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PSEUDO-CONVERGENT SEQUENCES AND PRÜFER DOMAINS OF INTEGER-VALUED POLYNOMIALS

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ABSTRACT. Let K be a field with rank one valuation and V the valuation domain of K. For a subset E of V, the ring of integer-valued polynomials on E is

$$\mathrm{Int}\,(E,V)=\{f\in K[x]\mid f(E)\subseteq V\}.$$

A question of interest regarding $\mathrm{Int}\,(E,V)$ is: for which E is $\mathrm{Int}\,(E,V)$ a Prüfer domain? In this paper, we contribute a partial answer to this question. We classify exactly when $\mathrm{Int}\,(E,V)$ is Prüfer in the case where the elements of E comprise a pseudo-convergent sequence in V. Our work expands on earlier results that apply when V is a discrete valuation domain.

1. Introduction. Let D be an integral domain (not a field) with quotient field K. We define the ring of integer-valued polynomials on D to be

$$\mathrm{Int}\,(D)=\{f(x)\in K[x]\mid f(D)\subseteq D\}.$$

Serious work on integer-valued polynomials began in 1919 with papers by Ostrowski **[12]** and Pólya **[13]**. These papers both focused on the D-module structure of $\mathrm{Int}\,(D)$. More recently, $\mathrm{Int}\,(D)$ has been studied as a ring. It was observed by Brizolis **[2]** that, if D is the ring of integers of an algebraic number field, then $\mathrm{Int}\,(D)$ is a Prüfer domain. The question of classifying all domains D such that $\mathrm{Int}\,(D)$ is Prüfer then became of interest. Chabert **[5]** and McQuillan **[9]** proved, independently of one another, that when D is Noetherian, $\mathrm{Int}\,(D)$ is Prüfer if and only if D is a Dedekind domain with all residue fields finite. For a general domain D, the question of when $\mathrm{Int}\,(D)$ is Prüfer was completely resolved in **[8]**.

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A construction related to that of Int (D) is the ring

$$\text{Int}\,(S, D) = \{f(x) \in K[x] \mid f(S) \subseteq D\},$$

where $S \subseteq D$. We call this a ring of integer-valued polynomials on a subset. The classification of when Int (S, D) is a Prüfer domain does not follow immediately from the classification result for Int (D). In fact, it seems to be significantly harder. It is easy to show that a necessary condition for Int (S, D) to be a Prüfer domain is that D be a Prüfer domain. McQuillan **[10]** proved that, if S is finite, then this condition is also sufficient.

For an infinite subset S of a valuation domain V, Cahen, Chabert and Loper **[4]** examined the question of when Int (S, V) is a Prüfer domain. Even in this special case there is no general classification result. If V is one-dimensional, then clearly the corresponding valuation induces a metric on the quotient field of V. We can then consider the completion of V with respect to this metric. We call a subset of V *precompact* if its completion is compact. It is proven in **[4]** that Int (S, V) is a Prüfer domain, provided S is precompact, and that this condition is necessary if the valuation is discrete. The question of the necessity of the precompactness condition for a general valuation domain was left open.

In this note, we show that precompactness is not necessary in general. To do so, we recall Ostrowski's notions of pseudo-convergent sequences and pseudo-limits (both defined below). Let V be a one-dimensional valuation domain, and let $E = \{\alpha_0, \alpha_1, \alpha_2, \ldots\}$ be a sequence of elements of V. Then Int (E, V) can be a Prüfer domain if E is pseudo-convergent and V does not contain a pseudo-limit for the sequence. Because E is not necessarily precompact in this case, our work contributes new information regarding subsets E of V for which Int (E, V) is Prüfer.

The paper is organized as follows. Section 2 reviews the definition and basic properties of pseudo-convergent sequences. Section 3 discusses the maximal spectrum of Int (E, V), and Sections 4 and 5 investigate the localizations of Int (E, V) at these maximal ideals. Our main result, Theorem 5.2, classifies, for E pseudo-convergent, exactly when Int (E, V) is Prüfer. We then close the paper with some examples, ultimately demonstrating (Example 5.12) that the precompactness of E is not necessary for Int (E, V) to be Prüfer.

