ and $\mathrm{ht}I' = n$, we have $\dim R/I' = 0$. Hence, applying Proposition 2.3, Lemma 2.4 and Lemma 2.5, after performing an elementary transformation on the generators of I', we can assume that

(1) $\mathrm{ht}(f_1, \ldots, f_{n-1}) = n - 1$;
(2) $\dim R/(f_1, \ldots, f_{n-1}) \leq 1$; and
(3) $f_n = 1 \mod J^2$. □

Write $C = R[Y]$, $K_1 = (f_1, \ldots, f_{n-1}, Y + f_n)$, $K_2 = IC$ and $K_3 = K_1 \cap K_2$.

Claim 4.7. *There exists a surjection* $\Delta(Y) : P[Y] \twoheadrightarrow K_3$ *such that* $\Delta(0) = \Phi$.

First, we show that the theorem follows from the claim. Specializing $\Delta(Y)$ at $Y = 1 - f_n$, we obtain a surjection $\Delta_1 : P \twoheadrightarrow I$. Since $1 - f_n \in J^2 \subset I^2$, $\Delta_1 = \Phi \mod I^2$. Therefore, Δ_1 is a lift of ψ. This proves the result.

Proof of Claim 4.7. λ induces an isomorphism $\delta : (R/I')^n \xrightarrow{\sim} P/I'P$ such that $\wedge^n(\delta) = \chi \otimes R/I'$. Also, $(\Phi \otimes R/I')\delta = w_{I'}$. Since $\dim C/K_1 = \dim R/(f_1, \ldots, f_{n-1}) \leq 1$, and P has trivial determinant, by Lemma 2.1, $P[Y]/K_1P[Y]$ is free of rank n. Choose an isomorphism $\Gamma(Y) : (C/K_1)^n \xrightarrow{\sim} P[Y]/K_1P[Y]$ such that $\wedge^n(\Gamma(Y)) = \chi \otimes C/K_1$.

Since $\wedge^n(\delta) = \chi \otimes R/I'$, $\Gamma(0)$ and δ differs by an element of $SL_n(R/I')$. Since $\dim R/I' = 0$, $SL_n(R/I') = E_n(R/I')$. Therefore, we can alter $\Gamma(Y)$ by an element of $SL_n(C/K_1)$ and assume that $\Gamma(0) = \delta$.

Let $\Lambda(Y) : (C/K_1)^n \twoheadrightarrow K_1/K_1^2$ be the surjection induced by the set of generators $(f_1, \ldots, f_{n-1}, Y + f_n)$ of K_1. Thus, we get a surjection $\Delta(Y) = \Lambda(Y)\Gamma(Y)^{-1} : P[Y]/K_1P[Y] \twoheadrightarrow K_1/K_1^2$. Since $\Gamma(0) = \delta$, $\Phi \otimes R/I' = w_{I'}\delta^{-1}$ and $\Lambda(0) = w_{I'}$, we have $\Delta(0) = \Phi \otimes R/I'$. By Lemma 2.12, we get a surjection $\Delta : P[Y] \twoheadrightarrow K_3$ such that $\Delta(0) = \Phi$. This proves the claim. □

Lemma 4.8. *Let P be a projective R-module of rank n having trivial determinant and $\chi : R \xrightarrow{\sim} \wedge^n P$. Let $e(P, \chi) = (I, w_I)$ in $E(R)$, where I is an ideal R of height n. Then, there exists a surjection $\Delta : P \twoheadrightarrow I$ such that (I, w_I) is obtained from (Δ, χ).*

Proof. Since $\dim R/I \leq 1$ and P has trivial determinant, by Lemma 2.1, P/IP is a free R/I-module of rank n. Choose an isomorphism $\lambda : (R/I)^n \xrightarrow{\sim} P/IP$ such that $\wedge^n(\lambda) = \chi \otimes R/I$. Let $\gamma = w_I\lambda^{-1} : P/IP \twoheadrightarrow I/I^2$.

