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IDEAL ZERO-DIVISOR COMPLEX

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ABSTRACT. Using discrete Morse theory for simplicial complexes we determine the homotopy type of ideal zero-divisor complex for finite rings and for rings with infinitely many maximal ideals.

1. Introduction. In [**3**], the authors studied the set of zero-divisors in a commutative ring with identity R by associating a graph to that ring, the so called *zero-divisor graph* Γ_R, with vertices being zero-divisors and xy being an edge if and only if $xy = 0$. Akhtar and Lee [**1**] used a different approach, replacing zero-divisors of R by proper ideals and studying homology. They were mainly concerned with calculations concerning 0th homology for general rings and the first homology group and Euler characteristic for the ring $\mathbb{Z}/p^r\mathbb{Z}$, for a prime number p. For a more thorough account of zero-divisor graphs for commutative rings, see [**2**].

Inspired by the approach in [**1**], in this paper, we investigate the ideal zero-divisor simplicial complex (which was implicit in [**1**]) and analyze its topology for the case when a ring R has infinitely many maximal ideals and for the case when R is a finite ring.

When analyzing topology of a simplicial complex, one of the most widely used tools is discrete Morse theory for simplicial complexes that was introduced by Forman [**4**] in the 1990s as a combinatorial analogue to the original Morse theory. It was developed as a powerful tool useful in reducing the size of simplicial complexes while preserving their homotopy type. In this paper, when R is a finite ring, we use discrete Morse theory as a tool in determining the homotopy type, and

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when the ring R has infinitely many maximal ideals, we can obtain the results directly.

For another example of associating simplicial complexes to commutative rings, the reader may wish to consult [**9**], where the authors associated order complex with a general commutative ring via chains of ideals and they had determined the homotopy type of that complex.

2. Notation, definitions and previous results. This section is divided into two subsections. In the first, we give necessary background and definitions regarding simplicial complexes, while, in the second, we present discrete Morse theory for simplicial complexes.

2.1. Simplicial complex. For further information concerning simplicial complexes, geometric realization, etc., the reader is referred to [**8, 10**]. For information concerning necessary notions and results from homotopy theory, the reader is referred to [**6**]. Here, we present only the basic notions, mainly to fix notation and terminology.

Let $a_0, a_1, \dots, a_n$ be elements in some $\mathbb{R}^N$. They are said to be *geometrically* (or *affinely*) independent if, from

$$\sum_{i=0}^{n} \lambda_i a_i = 0, \quad \text{where } \textstyle\sum_{i=0}^{n} \lambda_i = 0,$$

it follows that $\lambda_0 = \lambda_1 = \cdots = \lambda_n = 0$. If these points are geometrically independent, then the convex hull σ forms a *geometric* n-simplex. Convex hulls of subsets of $\{a_0, a_1, \dots, a_n\}$ form *faces* of σ, and a_i are its vertices. The *standard* geometric n-simplex Δ^n is given by:

$$\Delta^n := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}_+^{n+1} : x_0 + x_1 + \cdots + x_n = 1\},$$

where $\mathbb{R}_+$ is the set of all non-negative real numbers. Any geometric n-simplex is homeomorphic to the standard geometric n-simplex. In what follows, we will usually simply state simplex instead of geometric simplex.

An *abstract simplicial complex* K is a collection of finite non-empty sets such that, if $A \in K$, and $\emptyset \neq B \subseteq A$, then $B \in K$. The union $\cup K$ is the set of all vertices of K. If $A \in K$, and A has $n+1$ elements, we refer to A as an n-simplex of K. A non-empty subset B of A is called a face of A.

We can associate to any abstract simplicial complex K a topological space $|K|$, the *geometric realization* of K. This space lies in an Euclidean space (of possibly infinite dimension). Simply, to every n-simplex A of K is associated a geometric n-simplex σ_A in such a way that faces of σ_A are associated to faces of A. If A and B are disjoint the associated geometric simplices are also disjoint. Union of all of these geometric simplices is the required geometric realization. The topology on $|K|$ is given as follows: a set F is closed in $|K|$ if and only if $F \cap \sigma_A$ is closed in σ_A for every $\sigma \in K$ (σ_A itself has the subspace topology induced by the n-dimensional plane determined by its vertices). Space $|K|$ is determined up to a homeomorphism.

In what follows, we use the same letter to denote simplex of K and the corresponding geometric simplex in a geometric realization. Note that, when we refer to the topological properties of the simplicial complex K, we are always actually referring to the topological space $|K|$.

We refer to the ideal zero-divisor complex for a commutative ring with identity R as $\Delta(R)$, the definition of which will be given in Section 3, or simply write Δ. Furthermore, for a simplex $\sigma = \{I_0, \dots, I_n\} \in \Delta$, we use the notation $\sigma \setminus I_i = \{I_0, \dots, \widehat{I_i}, \dots, I_n\}$ and $\sigma \times J = \{I_0, \dots, I_n, J\}$. Also, $\alpha^{(p)}$ denotes a p-dimensional simplex.

2.2. Discrete Morse theory for simplicial complexes. Here, we present some basic notions from discrete Morse theory applied to finite simplicial complexes, which we will use as a tool for determining the homotopy type of the ideal zero-divisor complex for finite rings. For a more thorough background, we encourage the reader to consult [**4**], which is Forman's original paper dealing with discrete Morse theory for simplicial complexes, as well as his guide [**5**]. Furthermore, Jonsson's book on simplicial complexes of graphs [**7**], in which an algebraic version of discrete Morse theory is presented, provides some very useful theorems.