2. Pseudo-convergent sequences. Throughout, K denotes a field with rank one valuation v, the ring

$$V = \{a \in K \mid v(a) \geq 0\}$$

is the valuation domain of K, and

$$\mathfrak{m} = \{a \in K \mid v(a) > 0\}$$

is the maximal ideal of V. A sequence $(a_i)_{i\in\mathbb{N}}$ of elements of K is called *pseudo-convergent* if, for all $i > j > k$, we have $v(a_i - a_j) > v(a_j - a_k)$. An element $a \in K$ is a *pseudo-limit* of the pseudo-convergent sequence $(a_i)_{i\in\mathbb{N}}$ if, for all $i > j$, we have $v(a - a_i) > v(a - a_j)$.

We let

$$E = \{\alpha_0, \alpha_1, \alpha_2, \ldots\} \subseteq V$$

be such that the sequence $(\alpha_i)_{i\in\mathbb{N}}$ is pseudo-convergent. We associate E with the sequence $(\alpha_i)_{i\in\mathbb{N}}$ so that terminology for pseudo-convergent sequences carries over to E (e.g., we may say that E is pseudo-convergent). Given a rational function $\phi \in K(x)$, we let $v_i(\phi) = v(\phi(\alpha_i))$ for each $i \in \mathbb{N}$.

Our first lemma lists some fundamental properties of pseudo-convergent sequences. These properties will be used frequently throughout this paper.

Lemma 2.1. *Let $(a_i)_{i\in\mathbb{N}}$ be a pseudo-convergent sequence in K, and let $f \in K[x]$.*

- (i) [**7,** Lemma 1]. *Either*
  - (a) *$v(a_i) > v(a_j)$, for all $i > j$, or*
  - (b) *there exists $n \in \mathbb{N}$ such that $v(a_i) = v(a_n)$ for all $i \geq n$.*
- (ii) [**7,** Lemma 2]. *For all $i > j$, we have $v(a_i - a_j) = v(a_{j+1} - a_j)$.*
- (iii) [**11,** page 371]. *The sequence $(f(a_i))_{i\in\mathbb{N}}$ is eventually pseudo-convergent, that is, there exists $n \in \mathbb{N}$ such that, whenever $i > j > k \geq n$, $v(f(a_i) - f(a_j)) > v(f(a_j) - f(a_k))$. Consequently, either*
  - (a) *$v(f(a_i)) > v(f(a_j))$ for all $i > j \geq n$, or*
  - (b) *there exists $j' \in \mathbb{N}$, $j' \geq n$, such that $v(f(a_i)) = v(f(a_{j'}))$ for all $i \geq j'$.*

The conditions in parts (i) and (ii) of Lemma 2.1 are important enough to warrant their own terminology.

Definition 2.2. Let $(a_i)_{i\in\mathbb{N}}$ be a pseudo-convergent sequence in K. If $v(a_i) > v(a_j)$ for all $i > j$, then we say that $(a_i)_{i\in\mathbb{N}}$ is *increasing*. If there exists $n \in \mathbb{N}$ such that $v(a_i) = v(a_n)$ for all $i \geq n$, then we say that $(a_i)_{i\in\mathbb{N}}$ *stabilizes* or is *stable*.

Let $f \in K[x]$, and let n be as in Lemma 2.1 (iii). If $v(f(a_i)) > v(f(a_j))$ for all $i > j \geq n$, then we say that $(f(a_i))_{i\in\mathbb{N}}$ is *eventually increasing*. If there exists $j' \in \mathbb{N}$, $j' \geq n$, such that $v(f(a_i)) = v(f(a_{j'}))$ for all $i \geq j'$, then we say that $(f(a_i))_{i\in\mathbb{N}}$ *eventually stabilizes* or is *eventually stable*.

3. The maximal spectrum of $\operatorname{Int}(E, V)$.

When $S \subseteq V$, the ring $\operatorname{Int}(S, V)$ of integer-valued polynomials on S is

$$\operatorname{Int}(S, V) = \{f \in K[x] \mid f(S) \subseteq V\}.$$

Our focus will be on the ring $\operatorname{Int}(E, V)$, where E comprises a pseudo-convergent sequence as in Section 2. The major question we investigate is: when is $\operatorname{Int}(E, V)$ a Prüfer domain? We will completely answer this question (Theorem 5.2) and give necessary and sufficient conditions in terms of E for $\operatorname{Int}(E, V)$ to be Prüfer.