Due to the fact that $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = (I\mathcal{R}, w_I \otimes \mathcal{R})$ in $E(\mathcal{R})$, by Lemma 2.11, there exists a surjection $\Gamma : P \otimes \mathcal{R} \twoheadrightarrow I\mathcal{R}$ such that $(I\mathcal{R}, w_I \otimes \mathcal{R})$ is obtained from the pair $(\Gamma, \chi \otimes \mathcal{R})$, i.e., Γ is a lift of $\gamma \otimes \mathcal{R}$. Applying Lemma 4.6, there exists a surjection $\Delta : P \twoheadrightarrow I$ such that Δ is a lift of γ. Since $(\Delta \otimes R/I)\lambda = w_I$ and $\wedge^n(\lambda) = \chi \otimes R/I$, (I, w_I) is obtained from the pair (Δ, χ). This completes the proof. □

The next lemma follows from Lemma 3.4.

Lemma 4.9. *Let $(I, w_I) \in E(R)$. Then, there exists an ideal $I_1 \subset R$ of height n and a local orientation w_{I_1} of I_1 such that $(I, w_I) + (I_1, w_{I_1}) = 0$ in $E(R)$. Further, I_1 can be chosen to be comaximal with any ideal $K \subset R$ of height ≥ 2.*

Corollary 4.10. *Let P be a projective R-module of rank n with trivial determinant and $\chi : R \xrightarrow{\sim} \wedge^n(P)$. Then, $e(P, \chi) = 0$ if and only if P has a unimodular element. In particular, if P has a unimodular element, then*

(1) *P maps onto any ideal of height n generated by n elements* (4.6).

(2) *Let $\beta : P \twoheadrightarrow I$ be a surjection, where I is an ideal R of height n. Then I is generated by n elements.*

Proof. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where I is an ideal R of height n. Let $e(P,\chi) = (I, w_I)$ in $E(R)$, where (I, w_I) is obtained from the pair (α, χ).

Assume that $e(P,\chi) = 0$ in $E(R)$. Then $(I, w_I) = 0$ in $E(R)$. By Lemma 4.9, there exists an ideal I' of height n such that $I' + \mathcal{J}(A) = R$ and a local orientation $w_{I'}$ of I' such that $(I, w_I) + (I', w_{I'}) = 0$ in $E(R)$. Since $(I, w_I) = 0$, $(I', w_{I'}) = 0$ in $E(R)$. Hence, without loss of generality, we can assume that $I + \mathcal{J}(A)R = R$.

By Lemma 4.5, I is generated by n elements, say $I = (f_1, \cdots, f_n)$. Since $I + \mathcal{J}(A)R = R$, $\dim R/I = 0$. Hence, applying Lemmas 2.3 and 2.4, after performing some elementary transformations on the generators of I, we can assume that dim $R/(f_1, \ldots, f_{n-1}) \leq 1$.

Let $C = R[Y]$ and $K = (f_1, \ldots, f_{n-1}, Y + f_n)$ be an ideal of C. We have two surjections $\alpha : P \twoheadrightarrow K(0)(= I)$ and $\phi : P[Y]/KP[Y] \twoheadrightarrow K/K^2$ such that $\phi(0) = \alpha \mod K(0)^2$, where ϕ is the composition of two maps, $\phi_1 : P[Y]/KP[Y] \xrightarrow{\sim} (C/K)^n$ with $\wedge^n(\phi_1) = \chi^{-1} \otimes C/K$ and $\phi_2 : (C/K)^n \twoheadrightarrow K/K^2$ defined by $(f_1, \ldots, f_{n-1}, Y + f_n)$. Applying Lemma 2.12, with $I_1 = K$ and $I_2 = C$, we get a surjection $\Phi : P[Y] \twoheadrightarrow K$. Since $\Phi(1 - f_n) : P \twoheadrightarrow R$, P has a unimodular element.

Conversely, we assume that P has a unimodular element. Applying Lemma 2.11, we have $(I\mathcal{R}, w_I \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. By Lemma 3.11, $(I, w_I) = 0 = e(P, \chi)$ in $E(R)$. $\square$

The next result is an analogue of [**1,** Theorem 4.13]. The proof is similar to the case $f(T) = T$ [**14,** Theorem 4.10].

Theorem 4.11. *Let A be a regular domain of dimension d, essentially of finite type over an infinite perfect field k and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let n be an integer such that $2n \geq d + 3$. Let $I \subset R$ be an ideal of height n, and let P be a projective A-module of rank n. Assume that I contains some special polynomial relative to $f(T)$, say $g(T)$, such that $g(0) = 1$. Then, any surjection $\phi : P \otimes R \twoheadrightarrow I/I^2$ can be lifted to a surjection $\Phi : P \otimes R \twoheadrightarrow R$.*

Remark 4.12. The referee suggested that the above result can be proved for any infinite field.

