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ON THE LOEWY LENGTH OF MODULES OF FINITE PROJECTIVE DIMENSION

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ABSTRACT. Let $(A,\mathfrak{m})$ be a Gorenstein local ring, and let M be an A module of finite length and finite projective dimension. We prove that the Loewy length of M is greater than or equal to the order of A.

1. Introduction. Let $(A,\mathfrak{m})$ be a Gorenstein local ring of dimension d and embedding dimension c. Let M be a finitely generated A-module, and let $\lambda(M)$ denote its length. The *order* of A is given by the formula

$$\operatorname{ord}(A) = \min\left\{ n \in \mathbb{N} \;\middle|\; \lambda(A/\mathfrak{m}^{n+1}) < \binom{n+c}{n} \right\}$$

if A is singular and if A is a regular set $\operatorname{ord} A = 1$. Note that, if A is singular, then $\operatorname{ord}(A) \geq 2$. The *Loewy length* of an A-module M is defined to be the number

$$\ell\ell(M) = \min\{i \geq 0 \mid \mathfrak{m}^i M = 0\}.$$

Notice that $\ell\ell(M)$ is finite if and only if $\lambda(M)$ is finite.

Let

$$G(A) = \bigoplus_{n\geq 0} \mathfrak{m}^n/\mathfrak{m}^{n+1}$$

be the associated graded ring of A, and let $G(A)_+$ denotes its irrelevant maximal ideal. Let $H^i(G(A))$ be the ith-local cohomology module of $G(A)$ with respect to $G(A)_+$. The *Castelnuovo-Mumford regularity* of $G(A)$ is

$$\operatorname{reg} G(A) = \max\{i + j \mid H^i(G(A))_j \neq 0\}.$$

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In the very nice paper [**2,** subsection 1.1] the authors proved that, if $G(A)$ is Cohen-Macaulay, then for each non-zero finitely generated A-module M of finite projective dimension,

$$\ell\ell(M) \geq \operatorname{reg} G(A) + 1 \geq \operatorname{ord}(A).$$

We should note that the real content of their result is the first inequality. The second inequality is elementary, see [**2,** subsection 1.6]. The hypothesis $G(A)$ is Cohen-Macaulay is quite strong, for instance, $G(A)$ need not be Cohen-Macaulay even if A is a complete intersection, see [**10,** subsection 2.3]. In this short paper, we show

Theorem 1.1. *Let $(A, \mathfrak{m})$ be a Gorenstein local ring, and let M be a non-zero finitely generated module of finite projective dimension. Then*

$$\ell\ell(M) \geq \operatorname{ord}(A).$$

The proof of Theorem 1.1 uses invariants of Gorenstein local rings defined by Auslander and studied by Ding. We also introduce a new invariant $\vartheta(A)$ which is useful in the case $G(A)$ is not Cohen-Macaulay.

In Section 2, we recall the notion of index of a local ring. In Section 3, we introduce our invariant $\vartheta(A)$. In Section 4, we prove Theorem 1.1.

2. The index of a Gorenstein local ring. Let $(A, \mathfrak{m})$ be a *complete* Gorenstein local ring, and let M be a finitely generated A-module. Let $\mu(M)$ denote the minimal number of generators of M. In this section, we recall the definition of the delta invariant of M and the definition of the index of A. A good reference for this topic is [**5**].

2.1. A *maximal Cohen-Macaulay approximation of M* is a short exact sequence

$$(*) \qquad 0 \longrightarrow Y_M \longrightarrow X_M \xrightarrow{f} M \longrightarrow 0,$$

where X_M is a maximal Cohen-Macaulay A-module and $\operatorname{projdim} Y_M < \infty$. If f can only be factored through itself by way of an automorphism of X_M, then the approximation is said to be *minimal.* Any module has a minimal approximation, and minimal approximations are unique up to non-unique isomorphisms. Suppose now that $(*)$ is a minimal approximation. Let

$$X_M = \overline{X_M} \oplus F$$

where $\overline{X_M}$ has no free summands and F is free. Then the *delta* invariant of M, denoted by $\delta_A(M)$, is defined to be the rank of F.