Definition 2.1. A function $f\colon K \to \mathbb{R}$ is a discrete Morse function if, for every $\alpha^{(p)} \in K$,

(1) $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$ for at most one $\beta^{(p+1)} \supset \alpha^{(p)}$, and
(2) $f(\gamma^{(p-1)}) \geq f(\alpha^{(p)})$ for at most one $\gamma^{(p-1)} \subset \alpha^{(p)}$

We say that $\alpha^{(p)}$ is a critical simplex if, for all $\beta^{(p+1)} \supset \alpha^{(p)}$, $f(\beta^{(p+1)}) > f(\alpha^{(p)})$, and, for all $\gamma^{(p-1)} \subset \alpha^{(p)}$, $f(\gamma^{(p-1)}) < f(\alpha^{(p)})$.

Forman's main theorem is as follows.

Theorem 2.2 ([**5**]). *Suppose K is a simplicial complex with a discrete Morse function. Then, K is homotopy equivalent to a* CW *complex with exactly one cell of dimension p for each critical simplex of dimension p.*

Writing a discrete Morse function for a simplicial complex is not difficult, for example, take $f(\alpha^{(p)}) = p$, but writing a good discrete Morse function that gives as little critical simplices as possible proves to be challenging. This is why Forman, instead of directly considering a discrete Morse function, looked at a discrete vector field which, under certain conditions, gives rise to a discrete Morse function.

Definition 2.3. A discrete vector field V on a finite simplicial complex K is a set of pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$ where $\alpha^{(p)} \subset \beta^{(p+1)}$, and each simplex is in at most one pair. We say that $\{\alpha^{(p)}, \beta^{(p+1)}\}$ is a matching in V, that is, simplices $\alpha^{(p)}$ and $\beta^{(p+1)}$ are matched in V, while simplex σ in K is critical or unmatched with respect to V if σ is not contained in any pair in V.

If we have a discrete Morse function f, then we can specify a discrete vector field by forming the pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$ whenever $f(\beta^{(p+1)}) \leq f(\alpha^{(p)})$ (notice that conditions (1) and (2) from Definition 2.1 cannot simultaneously occur for the same simplex $\alpha^{(p)}$). The converse is not always true; thus, we introduce a necessary and sufficient condition for V to be a discrete vector field of some Morse function f.

Definition 2.4. Given a discrete vector field V on a finite simplicial complex K, a V-path is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \alpha_2^{(p)}, \dots, \beta_{r+1}^{(p+1)}$$

such that, for each $i = 0, \dots, r$, $\{\alpha_i, \beta_i\} \in V$ and $\beta_i \supset \alpha_{i+1} \neq \alpha_i$. We say that such a path is non-trivially closed if $r > 0$ and $\alpha_0 = \alpha_{r+1}$.

Theorem 2.5 ([**5**]). *A discrete vector field V on a finite simplicial complex K is a discrete vector field of some Morse function if and only if there are no non-trivially closed V-paths.*

We say that V is acyclic if this condition holds. By Theorem 2.5, an acyclic discrete vector field V on K gives rise to a discrete Morse function on K whose critical simplices are exactly the critical (unmatched) simplices of V. Then, by Theorem 2.2, such a complex is homotopy equivalent to a CW complex with cells that correspond to critical simplices of V.

In order to determine the homotopy type of such a CW complex we need some further theoretical background.

Consider a discrete vector field V on a finite simplicial complex K. Let $D = D(K, V)$ be the digraph with the set of all simplices of K as its vertex set and with a directed edge from σ to τ if and only if one of the following holds:

(1) $\{\sigma, \tau\} \in V$;
(2) $\{\sigma, \tau\} \notin V$ and $\sigma = \tau \times x$ for some $x \notin \tau$.

Thus, every edge in D corresponds to an edge in Hasse diagram of K ordered by set inclusion; edges corresponding to pairs in V are directed from the smaller set to the larger set, while other edges are directed from the larger to the smaller set. We write $\sigma \to \tau$ if there is a directed path from σ to τ in D. It is easily proven that, if V is acyclic, then D is an acyclic digraph, that is, $\sigma \to \tau$ and $\tau \to \sigma$ implies $\sigma = \tau$. For details, see [**7, 11**].

We use the following construction from Jonsson, see [**7**], together with the Forman's theorems of simplicial Morse theory, in order to determine the homotopy type of ideal zero-divisor complex in Section 3.

For an acyclic discrete vector field V on a finite simplicial complex K, let $\mathcal{U}(K, V)$ be the family of critical simplices of K with respect to V. For a non-empty family of critical simplices $\mathcal{V} \subseteq \mathcal{U}(K, V)$, consider the following subcomplex of K:

$$K_{\mathcal{V}} = \{\tau \in K : \sigma \to \tau, \text{ for some } \sigma \in \mathcal{V}\}.$$

We assume that $\mathcal{V} \subseteq K_{\mathcal{V}}$. When $\mathcal{V}$ contains only one critical simplex σ, we denote it as K_σ. The following results are due to Jonsson [**7,** Proof of Theorem 4.11 and following corollary]. They will prove useful in determining the homotopy type of ideal zero-divisor complex for finite rings.

Theorem 2.6 ([**7**]). *Suppose that* $\sigma \in \mathcal{U}(K,V)$ *has the property that* $\Sigma = K_\sigma \cap K_{\mathcal{U}\setminus\{\sigma\}}$ *is not empty and contractible to a point. Then* K *is homotopy equivalent to* $K_\sigma \vee K_{\mathcal{U}\setminus\{\sigma\}}$.

This theorem yields the following result:

Corollary 2.7. *Let* $\mathcal{U}(K,V)$ *be a collection of* k *distinct critical simplices,* $\mathcal{U}(K,V) = \{\sigma_1,\dots,\sigma_k\}$ *with the property that* $\bigcap_{i=1}^k K_{\sigma_i}$ *is non-empty and contractible to a point. Then,* K *is homotopy equivalent to* $\bigvee_{i=1}^k K_{\sigma_i}$.

