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INTEGRAL DOMAINS IN WHICH EVERY IDEAL IS PROJECTIVELY EQUIVALENT TO A PRIME IDEAL

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ABSTRACT. In this paper, we give complete characterizations of Noetherian domains and integrally closed domains in which every ideal is projectively equivalent to a prime ideal. We also characterize pullbacks satisfying this property and show how to construct integral domains in which every ideal is projectively equivalent to a prime ideal outside the context of Noetherian domains and integrally closed domains.

1. Introduction. Let R be a commutative ring with unit, and let I be a regular ideal of R. Recall that an ideal J is projectively equivalent to I if $(I^m)' = (J^n)'$ for some positive integers m and n (where A' denotes the integral closure of A in R, that is,

$$A' = \{z \in R \mid z \text{ satisfies an equation of the form } z^r + a_1 z^{r-1} + \cdots + a_r = 0 \text{ where } a_i \in A^i \text{ for each } i\}).$$

The concept of projective equivalence of ideals and the study of ideals projectively equivalent to an ideal I was introduced by Samuel [**16**] and further developed by Nagata [**13**]. Making use of the interesting work of Rees [**14**], McAdam, Ratliff and Sally [**12**] proved that the set $\mathcal{P}(I)$ of integrally closed ideals projectively equivalent to I is linearly ordered by inclusion and is discrete. Also, they proved that, if I and J are projectively equivalent, then the set Rees(I) of Rees valuation rings of I is equal to the set Rees(J) of Rees valuation rings of J, and the values of I and J with respect to these Rees valuation rings are proportional. Recently, Ciupreca, et al., defined and studied, in a

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series of papers, the notion of projectively full ideals, see for instance, [**3, 4**].

The purpose of this paper is to continue the investigation of projectively equivalent ideals from a pure commutative rings point of view. Particularly, in Section 2, we give a complete characterization of integrally closed domains in which every nonzero ideal is projectively equivalent to a prime ideal. This leads us to a new characterization of a discrete valuation domain (DVR), that is, a locally finite dimensional domain R is a DVR if and only if it is integrally closed and every nonzero ideal is projectively equivalent to a prime ideal (Corollary 2.10).

The third section deals with the Noetherian-like setting. We first characterize Noetherian domains for which every ideal is projectively equivalent to a prime ideal. It turns that such domains are exactly Noetherian domains with integral closure being a DVR (Theorem 3.1). We also prove that strong Mori domains in which every ideal is projectively equivalent to a prime ideal must be Noetherian domains; however, Mori domains with this property need not be Noetherian domains.

In Section 4, we characterize general pullbacks for which every nonzero ideal is projectively to a prime ideal. Precisely, we prove that, for a general pullback of type ($\square$), every ideal of R is projectively equivalent to a prime ideal if and only if every ideal of T is projectively equivalent to a prime ideal, $D = k$ is a field and K is algebraic over k (Theorem 4.1). This leads us to construct non-integrally closed (Noetherian and non-Noetherian) domains for which every nonzero ideal is projectively to a prime ideal. Throughout this paper, R denotes an integral domain which is not a field, R' its integral closure and $\mathrm{Spec}(R)$ its prime spectrum. Other notation will be standard as in [**9, 11**].

2. Integrally closed domains. We start this section with a characterization of integrally closed domains in which every ideal is projectively equivalent to an invertible (respectively, a principal) ideal.

Theorem 2.1. *Let R be an integrally closed domain. The following are equivalent*:

(1) *every ideal is projectively equivalent to an invertible (respectively, a principal) ideal.*

(2) *Every prime ideal is projectively equivalent to an invertible (respectively, a principal) ideal.*

(3) *R is a Dedekind domain (respectively, Dedekind domain with torsion class group).*

The proof of this theorem requires the next lemma.

Lemma 2.2. *Let R be an integrally closed domain and I a nonzero ideal of R.*

(1) $(I : I) \subseteq (I' : I')$.

(2) *For every $x \in I^{-1}$, $(xI)' = xI'$.*

Proof.

(1) Let $x \in (I : I)$ and $z \in I'$. Then z satisfies an equation of the form

$$z^r + a_1 z^{r-1} + \cdots + a_r = 0,$$

where $a_i \in I^i$ for each i. Thus,

$$(xz)^r + xa_1(xz)^{r-1} + \cdots + x^r a_r = 0.$$

But, since $x^i a_i \in I^i(I : I) \subseteq I^i$ for each i and R is integrally closed, $xz \in R$, and thus, $xz \in I'$. Hence, $x \in (I' : I')$, as desired.

(2) Fix $0 \neq x \in I^{-1}$, and let $z \in (xI)'$. Clearly, xI is an integral ideal of R and z satisfies an equation of the form

$$z^r + a_1 z^{r-1} + \cdots + a_r = 0,$$

where $a_i \in (xI)^i = x^i I^i$ for each i. Write $a_i = x^i b_i$, $b_i \in I^i$ for each i. Then

$$z^r + xb_1 z^{r-1} + \cdots + x^r b_r = 0.$$

Now, dividing by x^r, we obtain

$$\left(\frac{z}{x}\right)^r + b_1\left(\frac{z}{x}\right)^{r-1} + \cdots + b_r = 0.$$

Since R is integrally closed and $b^i \in I^i$ for each i, $z/x \in R$, and so, $z/x \in I'$. Hence, $z \in xI'$, and therefore, $(xI)' \subseteq xI'$.