We give some alternate definitions of the delta invariant.

2.2. It can be shown that $\delta_A(M)$ is the smallest integer n such that there is an epimorphism $X \oplus A^n \to M$ with X a maximal Cohen-Macaulay module with no free summands, see [**14,** subsection 4.2]. This definition of the delta invariant is used by Ding [**3**].

The stable CM-trace of M is the submodule $\tau(M)$ of M generated by the homomorphic images in M of all MCM modules without a free summand. Then $\delta_A(M) = \mu(M/\tau(M))$, see [**14,** subsection 4.8]. This is the definition of the delta invariant in [**2**].

We collect some properties of the delta invariant that we need.

2.3. Let M and N be finitely generated A-modules.

(1) If N is an epimorphic image of M then $\delta_A(M) \geq \delta_A(N)$.
(2) $\delta_A(M \oplus N) = \delta_A(M) + \delta_A(N)$.
(3) $\delta_A(M) \leq \mu(M)$.
(4) If $\operatorname{projdim} M < \infty$ then $\delta_A(M) = \mu(M)$.
(5) Let $x \in \mathfrak{m}$ be $A \oplus M$ regular. Then $\delta_A(M) = \delta_{A/(x)}(M/xM)$.
(6) If A is zero-dimensional Gorenstein local ring and I is an ideal in A then $\delta_A(A/I) \neq 0$ if and only if $I = 0$.
(7) If A is not regular then $\delta_A(\mathfrak{m}^s) = 0$ for all $s \geq 1$.
(8) $\delta_A(k) = 1$ if and only if A is regular.
(9) $\delta_A(A/\mathfrak{m}^n) \geq 1$ for all $n \gg 0$.

Proofs and references. For (1), (2), (4), (8) and (9), see [**2,** subsection 1.2]. Notice that item (3) follows easily from the second definition of the delta invariant. Assertion (5) is proved in [**1,** subsection 5.1]. For item (6), note that A/I is maximal Cohen-Macaulay. Assertion (7) is due to Auslander. Unfortunately, this paper by Auslander is unpublished. However, there is an extension of the delta invariant to all Noetherian local rings due to Martsinkovsky [**6**]; he proves [**7,** Theorem 6] that $\delta_A(\mathfrak{m}) = 0$. We prove by induction that $\delta_A(\mathfrak{m}^s) = 0$ for all $s \geq 1$. For $s = 1$, this is true. Assume it is true for $s = j$. We prove it for $s = j + 1$. Let

$$\mathfrak{m}^{j+1} = \langle a_1 b_1, a_2 b_2, \ldots, a_m b_m \rangle,$$

where $a_i \in \mathfrak{m}^j$ and $b_i \in \mathfrak{m}$. Let

$$I_i = a_i\mathfrak{m} \quad \text{for } i = 1, \dots, m.$$

Note that $I_i \subseteq \mathfrak{m}^{j+1}$, and the natural map

$$\phi\colon \bigoplus_{i=1}^{m} I_i \longrightarrow \mathfrak{m}^{j+1}$$

is surjective. By assertions (1) and (2) it is enough to show that $\delta_A(I_i) = 0$ for all i. But this is clear as I_i is a homomorphic image of $\mathfrak{m}$.

2.4. The *index* of A is defined by Auslander to be the number

$$\operatorname{index}(A) = \min\{n \mid \delta_A(A/\mathfrak{m}^n) \geq 1\}.$$

It is positive by subsection 2.3 (4) and finite by subsection 2.3 (9). It equals 1 if and only if A is regular, see subsection 2.3 (8).