3. Ideal zero-divisor complex.

Definition 3.1. Let R be a commutative ring with identity, and let $I^*(R)$, the set of all proper non-zero ideals of R, be the vertex set. We define ideal zero-divisor complex $\Delta(R)$ as follows:

$$\{I_0, I_1, \dots, I_n\} \in \Delta(R) \text{ if and only if } I_0 I_1 \cdots I_n \neq 0.$$

First, we will look at a few examples of ideal zero-divisor complexes.

Example 3.2. Let $R = \mathbb{Z}/p^8\mathbb{Z}$. We will abuse notation and denote the ideal $I = \langle p^i \rangle$ by p^i. Geometric realization of this complex is represented in Figure 1.

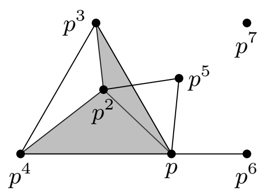

FIGURE 1. $|\Delta(\mathbb{Z}/p^8\mathbb{Z})|$.

Example 3.3. Let $R = \mathbb{Z}/p^2q^2\mathbb{Z}$. Again, we will abuse notation and let p^iq^j stand for the ideal $I = \langle p^iq^j \rangle$. Geometric realization of this complex is represented in Figure 2.

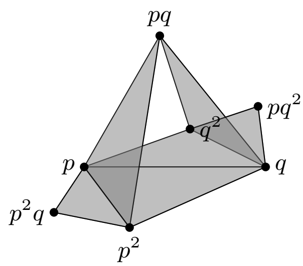

FIGURE 2. $|\Delta(\mathbb{Z}/p^2q^2\mathbb{Z})|$.

In the following subsections, we determine the homotopy type of this complex for commutative rings with identity that have either infinitely many maximal ideals or are finite.

3.1. Ideal zero-divisor complex for rings with infinitely many maximal ideals.

Lemma 3.4. *Suppose that R is such that* $\mathrm{Max}(R)$ *is infinite. If K_0 is a finite subcomplex of $\Delta(R)$, then there is a subcomplex K_1 such that K_0 is a subcomplex of K_1 and $|K_1|$ is contractible.*

Proof. Let K_0 be a finite subcomplex of $\Delta(R)$. For each simplex $\sigma_i = \{I_0, \ldots, I_n\} \in K_0$, let M_i be a maximal ideal in R such that $\mathrm{Ann}(I_0 \cdots I_n) \subseteq M_i$. Since there are finitely many, say k, maximal simplices in K_0, we can identify a finite collection of such maximal ideals $M_1, \ldots, M_k$ (not necessarily distinct). Now, choose a non-unit element $a \in R \backslash \bigcup_{i=1}^k M_i$ (since there are infinitely many maximal ideals in the ring, such an a always exists). Then, for any simplex $\sigma \in K_0$, if $\langle a \rangle$ is not already a vertex of σ, we have $\langle a \rangle I_0 \cdots I_n \neq 0$; thus, $\sigma \times \langle a \rangle \in \Delta(R)$. Now consider K_1 to be a subcomplex of $\Delta(R)$ which is the complex K_0 together with simplices $\sigma \times \langle a \rangle$ for all $\sigma \in K_0$ when $\langle a \rangle$ is not already a vertex of σ. Therefore, $|K_1|$ is a cone with apex $\langle a \rangle$, and hence, contractible. □

Proposition 3.5. *If* $\mathrm{Max}(R)$ *is infinite, then* $|\Delta(R)|$ *is contractible.*

Proof. Let us first show that the complex is connected. We proceed as in the previous proof. If I and J are any two non-zero proper ideals, let M_1 and M_2 be maximal ideals such that $\mathrm{Ann}(I) \subseteq M_1$ and $\mathrm{Ann}(J) \subseteq M_2$. If a is a non-unit element from $R \setminus (M_1 \cup M_2)$, then $\langle a \rangle$ is adjacent to I and J. The rest of the proof is the same as in **[9]**, and we give it here for the sake of completeness. Since $|\Delta(R)|$ has the homotopy type of a CW complex, we may use the Whitehead theorem. We only need to show that all homotopy groups of $|\Delta(R)|$ are trivial. Suppose that $n \geq 1$ and that $g\colon S^n \to |\Delta(R)|$ is a continuous map. Since the image $g[S^n]$ is compact, by **[10,** Lemma 2.5], there is a finite subcomplex K_0 such that $g[S^n] \subseteq |K_0|$. By Lemma 3.4, there is a subcomplex K_1 such that $K_0 \subset K_1$ and $|K_1|$ is contractible. So, the map g may be factored through the contractible space $|K_1|$, and it is homotopically trivial. We conclude that $\pi_n(|\Delta(R)|, *)$ is trivial. Since this holds for all n, by Whitehead's theorem, we obtain that $|\Delta(R)|$ is contractible. □

3.2. Ideal zero-divisor complex for finite rings. In subsection 3.2.1 we give an algorithm for constructing a discrete vector field V for ideal zero-divisor complex for finite rings. We prove that the vector field so obtained is acyclic, and we explicitly show which simplices remain unmatched (critical). Then, we determine the homotopy type of this complex.

3.2.1. Acyclic discrete vector field. Let R be a finite commutative ring with identity. We have finitely many maximal ideals, $|\mathrm{Max}(R)| = m$, which we denote by $M_1, \ldots, M_m$.

First, we present a discrete vector field for the case $m = 1$. Let $\sigma = \{I_0, \ldots, I_n\}$ be any simplex in $\Delta(R)$. We want to show that there exists $\tau^{n-1} \subset \sigma$ such that $\{\tau, \sigma\} \in V$ or that there exists $\rho^{n+1} \supset \sigma$ such that $\{\sigma, \rho\} \in V$, or that σ remains unmatched.