For the other inclusion, let $y \in I'$. Then $y \in R$, and y satisfies an equation

$$y^r + b_1 y^{r-1} + \cdots + b_r = 0,$$

where $b_i \in I^i$ for each i. Multiplying this equation by x^r, we obtain

$$(xy)^r + b_1 x(xy)^{r-1} + \cdots + b_r x^r = 0.$$

Since $b_i x^i \in x^i I^i = (xI)^i \subseteq R$ for each i and R is integrally closed, $xy \in R$, and thus, $xy \in (xI)'$. Hence, $xI' \subseteq (xI)'$ and therefore $xI' = (xI)'$. □

Proof of Theorem 2.1. (1) ⇒ (2) and (3) ⇒ (1) are trivial.

(2) ⇒ (3). Assume first that every prime ideal is projectively equivalent to a principal ideal, and let M be a maximal ideal of R. Then there exist a nonzero element $a \in R$ and a positive integer n such that

$$(M^n)' = (aR)' = aR$$

(since R is integrally closed). Since

$$aR = (M^n)' \subseteq M, \quad a \in M.$$

If $MM^{-1} \subsetneqq R$, then $M = MM^{-1}$, and so,

$$M^{-1} = (M : M) \subseteq (M^n : M^n) \subseteq ((M^n)' : (M^n)') = (aR : aR) = R,$$

by Lemma 2.2. Hence, $M^{-1} = (M : M) = R$ and, by induction, $M^{-n} = R$. But, since

$$M^n \subseteq (M^n)' = aR, \quad a^{-1}M^n \subseteq R,$$

and so, $a^{-1} \in M^{-n} = R$. Hence, $1 \in aR \subseteq M$, which is absurd. It follows that every maximal ideal M of R is invertible.

Now, suppose that R has a non-invertible prime ideal P. Then, there exists a maximal ideal M of R such that

$$P \subseteq PP^{-1} \subseteq M.$$

Since M is invertible, $P \subsetneqq M$, and so, $M^{-1} \subseteq (P : P)$. By hypothesis, there is a nonzero element $b \in R$ and a positive integer r such that

$(P^r)' = (bR)' = bR$. Thus,

$$M^{-1} \subseteq (P : P) \subseteq (P^r : P^r) \subseteq ((P^r)' : (P^r)') = (bR : bR) = R$$

(by Lemma 2.2). Hence, $M^{-1} = R$, and so, $R = MM^{-1} = M$, a contradiction. It follows that every nonzero prime ideal of R is invertible, and therefore, R is a Dedekind domain. Furthermore, since R is Prüfer, every ideal is integrally closed, [**9,** Theorem 27.7]. Hence, $M^n = (M^n)' = aR$ for every maximal ideal M of R, and thus, R has torsion class group.

Finally, assume that every nonzero prime ideal is projectively equivalent to an invertible ideal. Let M be a maximal ideal of R and PR_M a prime ideal of R_M where $P \subseteq M$. Let J be an invertible ideal of R and m, n positive integers such that $(P^m)' = (J^n)'$. By [**10,** Proposition 1.1.4],

$$(P^m R_M)' = (P^m)' R_M = (J^n)' R_M = (J^n R_M)'.$$

But, since J is invertible, JR_M is principal, and thus, PR_M is projectively equivalent to a principal ideal. Hence, R_M is a Dedekind domain, and therefore, R is an almost Dedekind domain. Thus, R is a one-dimensional Prüfer domain. So, for every maximal ideal M of R,

$$M^m = (M^m)' = (I^n)' = I^n$$

for some invertible, so finitely generated, ideal I of R. Hence, M^m is finitely generated and therefore invertible. Thus,

$$R = M^m M^{-m} \subseteq MM^{-1} \subseteq R.$$

Hence, $MM^{-1} = R$, and therefore, R is a Dedekind domain. □

The next example shows that Theorem 2.1 is not true if R is not integrally closed and a domain such that $(I : I) \subseteq (I' : I')$, for every nonzero ideal I, is not necessarily an integrally closed domain.

Example 2.3.

(1) Let k be a field, X an indeterminate over k and set

$$R = k[[X^2, X^3]].$$

Then R is a one-dimensional Noetherian local domain, $R' = k[[X]]$ and $\mathrm{Spec}(R) = \{(0), M\}$, where

$$M = (X^2, X^3) = X^2 k[[X]] = X^2 R'.$$

Set $J = X^2 R$. Then,

$$J' = X^2 R' \cap R = M \cap R = M.$$

Hence, M is projectively equivalent to J. However, R is not a Prüfer domain. In fact, as $R' = k[[X]]$ is the only valuation overring of R, $\mathrm{Rees}(I) = \{R'\}$ for every nonzero ideal I of R. Thus, I is projectively equivalent to M by [**4,** Theorem 3.4].

(2) Let I be a nonzero ideal of R. Since $R' = k[[X]]$ is the only proper overring of R, either $(I : I) = R$ or $(I : I) = k[[X]]$. If $(I : I) = R$, trivially $(I : I) \subseteq (I' : I')$. If $(I : I) = R' = k[[X]]$, then I is an ideal of $k[[X]]$. Thus, I is an integrally closed ideal of R' and a fortiori an integrally closed ideal of R. Hence, $(I : I) \subseteq (I' : I')$ for every ideal I of R; however, R is not integrally closed, as desired.

The next example shows that the inclusion in Lemma 2.2 (1) may be strict.