2.5. By [**2,** subsection 1.3], we have that, if $\operatorname{projdim} M$ is finite, then

$$\ell\ell(M) \geq \operatorname{index}(A).$$

3. The invariant $\vartheta(A)$. Throughout this section, $(A, \mathfrak{m})$ is a Cohen-Macaulay local ring of dimension d. We assume that k, the residue field of A, is infinite. In this section, we define an invariant $\vartheta(A)$. This will be useful when $G(A)$ is not Cohen-Macaulay.

3.1. Let a be a non-zero element of A. Then there exists $n \geq 0$ such that

$$a \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}.$$

Denote the image of a in $\mathfrak{m}^n/\mathfrak{m}^{n+1}$ by a^* and consider it as an element in $G(A)$. Also set $0^* = 0$. If $a \in \mathfrak{m}$ is such that a^* is $G(A)$-regular then $G(A/(a)) = G(A)/(a^*)$.

3.2. Recall that $x \in \mathfrak{m}$ is said to be A-*superficial* if there exists an integer $c > 0$ such that, for $n \gg 0$, we have

$$(\mathfrak{m}^{n+1}\colon x) \cap \mathfrak{m}^c = \mathfrak{m}^{n-1}.$$

Superficial elements exist if $d > 0$ as k is an infinite field. Since A is Cohen-Macaulay, it is easily shown that a superficial element is a non-zero divisor of A. Furthermore, we have

$$(\mathfrak{m}^{n+1} : x) = \mathfrak{m}^n \quad \text{for all } n \gg 0.$$

This enables the definition of the following two invariants of A and x:

$$\vartheta(A,x) = \inf\{n \mid (\mathfrak{m}^{n+1} : x) \neq \mathfrak{m}^n\},$$
$$\rho(A,x) = \sup\{n \mid (\mathfrak{m}^{n+1} : x) \neq \mathfrak{m}^n\}.$$

Note in [**9,** page 163] we defined an invariant $\rho^{\mathfrak{m}}(A,x)$. It is easily verified that $\rho^{\mathfrak{m}}(A,x) = \rho(A,x) + 1$.

3.3. Notice that $(\mathfrak{m}^{n+1} : x) = \mathfrak{m}^n$ for all $n \geq 0$ if and only if x^* is $G(A)$-regular. Therefore,

$$\vartheta(A,x) = +\infty \quad \text{and} \quad \rho(A,x) = -\infty.$$

If $\operatorname{depth} G(A) > 0$, then x^* is $G(A)$-regular, see [**4,** subsection 2.1]. Thus, in this case,

$$\vartheta(A,x) = +\infty \quad \text{and} \quad \rho(A,x) = -\infty.$$

If $\operatorname{depth} G(A) = 0$, then $(\mathfrak{m}^{n+1} : x) \neq \mathfrak{m}^n$ for some n. In this case,

$$0 \leq \vartheta(A,x) \leq \rho(A,x) < \infty.$$

By [**9,** subsections 2.7 and 5.1], we have

$$\rho(A,x) \leq \operatorname{reg} G(A) - 1.$$

3.4. A sequence

$$\mathbf{x} = x_1, \ldots, x_r \quad \text{in } \mathfrak{m}$$

with $r \leq d$ is said to be an A-*superficial sequence* if x_i is $A/(x_1,\ldots,x_{i-1})$-superficial for $i = 1, \ldots, r$. As the residue field of A is infinite, superficial sequences exist for all $r \leq d$. Since A is Cohen-Macaulay, it can easily be shown that superficial sequences are regular sequences, see [**12,** page 10].