Take $\{M_1\}$ to be an unmatched simplex. For any other simplex $\sigma = \{I_0, \ldots, I_n\}$ apply the algorithm for forming pairs in V, given in Figure 3.

Now, suppose that $m > 1$, and suppose that, if R has exactly two maximal ideals, then $M_1 M_2 \neq 0$. We will deal with the case $m = 2$

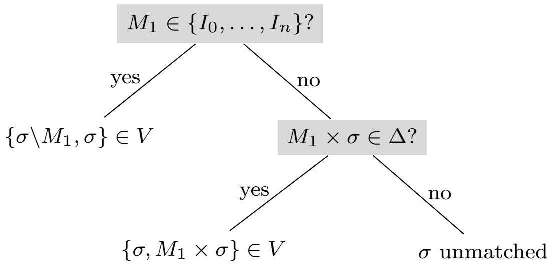

FIGURE 3. A discrete vector field for a finite local ring.

with $M_1 M_2 = 0$ later in subsection 3.2.2 and directly determine the homotopy type.

Take $\{M_1\}$ to be an unmatched simplex. For any other simplex $\sigma = \{I_0, \ldots, I_n\}$, apply the algorithm for forming pairs in V, given in Figure 4.

In order to show that this is a discrete vector field we will need the next lemma. It will also prove useful in the following subsections.

Lemma 3.6. *Suppose that* $I_0 \cdots I_n \neq 0$ *and* $M_i I_0 \cdots I_n = 0$ *for some maximal ideal* M_i *in* R. *Then, for any other maximal ideal* M_j *in* R, $j \neq i$, *we have* $M_j I_0 \cdots I_n \neq 0$.

Proof. Obviously, $M_i \subseteq \mathrm{Ann}(I_0 \cdots I_n)$ and, since M_i is maximal, we have $M_i = \mathrm{Ann}(I_0 \cdots I_n)$. Now, for any maximal ideal $M_j \neq M_i$, $M_j \nsubseteq M_i = \mathrm{Ann}(I_0 \cdots I_n)$; thus, $M_j I_0 \cdots I_n \neq 0$. □

In Figure 3, when the answer to the question whether $M_2 \in \{I_0, \ldots, I_n\}$ is no, by Lemma 3.6, we know that $M_2 I_0 \cdots I_n \neq 0$. Thus, having $\{\sigma, M_2 \times \sigma\} \in V$ is valid. Furthermore, we need to ensure that we are not matching an empty set in V, that is, that we do not have a pair $\{\sigma \backslash M_i, \sigma\} \in V$ where $\sigma = M_i$. Consider what simplices $\{M_i\}$, $i > 1$, get matched to. If $|\mathrm{Max}(R)| = 2$, we assume that $M_1 M_2 \neq 0$; thus, $\{\{M_2\}, \{M_1, M_2\}\} \in V$. Now, if $|\mathrm{Max}(R)| > 2$, we claim that $M_1 M_i \neq 0$, for $i > 1$. Suppose that M_j is some other maximal ideal, $j \neq 1$ and $j \neq i$, $M_j \supseteq M_1 M_i = 0$. Since maximal ideals are prime, we have $M_j \supseteq M_1$ or $M_j \supseteq M_i$, which is a contradiction. Therefore, for each maximal ideal M_i, $i > 1$, we have $\{\{M_i\}, \{M_1, M_i\}\} \in V$.

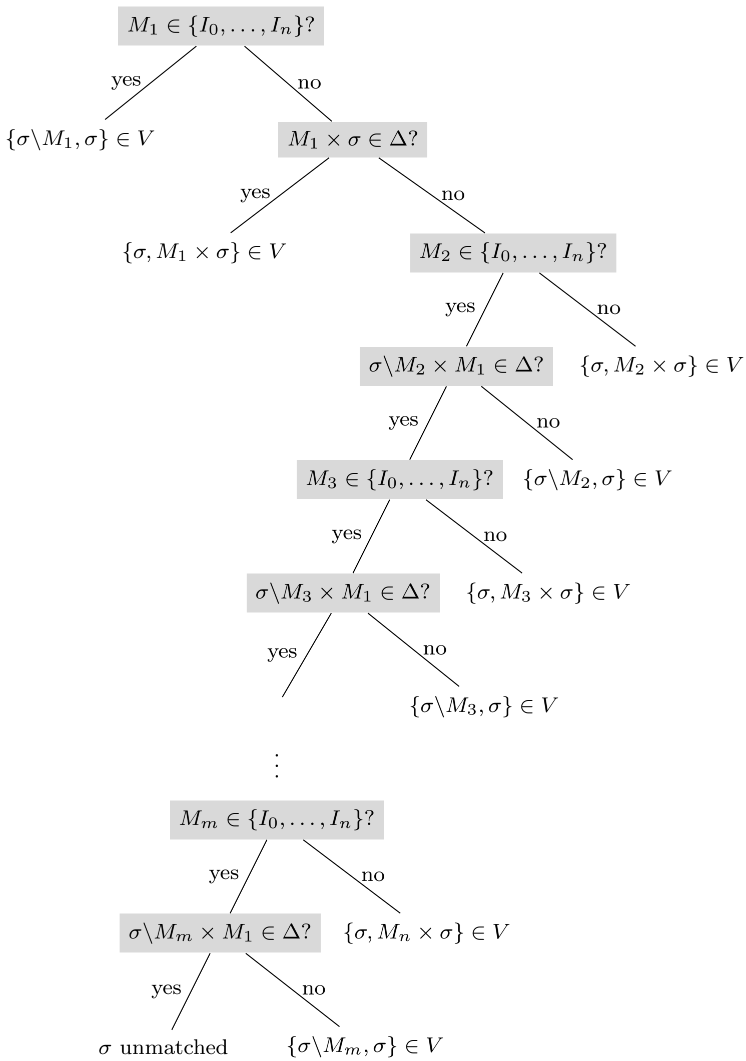

FIGURE 4. A discrete vector field for non-local finite ring not isomorphic to a product of two fields.