Example 2.4. Let k be a field, X and Y indeterminates over k and set

$$R = k + Yk(X)[[Y]],$$
$$T = k[X^2, X^3] + Yk(X)[[Y]]$$

and

$$I = YT.$$

Then R is an integrally closed PVD (pseudo-valuation domain),

$$I' = Y(k[X] + Yk(X)[[Y]]),$$

and

$$(I : I) = T \subsetneq k[X] + Yk(X)[[Y]] = (I' : I').$$

Corollary 2.5. *Let R be an integral domain with $(R : R') \neq (0)$ and such that every nonzero ideal is projectively equivalent to a principal ideal. Then, R' is a Dedekind domain with torsion class group.*

Proof. Let Q be a nonzero prime ideal of R', and let $0 \neq c \in (R : R')$. Let s be any positive integer, and let $z \in (Q^s)'_{R'}$, where $'_{R'}$ is the integral closure with respect to R'. Then $z \in R'$ and z satisfies an equation of the form

$$z^r + a_1 z^{r-1} + \cdots + a_r = 0,$$

where $a_i \in (Q^s)^i$ for each i. Thus $c^s z \in R$ and

$$(c^s z)^r + c^s a_1 (c^s z)^{r-1} + \cdots + c^{rs} a_r = 0.$$

But, since $(c^s)^i a_i \in ((cQ)^s)^i$, $c^s z \in ((cQ)^s)'$. Hence,

$$c^s (Q^s)'_{R'} \subseteq ((cQ)^s)'.$$

Since R' is integrally closed, by Lemma 2.2,

$$((cQ)^s)' \subseteq ((cQ)^s)'_{R'} = c^s (Q^s)'_{R'}.$$

Therefore,

$$((cQ)^s)' = ((cQ)^s)'_{R'}.$$

Now, since every nonzero ideal of R is projectively equivalent to a principal ideal, there exist a positive integer n and $a \in R$ such that

$$c^n (Q^n)'_{R'} = ((cQ)^n)'_{R'} = ((cQ)^n)' = (aR)' = aR' \cap R.$$

Since $a \in (aR)' = c^n (Q^n)'_{R'}$, $ac^{-n} \in (Q^n)'_{R'}$, and thus, $ac^{-n} R' \subseteq (Q^n)'_{R'}$. But, clearly, $(Q^n)'_{R'} \subseteq ac^{-n} R'$. Hence,

$$(Q^n)'_{R'} = ac^{-n} R',$$

and therefore, Q is projectively equivalent to a principal ideal in R'. By Theorem 2.1, R' is a Dedekind domain with torsion class group as desired. □

The next example shows that the converse is not true in general.

Example 2.6. Let k be a field of characteristic 0, X an indeterminate over k, and set $R = k[X^2, X^3]$. Clearly, $R' = k[X]$ is a *PID* and

$(R : R') = Q = (X^2, X^3) = X^2 R'$ is a maximal ideal of R. Let

$$N = (X+1)k[X] = (X+1)R'$$

and

$$M = N \cap R.$$

Then, M is a maximal ideal of R. Since $Q \nsubseteq M$, $R'_N = R_M$. Thus, for every positive integer r,

$$N^r \cap R = N^r R'_N \cap R = M^r R_M \cap R = M^r.$$

On the other hand, by [**10,** Proposition 1.6.1],

$$M^r \subseteq (M^r)' = M^r R' \cap R \subseteq N^r \cap R = M^r.$$

Hence,

$$M^r = (M^r)' = N^r \cap R = (X+1)^r k[X] \cap R.$$

We claim that M is not projectively equivalent to any principal ideal of R. Indeed, suppose that there is $f \in R$ and a positive integer n such that $M^n = (M^n)' = fR' \cap R$. Then

$$(X+1)^n k[X] \cap R = fR' \cap R.$$

Thus, $f = (X+1)^n g$ for some $g \in k[X]$. Since $X^r \in R$ for every $r \geq 2$,

$$(X+1)^n(1 - nX) = 1 + \sum_{p=2}^{n} \binom{n}{p} X^p - \sum_{p=1}^{n} n \binom{n}{p} X^{p+1} \in R.$$

Then

$$(X+1)^n(1-nX) \in (X+1)^n k[X] \cap R = fk[X] \cap R,$$

and so,

$$(X+1)^n(1-nX) = fh = (X+1)^n gh \quad \text{for some } h \in k[X].$$

Hence, $1 - nX = gh$, and so, $\deg(g) = 0$ or $\deg(g) = 1$. If $\deg(g) = 0$, then $g = c$ is a nonzero constant, and thus, $f = c(X+1)^n \in R$, which is absurd (since $X \notin R$). Hence, $\deg(g) = 1$, and so, $h = c \in k \setminus \{0\}$. Thus, $g = c^{-1}(1 - nX)$, and so,

$$f = c^{-1}(X+1)^n(1-nX).$$

Since

$$(X+1)^n(1-nX+X^2) \in (X+1)^n k[X] \cap R = fk[X] \cap R,$$

$(X+1)^n(1-nX+X^2) = c^{-1}(X+1)^n(1-nX)g$ for some $g \in k[X]$.