3.5. Let

$$\mathbf{x} = x_1, \ldots, x_d$$

be a maximal A-superficial sequence. Set $A_0 = A$ and

$$A_i = A/(x_1, \ldots, x_i) \quad \text{for } i = 1, \ldots, d.$$

Define

$$\vartheta(A, \mathbf{x}) = \inf\{\vartheta(A_i, x_{i+1}) \mid 0 \leq i \leq d-1\}.$$

Note that $G(A)$ is Cohen-Macaulay if and only if $x_1^*, \ldots, x_d^*$ is a $G(A)$-regular sequence, see [**4,** subsection 2.1]. It follows from subsection 3.3 that

$$\vartheta(A, \mathbf{x}) = +\infty \quad \text{if and only if} \quad G(A) \text{ is Cohen-Macaulay.}$$

Lemma 3.1. *With hypotheses as above, if $G(A)$ is not Cohen-Macaulay then $\vartheta(A, \mathbf{x}) \leq \operatorname{reg} G(A) - 1$.*

Proof. Suppose $\operatorname{depth} G(A) = i < d$. Then $x_1^*, \ldots, x_i^*$ is $G(A)$-regular, see [**4,** subsection 2.1]. Furthermore,

$$G(A_i) = G(A)/(x_1^*, \ldots, x_i^*).$$

Thus, $\operatorname{depth} G(A_i) = 0$. (Note that the case $i = 0$ is also included.)

By subsection 3.3, we obtain that

$$\vartheta(A_i, x_{i+1}) \leq \operatorname{reg}(G(A_i)) - 1.$$

It remains to note that, as $x_1^*, \ldots, x_i^*$ is a regular sequence of elements of degree 1 in $G(A)$, we have

$$\operatorname{reg} G(A_i) \leq \operatorname{reg} G(A). \qquad \square$$

3.6. We define

$$\vartheta(A) = \sup\{\vartheta(A, \mathbf{x}) \mid \mathbf{x} \text{ is a maximal superficial sequence in } A\}.$$

Note that, if $G(A)$ is not Cohen-Macaulay, then

$$\vartheta(A) \leq \operatorname{reg} G(A) - 1,$$

see subsection 3.1. If $G(A)$ is Cohen-Macaulay, then

$$\vartheta(A) = +\infty.$$

3.7. Let A be a singular ring, and let $x \in \mathfrak{m}$ be an A-superficial element. Let $t = \operatorname{ord}(A)$. The following fact is well known (for instance, see [**11,** page 295])

$$(\mathfrak{m}^{i+1} \colon x) = \mathfrak{m}^i \quad \text{for } i = 0, \ldots, t-1.$$

It follows that

$$\vartheta(A, x) \geq \operatorname{ord}(A)$$

for any superficial element x of A.

Notice that $\operatorname{ord}(A/(x)) \geq \operatorname{ord}(A)$ for any superficial element x of A (for instance, see [**11,** page 296]). Thus, if $\mathbf{x} = x_1, \ldots, x_d$ is a maximal A-superficial sequence, we have that

$$\vartheta(A_i, x_{i+1}) \geq \operatorname{ord}(A_i) \geq \operatorname{ord}(A) \quad \text{for all } i = 0, \ldots, d-1.$$

It follows that

$$\vartheta(A, \mathbf{x}) \geq \operatorname{ord}(A). \tag{3.1}$$

Strict inequality in equation (3.1) can hold.

Example 3.2. Let $(A, \mathfrak{m})$ be a one-dimensional stretched Gorenstein local ring, i.e., there exists an A-superficial element x such that, if $\mathfrak{n}$ is the maximal ideal of $B = A/(x)$, then $\mathfrak{n}^2$ is principal. For such rings, $\operatorname{ord}(B) = 2$. So, $\operatorname{ord}(A) = 2$. However, for stretched Gorenstein rings of dimension 1, $(\mathfrak{m}^3 \colon x) = \mathfrak{m}^2$; see [**13,** subsection 2.5]. (Note that $(\mathfrak{m}^{i+1} \colon x) = \mathfrak{m}^i$ for $i \leq 1$ for any Cohen-Macaulay ring A.) Thus, $\vartheta(A, x) \geq 3$.

See [**13,** Example 3] for an example of a one-dimensional stretched Gorenstein local ring A with $G(A)$ not Cohen-Macaulay.