With this background, the algorithm given in Figure 4 precisely gives a discrete vector field V for simplicial complex $\Delta(R)$ for finite ring R

such that $|\operatorname{Max}(R)| = m > 1$, and R is not isomorphic to a product of two fields. To illustrate this, in Figure 5, we give an example of the digraph D of such a discrete vector field V for the case $R = \mathbb{Z}/p^2q^2\mathbb{Z}$. Reverse arrows show which pairs are in V. Notice that the simplices which are shaded are critical, as there are no reverse arrows leading from them.

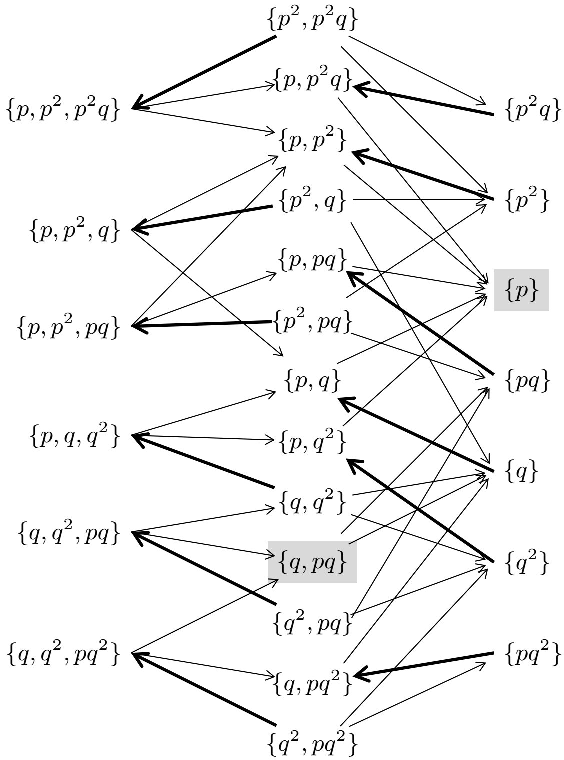

FIGURE 5. Digraph of discrete vector field for $R = \mathbb{Z}/p^2q^2\mathbb{Z}$.

In order to show that V is acyclic (in both cases when $m = 1$ and $m > 1$), we show that there are no non-trivial closed V-paths.

Note that the matchings in V for the case when $m > 1$ are:

(i) $\{\{I_0, \ldots, I_n\}, \{M_1, I_0, \ldots, I_n\}\}$; or

(ii) $\{\{I_0,\dots,I_n\},\{M_k,I_0,\dots,I_n\}\}$, which we can denote $\{\sigma\setminus M_k,\sigma\}$. The conditions which are satisfied when we have this type of matching are:
  - $M_1 \notin \{I_0,\dots,I_n\}$;
  - $M_1 I_0\cdots I_n = 0$;
  - $M_i \in \{I_0,\dots,I_n\}$ and $\sigma\setminus M_i \times M_1 \in \Delta$ for all $1 < i < k$; and
  - $\sigma\setminus M_k \times M_1 \notin \Delta$.

When R is a finite ring with one maximal ideal, matchings in V are only of the first type, so the proof of acyclicity follows the same argument as in the case when the number of maximal ideals is greater than one.

Now suppose that we have a V-path as described in subsection 2.2. Denote this path as $\alpha_0,\beta_0,\alpha_1,\beta_1,\dots,\alpha_r,\beta_r$, where each α_i is an n-simplex, each β_i is an $n+1$-simplex, $\{\alpha_i,\beta_i\}\in V$ and $\beta_i \supset \alpha_{i+1} \neq \alpha_i$ for each $0 \le i \le r$. We will show that this path cannot be nontrivially closed, that is, we cannot have $\alpha_0 = \alpha_{r+1}$ where $r > 0$.

Suppose that $\{\alpha_0,\beta_0\}\in V$ is a matching of the first type, that is, $\alpha_0 = \{I_0,\dots,I_n\}$ and $\beta_0 = \{M_1,I_0,\dots,I_n\}$. Since α_1 is a face of β_0 and $\alpha_1 \neq \alpha_0$, we must have $\alpha_1 = \{M_1,I_0,\dots,\widehat{I_i},\dots I_n\}$ for some $0 \le i \le n$. Now, in V this n-simplex gets matched to an $n-1$-simplex $\{I_0,\dots,\widehat{I_i},\dots I_n\}$ which cannot be β_1 (β_1 must be an $n+1$-simplex). Therefore, the above path ends at β_0.

Now suppose that $\{\alpha_0,\beta_0\}\in V$ is a matching of the second type, that is, $\alpha_0 = \{I_0,\dots,I_n\}$, $\beta_0 = \{M_k,I_0,\dots,I_n\}$, with conditions that are described above for this type of matching, which we denote $\{\sigma\setminus M_k,\sigma\}$. Since α_1 is a face of β_0 and $\alpha_1 \neq \alpha_0$, we must have $\alpha_1 = \{M_k,I_0,\dots,\widehat{I_i},\dots,I_n\}$ for some $0 \le i \le n$. β_1 is dependent upon which ideal from simplex β_0 is missing in α_1. First, suppose that the excluded ideal is one of the maximal ideals M_i, $1 < i < k$. Then, by the condition $\sigma\setminus M_i \times M_1 \in \Delta$, we must have that $\beta_1 = M_1 \times \alpha_1$. Then, by the same argument as for the first type of matching, α_2, which is a face of β_1 different from α_1, gets matched to $\alpha_2 \setminus M_1$ which is an $n-1$-simplex so that the path ends and it is not closed. Second, suppose that the excluded ideal is some other maximal ideal M_j, $j > k$, that might be among ideals $\{I_0,\dots,I_n\}$. Then, if $M_1 M_k I_0\cdots\widehat{I_i}\cdots I_n \neq 0$ we have $\beta_1 = M_1\times\alpha_1$ and the same argument as above applies, by which the path ends and is not closed. Otherwise, if