Then

$$1-nX+X^2 = c^{-1}(1-nX)g,$$

and so, $\deg(g) = 1$. Write $g = b + dX$ where $b, d \in k$ and $0 \neq d$. By identification, we obtain:

$$\left\{ \begin{array}{ccc} 1 & = & c^{-1}b \\ -n & = & c^{-1}(d-nb) \\ 1 & = & -c^{-1}nd \end{array} \right\}$$

Hence,

$$-n = c^{-1}d - nc^{-1}b = c^{-1}d - n,$$

and so $0 = c^{-1}d$, which is absurd. It follows that M is not projectively equivalent to any principal ideal of R.

The next proposition characterizes domains for which every nonzero finitely generated ideal is projectively equivalent to an invertible (respectively, a principal) ideal in the context of integrally closed domains.

Proposition 2.7. *Let R be an integrally closed domain. Then, every nonzero finitely generated ideal is projectively equivalent to an invertible* (*respectively, a principal*) *ideal if and only if R is a Prüfer domain* (*respectively, Prüfer with torsion Picard class group*).

Proof. Let I be a finitely generated ideal of R. Then there exist a positive integer n and a nonzero element $a \in R$ such that $(I^n)' = aR$. So

$$J = a^{-1}I^n \subseteq a^{-1}(I^n)' = R,$$

and thus, J is an integral ideal of R. On the other hand, since $a \in (I^n)'$, a satisfies an equation of the form

$$a^r + b_1a^{r-1} + \cdots + b_r = 0,$$

where $b_i \in (I^n)^i$ for each i. Dividing the equation by a^r, we obtain

$$1 + \frac{b_1}{a} + \frac{b_2}{a^2} + \cdots + \frac{b_r}{a^r} = 0.$$

Thus,

$$1 = -\left(\frac{b_1}{a} + \frac{b_2}{a^2} + \cdots + \frac{b_r}{a^r}\right).$$

But, since

$$\frac{b_i}{a^i} = (a^{-1})^i b_i \in (a^{-1}I^n)^i = J^i, \quad 1 \in J,$$

and thus, $J = R$. Hence, $I^n = aR$, and so,

$$R = I^n I^{-n} \subseteq II^{-1} \subseteq R.$$

Therefore, $II^{-1} = R$, and hence, R is a Prüfer domain. Also, since, for each finitely generated ideal I of R, $I^n = aR$ (for some positive integer n and $a \in R$), R has torsion Picard class group.

Now, suppose that every nonzero finitely generated ideal is projectively equivalent to an invertible ideal. Then, for every maximal ideal M of R, every finitely generated ideal of R_M is projectively equivalent to a principal ideal. Thus, R_M is a valuation domain, and therefore, R is Prüfer.

The converse is trivial. □

The next theorem characterizes locally finite-dimensional (LFD) domains for which every nonzero ideal is projectively equivalent to a prime ideal in case where the conductor $(R : R')$ is a nonzero ideal of R.

Theorem 2.8. *Let R be a locally finite-dimensional domain (LFD). If R' is a DVR, then every ideal of R is projectively equivalent to a prime ideal. The converse holds if $(R : R') \neq (0)$.*

The proof requires the next useful lemma.

Lemma 2.9. *Let R be a domain and I a nonzero ideal of R. Then*:

(1) *if Q is a prime ideal of R and Q is projectively equivalent to I, then Q is the unique minimal prime over I, in particular, $Q = \sqrt{I}$.*

(2) *If every nonzero ideal of* R *is projectively equivalent to a prime ideal, then* $\mathrm{Spec}(R)$ *is a chain, in particular,* R *is local.*

(3) *Assume that* R *is locally finite dimensional* (LFD) *and every ideal of* R *is projectively equivalent to a prime ideal. Then every prime ideal is the unique minimal prime over some principal ideal.*

Proof.

(1) Assume that Q is projectively equivalent to I. Then $(Q^m)' = (I^n)'$ for positive integers m and n. Then $I \subseteq Q$; and, if P is a prime ideal minimal over I, then

$$Q^m \subseteq (Q^m)' = (I^n)' \subseteq P.$$

Hence, $Q \subseteq P$, and therefore, $Q = P$. It follows that Q is the unique minimal prime over I as desired.

(2) Let P and Q be any nonzero prime ideals of R, and set $I = PQ$. Then, there is a prime ideal N of R such that I is projectively equivalent to N. Thus, $(I^m)' = (N^n)'$ for some positive integers m and n. Since, $(PQ)^m \subseteq (I^m)' = (N^n)' \subseteq N$, either $P \subseteq N$ or $Q \subseteq N$. Without loss of generality, we may assume that $P \subseteq N$. However,

$$N^n \subseteq (N^n)' = (I^m)' \subseteq I' \subseteq P \cap Q$$

implies that $N \subseteq P \cap Q$. Hence, $P = N \subseteq Q$. Therefore, any two nonzero prime ideals of R are comparable and so $\mathrm{Spec}(R)$ is a chain and R is local.

(3) First, recall that $\mathrm{Spec}(R)$ is a chain by (2). Let P be a nonzero prime ideal of R. Since R is LFD, there is a unique prime ideal $P' \subsetneq P$ of R such that $\mathrm{ht}\,(P/P') = 1$. Let $a \in P \setminus P'$. Then, aR is projectively equivalent to a prime ideal Q of R; and by (1), $Q = \sqrt{aR} = P$ is the unique minimal prime over aR. □

Proof of Theorem 2.8. Assume that R' is a DVR, and let N be its maximal ideal. Then R is local with maximal ideal $M = N \cap R$. Let I be a nonzero ideal of R. Then, $IR' = N^r$ and $MR' = N^s$ for some positive integers r and s. Hence, by [**10,** Proposition 1.6.1],

$$(I^s)' = I^s R' \cap R = (N^r)^s \cap R = (N^s)^r \cap R = M^r R' \cap R = (M^r)',$$

as desired.