The following result is crucial in the proof of our main result. We denote the multiplicity of A with respect to $\mathfrak{m}$ by $e(A)$.

Lemma 3.3. *Let $(A, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring with infinite residue field. Let*

$$\mathbf{x} = x_1, \ldots, x_d$$

be a maximal superficial sequence. Assume $G(A)$ is not Cohen-Macaulay. Let $s \leq \vartheta(A, \mathbf{x})$. Then

$$\mathfrak{m}^s \not\subseteq (\mathbf{x}).$$

Proof. Assume $\mathfrak{m}^n \subseteq (\mathbf{x})$ for some $n \leq \vartheta(A, \mathbf{x})$. Set $A_d = A/(\mathbf{x})$ and $A_{d-1} = A/(x_1, \ldots, x_{d-1})$. Let $\mathfrak{n}$ be the maximal ideal of A_{d-1}. By definition, $n \leq \vartheta(A_{d-1}, x_d)$.

We have an exact sequence

$$0 \longrightarrow \frac{(\mathfrak{n}^n : x_d)}{\mathfrak{n}^{n-1}} \longrightarrow \frac{A_{d-1}}{\mathfrak{n}^{n-1}} \xrightarrow{\alpha} \frac{A_{d-1}}{\mathfrak{n}^n} \longrightarrow \frac{A_{d-1}}{(\mathfrak{n}^n, x_d)} \longrightarrow 0.$$

Here, $\alpha(a + \mathfrak{n}^{n-1}) = a x_d + \mathfrak{n}^n$. Note that, as $\mathfrak{m}^n \subseteq (\mathbf{x})$, we have

$$A_{d-1}/(\mathfrak{n}^n, x_d) = A_d.$$

Recall that $(\mathfrak{n}^{i+1} : x_d) = \mathfrak{n}^i$ for all $i < \vartheta(A_{d-1}, x_d)$. In particular, we have $(\mathfrak{n}^n : x_d) = \mathfrak{n}^{n-1}$. Thus, we have obtained

$$\lambda(\mathfrak{n}^{n-1}/\mathfrak{n}^n) = \lambda(A_d).$$

Notice that $e(A) = e(A_{d-1}) = e(A_d) = \lambda(A_d)$, cf., [**8,** Corollary 11]. Furthermore, for all $i \geq 0$, we have

$$\lambda(\mathfrak{n}^i/\mathfrak{n}^{i+1}) = e(A_{d-1}) - \lambda(\mathfrak{n}^{i+1}/x_d\mathfrak{n}^i),$$

cf., [**8,** Proposition 13]. For $i = n-1$, our result implies that $\mathfrak{n}^n = x_d\mathfrak{n}^{n-1}$. It follows that $\mathfrak{n}^j = x_d\mathfrak{n}^{j-1}$ for all $j \geq n$. In particular, we have $(\mathfrak{n}^j : x_d) = \mathfrak{n}^{j-1}$ for all $j \geq n$. As $n \leq \vartheta(A_{d-1}, x_d)$, we obtain that $(\mathfrak{n}^j : x_d) = \mathfrak{n}^{j-1}$ for all $j \leq n$. It follows that x_d^* is $G(A_{d-1})$-regular. So $\operatorname{depth} G(A_{d-1}) = 1$. By Sally descent, see [**8,** Theorem 8], we obtain that $G(A)$ is Cohen-Macaulay. This is a contradiction. □

4. Proof of Theorem 1.1. In this section, we prove our main theorem. We will use the invariant $\vartheta(A)$ which is defined only when the residue field of A is infinite. We first show that, in order to prove our result, we can assume that the residue field of A is infinite.