One of the many equivalent conditions for a commutative domain D to be Prüfer is that the localization of D at each maximal ideal is a valuation domain (see [**6,** Theorem 22.1]). We will use this characterization of Prüfer domains in our work with $\operatorname{Int}(E, V)$. Hence, we require a description of all the maximal ideals of $\operatorname{Int}(E, V)$. This is the goal of the present section. The complete classification of the maximal spectrum of $\operatorname{Int}(E, V)$ is given in Corollary 3.9.

The maximal ideals of $\operatorname{Int}(E, V)$ come in two types: unitary and non-unitary. An ideal $\mathfrak{I}$ of $\operatorname{Int}(E, V)$ is unitary if $\mathfrak{I} \cap V \neq (0)$ and is non-unitary if $\mathfrak{I} \cap V = (0)$. When $\mathfrak{M}$ is a maximal ideal of $\operatorname{Int}(E, V)$, $\mathfrak{M} \cap V$ is a prime ideal of V and, since we are assuming that V is one-dimensional, $\mathfrak{M}$ being unitary is equivalent to having $\mathfrak{M} \cap V = \mathfrak{m}$.

When $\mathfrak{M}$ is non-unitary, we can use established theory to prove that $\operatorname{Int}(E, V)_{\mathfrak{M}}$ is a valuation domain.

Theorem 3.1. *The nonzero non-unitary prime ideals of* $\operatorname{Int}(E, V)$ *are in one-to-one correspondence with the irreducible polynomials of* $K[x]$. *To each irreducible* $q \in K[x]$, *we associate the prime ideal*

$$\mathfrak{P}_q := q(x)K[x] \cap \operatorname{Int}(E, V),$$

and every non-unitary prime ideal of $\operatorname{Int}(E, V)$ *has this form. Moreover, the localization of* $\operatorname{Int}(E, V)$ *at a non-unitary maximal ideal is a valuation domain.*

Proof. The characterization of the non-unitary prime ideals follows from [**3,** Proposition V.1.1] and the comment preceding it. For the statement about the localization, note first that $\operatorname{Int}(E, V)$ contains $\operatorname{Int}(V)$. Let $\mathfrak{M}$ be a non-unitary maximal ideal of $\operatorname{Int}(E, V)$, and let $\mathfrak{P} = \mathfrak{M} \cap \operatorname{Int}(V)$; then, $\mathfrak{P}$ is a nonzero non-unitary prime of $\operatorname{Int}(V)$. By [**3,** Corollary V.1.2], $\operatorname{Int}(V)_{\mathfrak{P}}$ is a valuation domain, and it is easy to see that $\operatorname{Int}(V)_{\mathfrak{P}}$ is contained in $\operatorname{Int}(E, V)_{\mathfrak{M}}$. Hence, $\operatorname{Int}(E, V)_{\mathfrak{M}}$ must also be a valuation domain. □

In light of Theorem 3.1, we can concentrate on the unitary maximal ideals of $\operatorname{Int}(E, V)$. This will remain our focus for the remainder of the paper.

Definition 3.2. For each $i \in \mathbb{N}$, let

$$\mathfrak{M}_i = \{f \in \operatorname{Int}(E, V) \mid f(\alpha_i) \in \mathfrak{m}\}.$$

We also let

$$\mathfrak{M}_\infty = \{f \in \operatorname{Int}(E, V) \mid f(\alpha_i) \in \mathfrak{m} \text{ for all but finitely many } i \in \mathbb{N}\}.$$

It is easy to see that $\mathfrak{M}_i$ is an ideal for each i, and that each $\mathfrak{M}_i$ is distinct. Moreover, the $\mathfrak{M}_i$ are all maximal because $\operatorname{Int}(E, V)/\mathfrak{M}_i \cong V/\mathfrak{m}$ via the map $f \mapsto f(\alpha_i) \bmod \mathfrak{m}$. The set $\mathfrak{M}_\infty$ is easily seen to be a prime ideal, but it is non-trivial to verify that it is maximal. For now, we can at least say that $\mathfrak{M}_\infty$ is distinct from all the $\mathfrak{M}_i$.