Now, suppose that $(R : R') \neq (0)$ and every ideal of R is projectively equivalent to a prime ideal. By Lemma 2.9, R is local, and so, it is of finite dimension since it is LFD. Also, $\mathrm{Spec}(R)$ is a finite chain and every prime ideal is projectively equivalent to a principal ideal. Thus, every ideal of R is projectively equivalent to a principal ideal, and, by Corollary 2.5, R' is a Dedekind domain with torsion class group. Hence, it suffices to show that R' is local.

If R is integrally closed, then $R' = R$ is local. Next, assume that $R' \neq R$. Since $\dim R = \dim R' = 1$,

$$\mathrm{Spec}(R) = \{(0) \subsetneqq M\},$$

and so, $M \subseteq J(R')$ where $J(R')$ is the Jacobson radical of R'. Set $A = (R : R')$. Then $(A^n)' = (M^m)'$ for positive integers m and n. But since A^n is an ideal of R' and R' is Prüfer, A^n is an integrally closed ideal of R', and, a fortiori, an integrally closed ideal of R. Hence, $A^n = (M^m)'$.

Now, let N and Q be maximal ideals of R'. Then NA is an ideal of both R and R'. Then, $((AN)^r)' = (M^s)'$ for some positive integers r and s. Again, since R' is Prüfer, $(AN)^r$ is an integrally closed ideal of R' and, a fortiori, an integrally closed ideal of R. Hence $(AN)^r = (M^s)'$. Thus,

$$\begin{aligned} N^{rm}A^{rm} &= (AN)^{rm} = ((AN)^{rm})' \\ &= (((AN)^r)^m)' = (((M^s)')^m)' \\ &= (M^{sm})' = (((M^m)')^s)' = A^{ns}. \end{aligned}$$

If $rm \geq ns$, then composing the two sides of the equality by $(R' : A^{ns})$ and using the fact that R' is a Dedekind domain (so $A^{ns}(R' : A^{ns}) = R'$), we obtain

$$N^{rm}A^{rm-ns} = N^{rm}A^{rm}(R' : A^{ns}) = A^{ns}(R' : A^{ns}) = R',$$

which is a contradiction. Hence, $rm < ns$, and so,

$$N^{rm} = N^{rm}A^{rm}(R' : A^{rm}) = A^{ns}(R' : A^{rm}) = A^{ns-rm} \subseteq M \subseteq Q.$$

Thus, $N \subseteq Q$, and hence, $N = Q$. It follows that R' is local, and therefore, R' is a DVR. $\square$

Corollary 2.10. *Let R be a locally finite dimensional domain which is integrally closed. Then, every nonzero ideal is projectively equivalent to a prime ideal if and only if R is a DVR.*

Question 2.11 (Open question)**.** *Beyond the context where $(R : R') \neq (0)$, we are not able to prove or disprove whether R' is a DVR if every ideal of R is projectively equivalent to a prime ideal. A more general question is about whether the integral closure R' of such a domain R inherits the property that every ideal is projectively equivalent to a prime ideal. If the answer is "yes," then R' would be a DVR by the integrally closed case. A weaker version of this question is regarding the integral closure of a domain in which every ideal is projectively equivalent to a prime (respectively, invertible, respectively, principal) ideal is a Prüfer domain (equivalently, R is quasi-Prüfer)?*

While the polynomial ring $R[X]$ never has the property that every ideal is projectively equivalent to a prime ideal (as it is never local), our next corollary shows that the power series rings $R[[X]]$ has this property only if R is a field.

Corollary 2.12. *Every nonzero ideal of $R[[X]]$ is projectively equivalent to a prime ideal if and only if R is a field.*

Proof. Assume that R is not a field. By Lemma 2.9, $R[[X]]$ is local and so is R. Let M be the maximal ideal of R, and let $0 \neq m \in M$. Set

$$Q = XR[[X]] \quad \text{and} \quad P_m = (X - m)R[[X]].$$

Again, by Lemma 2.9, P and Q are comparable. Without loss of generality, we may assume that $Q \subseteq P_m$. Then $X = (X - m)f$ for some

$$f = \sum_{n \geq 0} a_n X^n \in R[[X]].$$

Necessarily,

$$-a_0 m = 0 \quad \text{and} \quad a_0 - a_1 m = 1.$$

Thus, $-a_1 m = 1$, which is absurd. It follows that R is a field.

The converse follows immediately from Corollary 2.10 since $k[[X]]$ is a DVR. □

3. Noetherian-settings. The next theorem deals with Noetherian domains for which every nonzero ideal is projectively equivalent to a prime ideal. A combination of Lemma 2.9 and the Principal ideal theorem shows that such a domain is a one-dimensional local domain.

Theorem 3.1. *Let R be a Noetherian domain. The following are equivalent.*

(1) *Every nonzero ideal of R is projectively equivalent to a prime ideal.*

(2) *R is a one-dimensional local domain, and all nonzero ideals of R are projectively equivalent to a same principal ideal.*

(3) *R' is a DVR.*

Proof.