4.1. If the residue field of A is finite, then we consider the A-flat extension $B = A[X]_{\mathfrak{m}A[X]}$. Note that $\mathfrak{n} = \mathfrak{m}B$ is the maximal ideal of B and $B/\mathfrak{n} = k(X)$ is an infinite field. Let M be a finitely generated A-module. The following facts can easily be proved:

(1) $\lambda_B(M \otimes_A B) = \lambda_A(M)$.
(2) $\mathfrak{m}^i \otimes B = \mathfrak{n}^i$ for all $i \geq 1$.
(3) $\lambda_B(B/\mathfrak{n}^{i+1}) = \lambda_A(A/\mathfrak{m}^{i+1})$ for all $i \geq 0$.
(4) $\operatorname{ord}(B) = \operatorname{ord}(A)$.

(5) $\operatorname{projdim}_A M = \operatorname{projdim}_B M \otimes_A B$.
(6) $\mathfrak{m}^i M = 0$ if and only if $\mathfrak{n}^i(M \otimes_A B) = 0$.
(7) $\ell\ell_A(M) = \ell\ell_B(M \otimes_A B)$.

The next result is due to Ding, see [**3,** subsections 1.5, 2.2, 2.3].

Lemma 4.1. *Let $(A, \mathfrak{m})$ be a Noetherian local ring and s an integer. Suppose that $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ is A-regular and the induced map $\overline{x}\colon \mathfrak{m}^{i-1}/\mathfrak{m}^i \to \mathfrak{m}^i/\mathfrak{m}^{i+1}$ is injective for $1 \le i \le s$. Then*

(1) *$A/\mathfrak{m}^s$ is an epimorphic image of $(\mathfrak{m}^s, x)$.*
(2) *There is an A-module decomposition*

$$\frac{(\mathfrak{m}^s, x)}{x(\mathfrak{m}^s, x)} \cong \frac{A}{(\mathfrak{m}^s, x)} \oplus \frac{(\mathfrak{m}^s, x)}{(x)}.$$

We now give

Proof of Theorem 1.1. By subsection 4.1, we may assume that the residue field of A is infinite. Also note that $\operatorname{ord}(A) = \operatorname{ord}(\widehat{A})$ and $\ell\ell_A(M) = \ell\ell_{\widehat{A}}(\widehat{M})$. Thus, we may assume that A is complete.

If $G(A)$ is Cohen-Macaulay, then the result holds by [**2,** Theorem 1.1]. So assume that $G(A)$ is not Cohen-Macaulay. We prove $\operatorname{index}(A) \ge \vartheta(A)$. By 2.5 and (3.1), the result is implied.

Let $\mathbf{x} = x_1, \ldots, x_d$ be an A-superficial sequence with $\vartheta(A) = \vartheta(A, \mathbf{x})$. Assume that $\operatorname{index}(A) < \vartheta(A, \mathbf{x})$. By definition, $\delta_A(A/\mathfrak{m}^s) \ge 1$ for some $s < \vartheta(A, \mathbf{x})$. Set $A_0 = A$ and $A_i = A/(x_1, \ldots, x_i)$ for $1 \le i \le d$. Let $\mathfrak{m}_i$ be the maximal ideal of A_i. We prove by descending induction that

$$\delta_{A_i}(A_i/\mathfrak{m}_i^s) \ge 1 \quad \text{for all } i, 0 \le i \le d.$$

For $i = 0$, this is our assumption. Now assume that this is true for i, and we will prove it for $i+1$. We first note that $s < \vartheta(A, \mathbf{x}) \le \vartheta(A_i, x_{i+1})$. Therefore, $(\mathfrak{m}_i^{j+1} \colon x_{i+1}) = \mathfrak{m}_i^j$ for all $j \le s$. So, by Lemma 4.1, we obtain that $A_i/\mathfrak{m}_i^s$ is an epimorphic image of $(\mathfrak{m}_i^s, x_{i+1})$. Thus, $\delta_{A_i}((\mathfrak{m}_i^s, x_{i+1})) \ge 1$. We also have an A_i-module decomposition

$$(\dagger) \qquad \frac{(\mathfrak{m}_i^s, x_{i+1})}{x_{i+1}(\mathfrak{m}_i^s, x_{i+1})} \cong \frac{A_i}{(\mathfrak{m}_i^s, x_{i+1})} \oplus \frac{(\mathfrak{m}_i^s, x_{i+1})}{(x_{i+1})}.$$