The following result is a consequence of 4.11.

Theorem 4.13. *Let A be a regular domain of dimension $n \geq 3$, essentially of finite type over an infinite perfect field k. Let $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. Let $(I, w_I) \in E(R)$. Assume that I contains a special polynomial relative to $f(T)$. Then, $(I, w_I) = 0$.*

Let A be a ring of dimension n containing an infinite field, and let P be a projective $A[T]$-module of rank n. In [**5**], it is proved that, if $P_{g(T)}$ has a unimodular element for some monic polynomial $g(T) \in A[T]$, then P has a unimodular element. We will prove the analogous result for $A[T, 1/f(T)]$. The case $f(T) = T$ is proved in [**14,** Theorem 4.13].

Theorem 4.14. *Let P be a projective R-module of rank n with trivial determinant. If $P_{g(T)}$ has a unimodular element, where $g(T)$ is special polynomial relative to $f(T)$, then P has a unimodular element.*

Proof. Let χ be an orientation of P. Since P_g has a unimodular element, $e(P \otimes \mathcal{R}, \chi \otimes \mathcal{R}) = 0$ in $E(\mathcal{R})$. By Remark 4.4, $e(P, \chi) = 0$ in $E(R)$. Hence, by Lemma 4.10, P has a unimodular element. This completes the proof. □

5. Weak Euler class group of $A[T, 1/f(T)]$**.** Let A be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension $n \geq 3$ with $\text{ht}\mathcal{J}(A) \geq 2$ and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic. We define the nth *weak Euler class group* $E_0^n(R)$ of R as follows.

Let S be the set of ideals of R with the properties:

(1) $\text{ht}I = n$ and I/I^2 is generated by n elements, and
(2) $\text{Spec}(R/I)$ is connected. Let G be a free abelian group on S.

Let $I \subset R$ be an ideal of height n such that I/I^2 is generated by n elements. Then, I can be decomposed as $I = I_1 \cap \cdots \cap I_r$, where the I_is are pairwise comaximal ideals of height n and $\text{Spec}(R/I_i)$ is connected for each i. We have seen that such a decomposition of I is unique. By (I), we denote the element $\sum(I_k, w_{I_k})$ of G.

Let H be the subgroup of G generated by elements of the type (I), where $I \subset R$ is an ideal of height n such that I is generated by n elements. We define $E_0^n(R) = G/H$. In what follows, by abuse of notation, we will write $E_0(R)$ for $E_0^n(R)$. Note that there is a canonical surjective homomorphism from $E(R)$ to $E_0(R)$ obtained by forgetting the orientations. For the rest of this section, assuming the following assumption, we obtain results similar to those in [**14,** Section 5].

Assumption 5.1. *Let A be a commutative Noetherian ring containing $\mathbb{Q}$ of dimension $n \geq 3$ with $\mathrm{ht}\mathcal{J}(A) \geq 2$ and $R = A[T, 1/f(T)]$, where $f(T) \in A[T]$ is monic.*

Notation 5.2. Let $I \subset R$ be an ideal of height n, and let $w_I : (R/I)^n \twoheadrightarrow I/I^2$ be a local orientation of I. Let $\theta \in GL_n(R/I)$ be such that $\det\theta = \overline{g}$, where $\overline{g} \in R/I$ is unit. Then $w_I\theta$ is another orientation of I, which we denote by $\overline{g}w_I$.

Remark 5.3. If w_I and $\widetilde{w}_I$ are two local orientations of I, then by [**2,** Lemma 2.2], it is easy to see that $\widetilde{w}_I = \overline{g}w_I$ for some unit $\overline{g} \in R/I$.

The proof of the next result is essentially contained in [**2,** 2.7, 2.8, 5.1].