(1) $\Rightarrow$ (2). Since R is an LFD, by Lemma 2.9, R is local with maximal ideal M, and M is the unique minimal prime over a certain principal ideal aR. Thus, by the Principal ideal theorem, $\dim R = \operatorname{ht} M = 1$. Hence,

$$\operatorname{Spec}(R) = \{(0) \subsetneq M,$$

and so all nonzero ideals of R are projectively equivalent to M, and therefore projectively equivalent to aR, as desired.

(2) $\Rightarrow$ (3). Assume that all ideals of R are projectively equivalent to the same principal ideal $I = aR$. Since R is a one-dimensional Noetherian domain, R' is a Dedekind domain, and thus, it suffices to show that R' is local. Since R is a one-dimensional local domain,

$$I \subseteq M \subseteq J(R').$$

By [**3,** Example 3.5],

$$\operatorname{Rees}(I) = \{R'_N \mid N \in \operatorname{Max}(R')\}.$$

Moreover, If $|\operatorname{Rees}(I)| \geq 2$, there exists an ideal J of R with $\operatorname{Rees}(I) = \operatorname{Rees}(J)$, but J is not projectively equivalent to I, which is absurd. Hence, $|\operatorname{Rees}(I)| = 1$, and therefore, R' is local, as desired.

(3) $\Rightarrow$ (1). Follows from Theorem 2.8. $\square$

Recall that a domain R is said to be strong Mori domain if R satisfies the *acc* on w-ideals. Noetherian and strong Mori domains are Mori.

Proposition 3.2. *Let R be a strong Mori domain. If every ideal of R is projectively equivalent to a prime ideal, then* $\dim R = 1$*, and so R is Noetherian.*

Proof. Suppose that $\dim R \geq 2$, and let

$$(0) \subsetneqq P \subsetneqq M$$

be a chain of prime ideals of R. Let $a \in M \setminus P$. Then there exists a prime ideal Q of R such that aR is projectively equivalent to Q. Set $(a^n R)' = (Q^m)'$ for some positive integers m and n. By Lemma 2.9, P and Q are comparable. But, since $a \notin P$, $P \subsetneqq Q$. On the other hand, Q is minimal over aR. But since R is strong Mori, by [**5**, Corollary 1.11], $\operatorname{ht} Q = 1$, which is a contradiction. Hence, $\dim R = 1$, and again, by [**5**, Corollary 1.10], R is Noetherian. □

Recall that a commutative ring R with identity is compactly packed by primes (CP-ring for short) if, whenever an ideal I of R is contained in the union of a family of prime ideals of R, then I is actually contained in one of the primes of the family [**15**]; equivalently, every prime ideal is the radical of a principal ideal [**17**].

The next proposition shows that a Mori domain in which every ideal is projectively equivalent to a prime ideal is a CP-ring. However, we are not able to prove or disprove that R must be a one-dimensional domain. We recall that a domain R is semi-normal if, for every $x \in qf(R)$, $x^2, x^3 \in R$, implies that $x \in R$.

Proposition 3.3. *Let R be a Mori domain. If every ideal of R is projectively equivalent to a prime ideal, then R is a CP-ring with finite prime spectrum. In particular, if R satisfies the* PIT (*Principal ideal theorem*) *or* $(R : R') \neq (0)$ *or R is semi-normal, then* $\dim R = 1$.

Proof. We claim that $\operatorname{Spec}(R)$ is finite. By way of contradiction, let $\{P_n\}_{n \geq 0}$ be an infinite chain of prime ideals of R. (Note that, by Lemma 2.9, $\operatorname{Spec}(R)$ is a chain). For each $n \geq 1$, let

$$a_n \in P_n \setminus P_{n-1}.$$

Then there is a prime ideal Q_n of R such that Q_n is projectively equivalent to a_nR. By Lemma 2.9, Q_n is the unique prime minimal over a_nR, and hence, Q_n is a t-prime ideal.

On the other hand, since Q_n, P_n and P_{n-1} are comparable (Lemma 2.9) and $a_n \in Q_n \setminus P_{n-1}$,

$$P_{n-1} \subsetneqq Q_n \subseteqq P_n.$$

Thus, we construct a chain

$$P_0 \subsetneqq Q_1 \subseteqq P_1 \subsetneqq Q_2 \subseteqq P_2 \cdots \subseteqq P_{n-1} \subsetneqq Q_n \subseteqq P_n \cdots ,$$

from which we extract the infinite chain

$$Q_1 \subsetneqq Q_2 \subsetneqq Q_3 \cdots \subsetneqq Q_{n-1} \subsetneqq Q_n \subsetneqq Q_{n+1} \cdots$$

of t-prime ideals, which is a contradiction since R is a Mori domain. Hence, $\operatorname{Spec}(R)$ is finite. Set

$$\operatorname{Spec}(R) = \{(0) = P_0 \subsetneqq P_1 \subsetneqq \cdots \subsetneqq P_n = M\},$$

where M is the maximal ideal of R. By Lemma 2.9, each prime of R is the radical of a principal ideal, and therefore, R is a CP-ring. Now, if $(R : R') \neq (0)$, by Theorem 2.8, R' is a DVR, and so, $\dim R = \dim R' = 1$. If R satisfies the PIT, then $\operatorname{ht} M = 1$, as M is minimal over a principal ideal. Finally if R is semi-normal and $\operatorname{ht} M \geq 2$, then M must contain infinitely many height 1 prime ideals [**1,** Theorem 2.6], which is a contradiction. Hence, $\dim R = 1$, as desired. $\square$

4. Pullbacks. Let T be an integral domain, M a nonzero ideal of T (not necessarily maximal), D an integral domain contained in $K = T/M$, $\phi : T \to T/M$ the canonical homomorphism and R the pullback of the diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\phi} & K = T/M \end{array}$$

We assume that $R \subsetneq T$, and we refer to this diagram as of type (Δ). If M is a maximal ideal of T and $K = T/M$ its residue field, we refer to the above diagram as of type $(\square)$; and if $T = V$ is a valuation domain with maximal ideal M, we shall refer to this as a classical diagram of type ($\square$). (For more details on classical diagrams and diagrams of type ($\square$), we refer to **[2, 6, 7, 8]**). Recall that M is a prime ideal of R as $R/M \simeq D$.