Lemma 3.3. *For each $i \in \mathbb{N}$, let*

$$H_i(x) = \Bigg[\prod_{\substack{0\le \ell\le i+1\\ \ell\ne i}} (x-\alpha_\ell)\Bigg] \Bigg/ \Bigg[\prod_{\substack{0\le \ell\le i+1\\ \ell\ne i}} (\alpha_i-\alpha_\ell)\Bigg].$$

Then, $H_i \in \mathrm{Int}\,(E,V)$ and has the following properties:

(i) $v_j(H_i) = \infty$ *for* $0 \le j \le i-1$ *and* $j = i+1$,
(ii) $v_i(H_i) = 0$,
(iii) *there exists* $\rho > 0$ *such that* $v_j(H_i) = \rho$ *for all* $j > i+1$.

Consequently, each $H_i \in \mathfrak{M}_\infty \setminus \mathfrak{M}_i$, and so $\mathfrak{M}_\infty \not\subseteq \mathfrak{M}_i$.

Proof. Properties (i) and (ii) are clear, since $H_i(\alpha_j) = 0$ for the values of j specified in (i), and $H_i(\alpha_i) = 1$.

For (iii), let $\rho = v_{i+2}(H_i)$. Then $\rho > 0$ because, for each $0 \le \ell \le i-1$, Lemma 2.1 says that $v(\alpha_{i+2}-\alpha_\ell) = v(\alpha_i - \alpha_\ell)$, and $v(\alpha_{i+2}-\alpha_{i+1}) > v(\alpha_{i+1}-\alpha_i)$ because E is pseudo-convergent. Finally, when $j > i+1$, another appeal to Lemma 2.1 gives $v(\alpha_j - \alpha_\ell) = v(\alpha_{i+2}-\alpha_\ell)$ for $0 \le \ell \le i-1$ and $\ell = i+1$, so $v_j(H_i) = \rho$. The fact that $H_i \in \mathfrak{M}_\infty \setminus \mathfrak{M}_i$ now follows. □

In Theorem 3.8 below, we will prove that $\mathfrak{M}_\infty$ is maximal and that the $\mathfrak{M}_i$ and $\mathfrak{M}_\infty$ comprise the full set of unitary maximal ideals of $\mathrm{Int}\,(E,V)$. Proving Theorem 3.8 requires several lemmas. In what follows, we say that $f \in \mathrm{Int}\,(E,V)$ is *unit-valued* on E if $f(\alpha_j) \in V^\times$, for each $j \in \mathbb{N}$; equivalently, $v_j(f) = 0$ for each j.

Lemma 3.4. *Let $\mathfrak{I}$ be an ideal of $\mathrm{Int}\,(E,V)$ such that $\mathfrak{I} \not\subseteq \mathfrak{M}_\infty$ and $\mathfrak{I} \not\subseteq \mathfrak{M}_i$ for all $i \in \mathbb{N}$. Then, $\mathfrak{I}$ contains a polynomial that is unit-valued on E.*

Proof. Since $\mathfrak{I} \not\subseteq \mathfrak{M}_\infty$, there exists $f \in \mathfrak{I}$ such that $v_j(f) = 0$ for infinitely many $j \in \mathbb{N}$. By Lemma 2.1, $(v_j(f))_{j\in\mathbb{N}}$ either eventually increases or is eventually stable. Since infinitely many $v_j(f)$ are 0, $(v_j(f))_{j\in\mathbb{N}}$ must stabilize at 0. Thus, there exists $n \in \mathbb{N}$ such that $v_j(f) = 0$ for all $j \ge n$.

We would be finished if $v_j(f) = 0$ for $0 \le j \le n-1$, but this need not occur in general. However, we can use f to produce another polynomial that is definitely unit-valued on E.

First, for $0 \le i \le n-1$, define

$$G_i(x) = \Bigg[\prod_{\substack{0 \le \ell \le n \\ \ell \ne i}} (x - \alpha_\ell) \Bigg] \Bigg/ \Bigg[\prod_{\substack{0 \le \ell \le n \\ \ell \ne i}} (\alpha_i - \alpha_\ell) \Bigg].$$

Then, $v_j(G_i) = \infty$ for $0 \le j \le n$, $j \ne i$, and $v_i(G_i) = 0$. By Lemma 2.1 (ii), $v_j(G_i) = v_{n+1}(G_i)$ for all $j \ge n+1$, and $v_{n+1}(G_i) > 0$, because

$$v(\alpha_{n+1} - \alpha_\ell) = \begin{cases} v(\alpha_i - \alpha_\ell) & 0 \le \ell < i \\ v(\alpha_{\ell+1} - \alpha_\ell) & \ell > i \end{cases}$$

and $v(\alpha_{\ell+1} - \alpha_\ell) > v(\alpha_i - \alpha_\ell)$ when $\ell > i$. Hence, each $G_i \in \operatorname{Int}(E, V)$.