$M_1 M_k I_0 \cdots \widehat{I_i} \cdots I_n = 0$ by Lemma 3.6, $M_1 M_j I_0 \cdots \widehat{I_i} \cdots I_n \neq 0$ which exactly gives $\sigma \setminus M_k \times M_1 \in \Delta$, a contradiction to the conditions for this type of matching. Therefore, the excluded ideal I_i from β_0 to α_1 cannot be any maximal ideal. This means that we cannot obtain a closed V-path, and $\alpha_0 = \alpha_{r+1}$ since we it is not possible to reintroduce the deleted ideal I_i as we do not have a matching of the type $\{\tau, \tau \times I_i\} \in V$ for ideal I_i, which is not maximal.

We showed that there are no non-trivial closed V-paths; hence, by Theorem 2.5, V is a discrete vector field of some Morse function.

From Figures 3 and 4, note that:

(i) for the case $m = 1$, the unmatched (critical) simplices are $\{M_1\}$ and simplices of the form $\{I_0, \ldots, I_n\}$ where $M_1 \notin \{I_0, \ldots, I_n\}$ and $M_1 I_0 \cdots I_n = 0$;
(ii) for the case $m > 1$, the unmatched (critical) simplices are $\{M_1\}$ and simplices of the form $\{M_2, \ldots, M_m, I_0, \ldots, I_n\}$ where $M_1 \notin \{I_0, \ldots, I_n\}$, $M_1 M_2 \cdots M_m I_0 \cdots I_n = 0$ and the product $M_1 M_2 \cdots \widehat{M_i} \cdots M_m I_0 \cdots I_n \neq 0$ for all $2 \leq i \leq m$.

3.2.2. Homotopy type. In order to determine the homotopy type of the ideal zero-divisor complex for finite rings, we will consider several cases of finite rings.

Theorem 3.7. *Let R be a finite local ring with maximal ideal M. If* $\mathrm{Ann}(M) = 0$*, then $\Delta(R)$ is contractible to a point. If* $\mathrm{Ann}(M) = M$*, then $\Delta(R)$ is homotopy equivalent to a disjoint union of a finite number of points. Finally, if* $\mathrm{Ann}(M) \neq 0$ *and* $\mathrm{Ann}(M) \neq M$*, then $\Delta(R)$ is homotopy equivalent to a disjoint union of a finite number of points and a connected complex that is homotopy equivalent to a wedge of spheres.*

Proof. If $\mathrm{Ann}(M) = 0$, then $\Delta(R)$ is obviously a cone with apex M, and hence, contractible to a point. Suppose $\mathrm{Ann}(M) \neq 0$. $\mathrm{Ann}(M)$ is a proper ideal in R, and obviously, for any other non-zero ideal I in R, we have $I \subseteq M$. Thus, $\mathrm{Ann}(M) \cdot I = 0$, and hence, $\mathrm{Ann}(M)$ is an isolated point. By the same argument, any ideal $I \subset \mathrm{Ann}(M)$ is an isolated point.

When $\mathrm{Ann}(M) \neq M$, the complex $|\Delta(R)|$ is the union of finite number of isolated points and the subcomplex $|\widetilde{\Delta}(R)|$ whose vertices are ideals $I \nsubseteq \mathrm{Ann}(M)$. For each vertex I, we have $I \cdot M \neq 0$, which shows that this subcomplex is connected. We apply the discrete vector field for the case ring, finite with one maximal ideal, to this subcomplex, and denote the set of critical simplices with respect to this vector field as $\mathcal{U}(\widetilde{\Delta}, V)$. The critical simplices are $\{M\}$ and $\{I_0, \dots, I_n\}$ where $M \notin \{I_0, \dots, I_n\}$ and $MI_0 \cdots I_n = 0$. Since the vertices are only the ideals which are not in $\mathrm{Ann}(M)$, M is the only critical 0-simplex. Let $\sigma \in \mathcal{U}(\widetilde{\Delta}, V)$. Consider the following subcomplex of $|\widetilde{\Delta}(R)|$, introduced in subsection 2.2:

$$\Delta_\sigma = \{\tau \in \Delta : \sigma \to \tau\}.$$

If $\sigma = \{M\}$, then $\Delta_\sigma = \{M\}$. Let $\sigma = \{I_0, \dots, I_n\} \in \mathcal{U}(\widetilde{\Delta}, V)$. Note that $MI_0 \cdots \widehat{I_i} \cdots I_n \neq 0$ for all $0 \leq i \leq n$, since each $I_i \subset M$ and $I_0 \cdots I_n \neq 0$. Therefore, for each $0 \leq i \leq n$, we have pairs $\{\{I_0, \dots, \widehat{I_i}, \dots, I_n\}, \{M, I_0, \dots, \widehat{I_i}, \dots, I_n\}\} \in V$. Consequently, apart from the faces of σ, the only $\tau \in \widetilde{\Delta}$ such that $\sigma \to \tau$ are simplices $\{M, I_0, \dots, \widehat{I_i}, \dots, I_n\}$ and, naturally, their faces. Hence, $|\Delta_\sigma|$ is the boundary of an n-simplex, which is homotopy equivalent to S^{n-1}.