The next theorem characterizes diagrams of type ($\square$), in which every ideal is projectively equivalent to a prime ideal and shows how to construct non-integrally closed (Noetherian and non-Noetherian) domains with this property.

Theorem 4.1. *For a diagram of type* ($\square$) *assume that T and D are LFDs. Then, every ideal of R is projectively equivalent to a prime ideal of R if and only if every ideal of T is projectively equivalent to a prime ideal of T, $D = k$ is a field and K is algebraic over k.*

The proof of Theorem 4.1 requires the next lemma.

Lemma 4.2. *Let $R \subseteq T$ be an extension of integral domains such that the conductor $A = (R : T)$ is a nonzero proper integrally closed ideal of R.*

(1) *If I is an ideal of R which is also an ideal of T, then I' is an ideal of T.*

(2) *If A is projectively equivalent to a principal ideal of R, then T is integral over R, that is, $T \subseteq R'$.*

Proof.

(1) Let I be an ideal of both R and T, and let $x \in I'$ and $b \in T$. Since $IT = I \subseteq R$, $I \subseteq A$, and so, $I' \subseteq A' = A$. Hence, $x \in A$, and so, $xb \in A \subseteq R$. Moreover, x satisfies an equation of the form

$$x^r + a_1 x^{r-1} + \cdots + a_r = 0,$$

where $a_i \in I^i$ for each $i \in \{1, \ldots, r\}$. Thus,

$$(xb)^r + ba_1(xb)^{r-1} + \cdots + b^r a_r = 0.$$

But, since $b_i a_i \in I^i T = I^i$ for each i, $xb \in I'$. Hence, I' is an ideal of T, as desired.

(2) Assume that A is projectively equivalent to aR for some nonzero element $a \in R$. Then, there exist positive integers m and n such that

$$(A^n)' = (a^m R)' = a^m R' \cap R.$$

Since

$$a^m R \subseteq (a^m R)' = (A^n)',$$

and $(A^n)'$ is an ideal of T by (1),

$$a^m T \subseteq (A^n)' = a^m R' \cap R.$$

Thus, $a^m T \subseteq a^m R'$, and hence, $T \subseteq R'$, as desired. $\square$

Proof of Theorem 4.1. First note that R is LFD. Assume that every ideal of R is projectively equivalent to a prime ideal. By Lemma 2.9, R is local and so T must be local with maximal ideal M. Now, let J be a nonzero ideal of T. Since $J \subseteq M$, J is an ideal of R. Hence,

$$(J^n)'_T = (J^n)'_R = (Q^m)'_R$$

for some prime ideal Q of R and positive integers m and n. But, since $J \subseteq M$, $Q \subseteq M$ and so Q is a prime ideal of T. Hence,

$$(J^n)'_T = (J^n)'_R = (Q^m)'_R = (Q^m)'_T,$$

as desired.

Now, since $(R:T) = M$ is a prime ideal of R, M must be projectively equivalent to a principal ideal of R (Lemma 2.9 (3)). By Lemma 4.2, T is integral over R. Hence, $D = k$ is a field and K is algebraic over k.

Conversely, by Lemma 2.9, T is local. Hence, R is local and $\mathrm{Spec}(R) = \mathrm{Spec}\, t(T)$. Let I be a nonzero ideal of R. Since R is local with maximal ideal M, $I \subseteq M$. Thus, IT is a nonzero proper ideal of T (as $IT \subseteq M$), and so

$$((IT)^n)'_T = (Q^m)'_T$$

for some positive integers m and n and a prime ideal Q of T. Since $\mathrm{Spec}(R) = \mathrm{Spec}(T)$, Q is a prime ideal of R. By [**10,** Proposition 1.6.1],

$$(I^n)' = (I^n T)'_T \cap R = ((IT)^n)'_T \cap R = (Q^m)'_T \cap R = (Q^m)'.$$

Hence, I is projectively equivalent to Q, as desired. $\square$

In view of Theorem 2.10, a valuation domain V of finite dimension has the property that every ideal is projectively equivalent to a prime ideal if and only if it is a DVR. Combining with Theorem 4.1, we obtain the following characterization of classical diagram of type ($\Box$).

Corollary 4.3. *For the pullback of the classical diagram of type* ($\Box$), *assume that V is of finite dimension and D is LFD. The following are equivalent*:

(1) *Every ideal of R is projectively equivalent to a prime ideal.*

(2) *Every ideal of R is projectively equivalent to a principal ideal.*

(3) *V is a DVR, $D = k$ is a field and K is algebraic over k.*

Proof.

(1) $\Leftrightarrow$ (3). Follows from Theorem 4.1 and Corollary 2.10.

(2) $\Rightarrow$ (3). By Corollary 2.5, R' is a Dedekind domain. Hence, $R' = V$ and so V is a DVR, $D = k$ is a field and K is algebraic over k, as desired.