By subsection 2.3 (5), we have that

$$\delta_{A_{i+1}}\left(\frac{(\mathfrak{m}_i^s, x_{i+1})}{x_{i+1}(\mathfrak{m}_i^s, x_{i+1})}\right) = \delta_{A_i}((\mathfrak{m}_i^s, x_{i+1})) \geq 1.$$

Also note that

$$\delta_{A_{i+1}}\left(\frac{(\mathfrak{m}_i^s, x_{i+1})}{(x_{i+1})}\right) = \delta_{A_{i+1}}(\mathfrak{m}_{i+1}^s) = 0,$$

by subsection 2.3 (7). By (†) and subsection 2.3 (2), it follows that

$$1 \leq \delta_{A_{i+1}}\left(\frac{A_i}{(\mathfrak{m}_i^s, x_{i+1})}\right) = \delta_{A_{i+1}}\left(\frac{A_{i+1}}{\mathfrak{m}_{i+1}^s}\right).$$

This proves our inductive step. So we have $\delta_{A_d}(A_d/\mathfrak{m}_d^s) \geq 1$. By subsection 2.3 (6), we have that $\mathfrak{m}_d^s = 0$. It follows that $\mathfrak{m}^s \subseteq (\mathbf{x})$. This contradicts Lemma 3.3. □

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REFERENCES

1. M. Auslander, A. Ding and O. Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), 273-317.

2. L. Avramov, R-O. Buchweitz, S.B. Iyengar and C. Miller, *Homology of perfect complexes*, Adv. Math. **223** (2010), 1731-1781.

3. S. Ding, *The associated graded ring and the index of a Gorenstein local ring*, Proc. Amer. Math. Soc. **120** (1994), 1029-1033.

4. S. Huckaba and T. Marley, *Hilbert coefficients and the depths of associated graded rings*, J. Lond. Math. Soc. **56** (1997), 64-76.

5. G.J. Leuschke and R. Wiegand, *Cohen-Macaulay representations*, Math. Surv. Mono. **181**, American Mathematical Society, Providence, RI, 2012.

6. A. Martsinkovsky, *New homological invariants for modules over local rings*, I, J. Pure Appl. Alg. **110** (1996), 1-8.

7. ______, *A remarkable property of the (co) syzygy modules of the residue field of a nonregular local ring*, J. Pure Appl. Alg. **110** (1996), 9-13.

8. T.J. Puthenpurakal, *Hilbert-coefficients of a Cohen-Macaulay module*, J. Algebra **264** (2003), 82-97.

9. ______, *Ratliff-Rush filtration, regularity and depth of higher associated graded modules*, J. Pure Appl. Alg. **208** (2007), 159-176.

10. ______, *The Hilbert function of a maximal Cohen-Macaulay module, Part II*, J. Pure Appl. Alg. **218** (2014), 2218-2225.

11. M.E. Rossi and G. Valla, *Cohen-Macaulay local rings of dimension two and an extended version of a conjecture of J. Sally*, J. Pure Appl. Alg. **122** (1997), 293-311.

12. J. D. Sally, *Number of generators of ideals in local rings*, Lect. Notes Pure Appl. Math. **35**, Dekker, New York, 1978.

13. ______ *Stretched Gorenstein rings*, J. Lond. Math. Soc. **20** (1979), 19-26.

14. A-M. Simon and J.R. Strooker, *Reduced Bass numbers, Auslander's δ-invariant and certain homological conjectures*, J. reine angew. Math. **551** (2002), 173-218.

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