Note that

$$\bigcap_{i=1}^{k} \Delta_{\sigma_i} = \{M\};$$

thus, using Corollary 2.7 from subsection 2.2, $|\widetilde{\Delta}(R)|$ is homotopy equivalent to

$$\bigvee_{i=1}^{k} |\Delta_{\sigma_i}|. \qquad \square$$

Proposition 3.8. *Let R be a ring with two maximal ideals M_1 and M_2 such that $M_1M_2 = 0$. Then, R is isomorphic to a product of two fields, and $|\Delta(R)|$ is homotopy equivalent to two isolated points.*

Proof. By the Chinese remainder theorem, $R \cong R/M_1 \times R/M_2$. For any ideals $I \subseteq M_1$ and $J \subseteq M_2$, we have that $IJ = 0$. Hence, $\Delta(R)$ is the disjoint union of two subcomplexes: $\Delta(M_1)$ as the subcomplex generated by all ideals that are contained in M_1 and $\Delta(M_2)$ as the subcomplex generated by all ideals that are contained in M_2. Now, let $\{I_0, \dots, I_n\}$ be any simplex in $\Delta(M_1)$. If $I_0 \cdots I_n M_1 = 0$, then $M_1 = \mathrm{Ann}(I_0 \cdots I_n)$, but we also have $M_2 \subseteq \mathrm{Ann}(I_0 \cdots I_n)$ since

$M_1M_2 = 0$. Hence, we have arrived at a contradiction. Therefore, $\Delta(M_1)$ is a cone with apex M_1, and hence, contractible to a point. By symmetry, the same argument holds for $\Delta(M_2)$. □

In order to determine the homotopy type of the ideal zero-divisor complex for the final case, we need the next lemma.

Lemma 3.9. *Let R be a non-local finite ring which is not isomorphic to a product of two fields. Let $\sigma_i, \sigma_j \in \mathcal{U}(\Delta, V)$, $i \neq j$, and $\sigma_i, \sigma_j \neq \{M_1\}$. Then, $\sigma_i \not\supset \sigma_j$.*

Proof. Note that, if $\sigma_j = \{M_1\}$, $\sigma_i \not\supset \sigma_j$ for any critical simplex σ_i. Suppose that we have $\sigma_j \subset \sigma_i$. Denote $\sigma_j = \{M_2, \ldots, M_m, I_0, \ldots, I_r\}$ and $\sigma_i = \{M_2, \ldots, M_m, I_0, \ldots, I_r, I_{r+1}, \ldots, I_n\}$ with the following conditions:

- $M_1 \notin \{I_0, \ldots, I_n\}$;
- $M_1M_2 \cdots M_mI_0 \cdots I_r = 0$; and
- $M_1M_2 \cdots \widehat{M_i} \cdots M_mI_0 \cdots I_n \neq 0$ for all $2 \leq i \leq m$.

Consider the ideal I_{r+1}. Having $I_{r+1} \subset M_1 = \mathrm{Ann}(M_2 \cdots M_mI_0 \cdots I_r)$ would contradict the fact that $M_2 \cdots M_mI_0 \cdots I_rI_{r+1} \neq 0$ (note that $\{M_2, \ldots, M_n, I_0, \ldots, I_{r+1}\}$ is a face of σ_i). Hence, $I_{r+1} \subset M_i$ for some $2 \leq i \leq m$. Now, since we have $M_1M_2 \cdots \widehat{M_i} \cdots M_mI_0 \cdots I_r \neq 0$ and $M_1M_2 \cdots M_mI_0 \cdots I_r = 0$, then $\mathrm{Ann}(M_1M_2 \cdots \widehat{M_i} \cdots M_mI_0 \cdots I_r) = M_i$. Having $I_{r+1} \subset M_i$ gives $M_1M_2 \cdots \widehat{M_i} \cdots M_mI_0 \cdots I_rI_{r+1} = 0$, which is a contradiction to the condition $M_1M_2 \cdots \widehat{M_i} \cdots M_mI_0 \cdots I_n \neq 0$. □

Theorem 3.10. *Let R be a finite ring such that $|\mathrm{Max}(R)| = m > 1$, and suppose that R is not isomorphic to a product of two fields. Then $|\Delta(R)|$ is homotopy equivalent to $\bigvee_{i=1}^{k} S^{d_i}$, where d_i is the dimension of the critical simplex σ_i with respect to the discrete vector field V.*

Proof. For any nonzero proper ideal $I \neq M_1$ in R, if $M_1I = 0$, then $M_2I \neq 0$ by Lemma 3.6 from subsection 3.2.1. This shows that the complex is connected since we have that $M_1M_2 \neq 0$. Furthermore, it also shows that we have $\{\{I\}, \{M_i, I\}\} \in V$ where $i = 1$ or $i = 2$, so

$\{M_1\}$ is the only critical 0-simplex with respect to the discrete vector field V.

Let $\mathcal{U}(\Delta, V)$ be the set of critical simplices of Δ with respect to V, and let D be the corresponding digraph, as described in Section 2. Let $\sigma \in \mathcal{U}(\Delta, V)$, and:

$$\Delta_\sigma = \{\tau \in \Delta : \sigma \to \tau\}.$$

If $\sigma = \{M_1\}$, then $\Delta_\sigma = \{M_1\}$. Let $\sigma = \{M_2, \dots, M_m, I_0, \dots, I_n\} \in \mathcal{U}(\Delta, V)$. We want to show that $\sigma \setminus J \times M_1 \in \Delta(R)$ for J being each of the ideals $\{M_2, \dots, M_m, I_0, \dots, I_n\}$. If $J \in \{M_2, \dots, M_m\}$, or if $J \subset M_i$ for some $2 \leq i \leq n$, the stated claim is true by the condition $M_1 M_2 \cdots \widehat{M_i} \cdots M_m I_0 \cdots I_n \neq 0$ which holds for critical simplex σ. Suppose that $J \subset M_i$. If we were to have $\sigma \setminus J \times M_1 \notin \Delta(R)$, then $\sigma \setminus J$ would be a critical simplex which is contradictory to Lemma 3.9. Therefore, apart from the faces of σ, the only $\tau \in \Delta$ such that $\sigma \to \tau$ are simplices $\sigma \setminus J \times M_1$ for $J \in \{M_2, \dots, M_m, I_0, \dots, I_n\}$ and, naturally, their faces.