(3) $\Rightarrow$ (2). Assume that V is a DVR and K is algebraic over k. Then $R' = V$. Let I be a nonzero ideal of R. If I is an ideal of V, then I is integrally closed in both R and V and $I = M^n$ for some $n \geq 1$. If I is not an ideal of V, then $I = a\phi^{-1}(W)$, where W is a k-subvector space of K with $k \subseteq W \subsetneq K$. Since $aR \subseteq I \subseteq IV = aV$,

$$aV = aR' = aR' \cap R = (aR)' \subseteq I' \subseteq (aV)' = aV.$$

Thus, $I' = aV = aR' = (aR)' = M^n$ for some $n \geq 1$, as desired. $\Box$

Theorem 4.4. *For the diagram of type* (Δ) *assume that T and D are LFDs and M is a prime principal ideal of T. Then, every nonzero ideal of R is projectively equivalent to a prime ideal if and only if M is a maximal ideal of T, every ideal of T is projectively equivalent to a prime ideal, $D = k$ is a field and K is algebraic over k.*

Proof. Set $M = aT$ and suppose that every ideal of R is projectively equivalent to a prime ideal.

Claim 1. For every nonzero ideal J of T, $(aJ)'_T = aJ'_T$. Indeed, let $x \in (aJ)'_T$. Then $x \in T$ and

$$x^r + b_1 x^{r-1} + \cdots + b_r = 0$$

where $b_i \in (aJ)^i = a^i J^i$. Since $aJ = JM \subseteq M$, $(aJ)'_T \subseteq M = aT$. Then, $x \in aT$, and so, $x/a \in T$. But, since

$$\left(\frac{x}{a}\right)^r + \frac{b_1}{a}\left(\frac{x}{a}\right)^{r-1} + \cdots + \frac{b_r}{a^r} = 0$$

and

$$\frac{b_i}{a^i} \in J^i, \qquad \frac{x}{a} \in J'_T,$$

and thus, $x \in aJ'_T$. Hence, $(aJ)'_T \subseteq aJ'_T$. Conversely, it is easy to see that $(aJ)'_T \supseteq aJ'_T$. Therefore $(aJ)'_T = aJ'_T$.

Claim 2. For every nonzero ideal J of T and for every positive integer n, $(a^n J)'_T = a^n J'_T$ by induction on n. If $n = 1$, the result is true by Claim 1. Assume the induction hypothesis for n. Then, applying Claim 1 and the induction hypothesis to the ideal $a^n J$, we obtain

$$(a^{n+1}J)'_T = (a.a^n J)'_T = a(a^n J)'_T = a.a^n J'_T = a^{n+1} J'_T,$$

as desired.

Claim 3. $(M^n)'_T = M^n$. If $n = 1$, we are done. Assume that $n \geq 2$. Applying Claim 2 to the ideal $J = M$, we obtain

$$(M^n)'_T = (a^n T)'_T = (a^{n-1}M)'_T = a^{n-1}M'_T = a^{n-1}M = M^n.$$

Now, let N be a maximal ideal of T such that $M \subseteq N$. Then aN is an ideal of both R and T. Since every nonzero ideal of R is projectively equivalent to a prime ideal, $((aN)^n)'_R = (Q^m)'_R$ for some prime ideal Q of R and positive integers n and m. Hence, $aN \subseteq Q \subseteq M$. Then,

$$M^2 = aM \subseteq aN \subseteq Q,$$

and so, $M \subseteq Q$. Hence, $Q = M$. Also, since $aN \subseteq M$,

$$((aN)^n)'_T = ((aN)^n)'_R.$$

By Claims 2 and 3,

$$\begin{aligned} a^n (N^n)'_T &= ((aN)^n)'_T = ((aN)^n)'_R \\ &= (Q^m)'_R = (M^m)'_R \\ &= (M^m)'_T = M^m = a^m T. \end{aligned}$$

Hence,

$$N^n \subseteq (N^n)'_T = a^{m-n} T = M^{m-n} \subseteq M,$$

and therefore, $N \subseteq M$. Thus, $M = N$ is a maximal ideal of T, as desired. The remaining conditions and the converse now follows from Theorem 4.1. □

Corollary 4.5. *Let $A \subsetneqq B$ be an extension of LFD domains, X an indeterminate over B, and set $R := A + XB[[X]]$. Then, every nonzero ideal of R is projectively equivalent to a prime ideal if and only if A and B are fields and B is algebraic over A.*

Proof. By Theorem 4.4, $M = XB[[X]]$ must be a maximal ideal of $T = B[[X]]$. Hence, B is a field, and, by Corollary 4.3, A is a field and B is algebraic over A. The converse also follows from Corollary 4.3. □

The next example shows how to construct a non Noetherain non-integrally closed Mori domain in which every ideal is projectively equivalent to a prime ideal.

Example 4.6. Let $\mathbb{Q}$ be the field of rational numbers, X an indeterminate over $\mathbb{Q}$ and

$$V = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots \sqrt{p}, \ldots)[[X]] = K + M,$$

where

$$K = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \ldots \sqrt{p}, \ldots)$$

and $M = XV$. Set $R = \mathbb{Q} + M$. By Corollary 4.3, every ideal of R is projectively equivalent to a prime ideal. However, R is a Mori domain [**8,** Theorem 4.18] which is not Noetherian [**8,** Theorem 4.12].

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