Lemma 5.4. *Let P be a projective R-module of rank n having trivial determinant and $\chi : R \xrightarrow{\sim} \wedge^n(P)$. Let $\alpha : P \twoheadrightarrow I$ be a surjection, where $I \subset R$ is an ideal of height n. Let (I, w_I) be obtained from (α, χ). Let $g \in R$ be a unit $\mod I$. Then there exists a projective R-module P_1 of rank n having trivial determinant with $\chi_1 : R \xrightarrow{\sim} \wedge^n(P_1)$ and a surjection $\beta : P_1 \twoheadrightarrow I$ such that*:

(1) $[P] = [P_1]$ *in* $K_0(R)$;
(2) $(I, \overline{g^{n-1}}w_I)$ *is obtained from* (β, χ_1).

The next lemma can be proved using [**2,** Lemmas 5.3, 5.4] and 3.11.

Lemma 5.5. *Let $(I, w_I) \in E(R)$ and $\overline{g} \in R/I$ be a unit. Then $(I, w_I) = (I, \overline{g^2} w_I)$ in $E(R)$.*

Adapting the proof of [**4,** Lemma 3.7] and using the Eisenbud-Evans theorem (Lemma 2.5) in place of "Swan's Bertini" theorem, the proof of the next lemma follows.

Lemma 5.6. *Let P be a stably free R-module of even rank $n \geq 4$, and let $\chi : R \xrightarrow{\sim} \wedge^n(P)$ be a trivialization. Suppose that $e(P, \chi) = (I, w_I)$ in $E(R)$. Then, $(I, w_I) = (I_1, w_{I_1})$ in $E(R)$ for some ideal $I_1 \subset R$ of height n generated by n elements. Moreover, I_1 can be chosen to be comaximal with any ideal of R of height* 2.

The following results can be proved by adapting the proofs of [**4,** 3.8, 3.9, 3.10, 3.11] and Lemma 5.6.

Proposition 5.7. *Let P be a projective R-module of even rank $n \geq 4$ with trivial determinant. Then we have the following*:

(1) *Let $I_1, I_2 \subset R$ be two comaximal ideals of height n and $I_3 = I_1 \cap I_2$. If any two of I_1, I_2, I_3 are surjective images of stably free R-modules of rank n, then so is the third.*
(2) *Let $(I, w_I) \in E(R)$. Then, $(I) = 0$ in $E_0(R)$ if and only if I is a surjective image of a stably free projective R-module of rank n.*
(3) *$e(P) = 0$ in $E_0(R)$ if and only if $[P] = [Q \oplus R]$ in $K_0(R)$ for some projective R-module Q of rank $n - 1$.*
(4) *Suppose that $e(P) = (I)$ in $E_0(R)$, where $I \subset R$ is an ideal of height n. Then there exists a projective R-module Q of rank n such that $[Q] = [P]$ in $K_0(R)$ and I is a surjective image of Q.*

The proof of the following result is the same as that of [**8,** Proposition 6.7].

Theorem 5.8. *Let n be an even integer ≥ 4. Let $(I, w_I) \in E(R)$ belong to the kernel of the canonical homomorphism $E(R) \twoheadrightarrow E_0(R)$. Then, there exists a stably free R-module P_1 of rank n and an isomorphism $\chi_1 : R \xrightarrow{\sim} \wedge^n P_1$ such that $e(P_1, \chi_1) = (I, w_I)$ in $E(R)$.*

Proof. Since $(I) = 0$ in $E_0(R)$, by Proposition 5.7 (2), there exist a stably free R-module P of rank n and a surjection $\alpha : P \twoheadrightarrow I$. Let $\chi : R \xrightarrow{\sim} \wedge^n(P)$ be an isomorphism. Suppose that (I, w_I) is obtained from (α, χ). By Remark 5.3, there exists a $g \in R$ such that $\overline{g} \in R/I$ is a unit and $\widetilde{w_I} = \overline{g}w_I$. By Lemma 5.4, there exists a projective R-module P_1 such that P_1 is stably isomorphic to P and an isomorphism $\chi : R \xrightarrow{\sim} \wedge^n(P_1)$ and such that $e(P_1, \chi_1) = (I, \overline{g^{n-1}}w_I)$ in $E(R)$. Since n is even, by Lemma 5.5, we have $(I, \overline{g^{n-1}}w_I) = (I, \overline{g}w_I)$ in $E(R)$. Hence, $e(P_1, \chi_1) = (I, w_I)$ in $E(R)$. $\square$

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