If σ is a simplex of dimension d (we have $d = n + m - 1$), we may conclude that Δ_σ is the boundary of an $(d+1)$-simplex given by vertices $\{M_1, M_2, \dots, M_m, I_0, \dots, I_n\}$, which is homotopy equivalent to S^d. Furthermore, the complex $\Delta(R)$ has $\{M_1\}$ as the only critical 0-simplex with respect to V, so we have

$$\bigcap_{i=1}^{k} \Delta_{\sigma_i} = \{M_1\}.$$

Now, using Corollary 2.7 from subsection 2.2, $|\Delta(R)|$ is homotopy equivalent to $\bigvee_{i=1}^{k} |\Delta_{\sigma_i}|$. Therefore, if we denote the dimension of σ_i with d_i, then we can conclude that $|\Delta(R)|$ is homotopy equivalent to $\bigvee_{i=1}^{k} S^{d_i}$. □

We now consider some results which are direct consequences of the above theorems.

Proposition 3.11. *When R is a finite ring isomorphic to a product of m fields, $m > 2$, $|\Delta(R)|$ is homotopy equivalent to S^{m-2}.*

Proof. We have that $M_1M_2\cdots M_m = 0$ so the only critical simplices with respect to V are $\{M_1\}$ and $\{M_2,\ldots,M_m\}$. Hence, by using Corollary 2.7, $|\Delta(R)|$ is homotopy equivalent to S^{m-2}. □

Example 3.12. Let $R = \mathbb{Z}/p^r\mathbb{Z}$ where p is prime and $r > 1$. Then $|\Delta(R)|$ is homotopy equivalent to the disjoint union of a point and a wedge of spheres $\bigvee_{i=1}^k S^{d_i}$, where the number of spheres of dimension n is the number of integer partitions of $r-1$ into $n+1$ distinct integers larger than 1.

Proof. The ideals in R are of the form $I = \langle p^i\rangle$. Note that $\mathrm{Ann}(\langle p\rangle) = \langle p^{r-1}\rangle$, which has no non-zero ideals as proper subsets; thus, the only isolated point in the complex is the ideal $\langle p^{r-1}\rangle$. As for the connected subcomplex, see Theorem 3.7, the critical simplices with respect to V are of the form $\{\langle p\rangle\}$ and $\{I_0,\ldots,I_n\}$ such that

$$\langle p\rangle \notin \{I_0,\ldots,I_n\}$$

and

$$\langle p\rangle I_0\cdots I_n = 0.$$

Therefore, the ideals $I_0,\ldots,I_n$ are such that $I_0\cdots I_n = \langle p^{r-1}\rangle$. Furthermore, ideals are of the form $I = \langle p^i\rangle$, $i \neq 1$, each of which must have a different power of p as the generator, and the sum of powers of all generators must be $r-1$. Therefore, the number of critical n-simplices is the number of integer partitions of $r-1$ into $n+1$ distinct integers larger than 1. □

Note that, in **[1]**, the authors studied the homology groups for the ring $R = \mathbb{Z}/p^r\mathbb{Z}$ and calculated the rank of $H_1(\mathbb{Z}/p^r\mathbb{Z})$ (we use notation from **[1]**) to be $(r-4)/2$ when r is even and $(r-5)/2$ when r is odd. This is exactly the number of integer partitions of $r-1$ into two distinct integers larger than 1, that is, the number of spheres of dimension 1 shown in Example 3.12.

Example 3.13. Let $R = \mathbb{Z}/a\mathbb{Z}$ where $a = p_1^{r_1}\cdots p_m^{r_m}$ for some prime numbers $p_1,\ldots,p_m$, $m > 1$. Suppose that R is not isomorphic to a product of two fields. Then, $|\Delta(R)|$ is homotopy equivalent to a wedge of spheres $\bigvee_{i=1}^k S^{d_i}$ where the number of spheres of dimension n is the number of ways in which we can write $p_1^{r_1-1} \cdots p_m^{r_m-1}$ as a product of $n-m+2$ distinct integers where none of the integers is from the set $\{p_1, \ldots, p_m\}$.

Proof. We can apply the algorithm for discrete vector field V which gives critical simplices $\{\langle p_1 \rangle\}$ and $\{\langle p_2 \rangle, \ldots, \langle p_m \rangle, I_0, \ldots, I_{n-m+1}\}$ where $\langle p_2 \rangle \cdots \langle p_m \rangle I_0 \cdots I_{n-m+1} = \langle a/p_1 \rangle$. Therefore, ideals $I_0, \ldots, I_{n-m+1}$ such that $I_0 \cdots I_{n-m+1} = p_1^{r_1-1} \cdots p_m^{r_m-1}$ give different critical simplices concludes the proof. $\square$

4. Concluding remarks. In the previous section, we proved that the ideal zero-divisor complex for rings with infinitely many maximal ideals is contractible. In addition, we presented a (powerful) tool for determining the homotopy type of the ideal zero-divisor complex for finite rings, namely, the complex is homotopy equivalent to a wedge of spheres (plus some finite number of points for the case where the ring has one maximal ideal), or in the case of a ring which is isomorphic to two fields where it is homotopy equivalent to two points. Forman's main theorem for simplicial Morse theory shows that the complex is homotopy equivalent to a CW complex with exactly one cell of dimension d for each critical simplex of dimension d, and in subsection 3.2.1, we stated an algorithm for discrete vector field V which precisely describes those critical simplices. We also proved that, each time, when we glue a critical simplex in the CW complex, we are gluing it to the critical 0-simplex, which then gives a sphere. Thus, the complex is homotopy equivalent to a wedge of spheres.

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