Next, for each $i \in \mathbb{N}$, let $f_i \in \mathfrak{I} \setminus \mathfrak{M}_i$. Let

$$S = \{0 \le s \le n-1 \mid v_s(f) > 0\},$$

and let

$$F = f + \sum_{s \in S} f_s G_s.$$

Then, $F \in \mathfrak{I}$, and we claim that F is unit-valued on E. Indeed, if $v_j(f) = 0$, then $j \notin S$, so for each $s \in S$ we have $v_j(f_s G_s) \ge v_j(G_s) > 0$; it follows that $v_j(F) = 0$. On the other hand, if $v_j(f) > 0$, then $j \in S$, so $v_j(f_j G_j) = 0$ while $v_j(f_s G_s) = \infty$ for $s \ne j$. Hence, $v_j(F) = 0$ in this case as well, and we conclude that F is unit-valued on E. □

Lemma 3.5. *Let $f \in \operatorname{Int}(E, V)$. Then, the set $f(E)$ mod $\mathfrak{m}$ is finite.*

Proof. Since $f \in \operatorname{Int}(E, V)$, we have $v(f(\alpha_i) - f(\alpha_j)) \ge 0$ for any choice of i and j. By Lemma 2.1 (iii), $(f(\alpha_i))_{i \in \mathbb{N}}$ is eventually pseudo-convergent. Hence, after a certain point, $v(f(\alpha_i) - f(\alpha_j)) > v(f(\alpha_j) - f(\alpha_k)) \ge 0$ whenever $i > j > k$. In other words, eventually $f(\alpha_i) - f(\alpha_j) \in \mathfrak{m}$ whenever $i > j$. So, the values of f on E, reduced modulo $\mathfrak{m}$, eventually stabilize. Consequently, $f(E)$ mod $\mathfrak{m}$ is finite. □

Lemma 3.6. *Assume* $f \in \mathrm{Int}\,(E, V)$ *is such that* $v_j(f) > 0$ *for all* $j \in \mathbb{N}$. *Then,* f *is in every ideal of* $\mathrm{Int}\,(E, V)$ *above* $\mathfrak{m}$ *and, since* V *is one-dimensional,* f *is in every unitary prime ideal of* $\mathrm{Int}\,(E, V)$.

Proof. Since each $v_j(f) > 0$ and the sequence $(v_j(f))_{j\in\mathbb{N}}$ is either eventually increasing or eventually stable, $(v_j(f))_{j\in\mathbb{N}}$ attains a minimum value. Let $\beta \in V$ be such that $v_j(f) \geq v(\beta) > 0$ for all $j \in \mathbb{N}$. Then, $f(x)/\beta \in \mathrm{Int}\,(E, V)$ and, since $\beta \in \mathfrak{m}$,

$$f(x) = (f(x)/\beta)\beta$$

is in each ideal of $\mathrm{Int}\,(E, V)$ containing $\mathfrak{m}$. $\square$

Proposition 3.7. *Let* $\mathfrak{P}$ *be a unitary prime ideal of* $\mathrm{Int}\,(E, V)$. *Then, either* $\mathfrak{P} \subseteq \mathfrak{M}_\infty$ *or* $\mathfrak{P} \subseteq \mathfrak{M}_i$ *for some* $i \in \mathbb{N}$.

Proof. Suppose that $\mathfrak{P} \not\subseteq \mathfrak{M}_\infty$ and $\mathfrak{P} \not\subseteq \mathfrak{M}_i$ for all $i \in \mathbb{N}$. Then, by Lemma 3.4, $\mathfrak{P}$ contains a polynomial F that is unit-valued on E. By Lemma 3.5, we can find finitely many units $u_1, u_2, \ldots, u_t \in V^\times$ to represent all the residues in $F(E)$ mod $\mathfrak{m}$.

Let

$$f = (F - u_1)(F - u_2)\cdots(F - u_t).$$

Then, $f(\alpha_j) \in \mathfrak{m}$ for all $j \in \mathbb{N}$. Since $\mathfrak{P} \cap V = \mathfrak{m}$, $f \in \mathfrak{P}$ by Lemma 3.6. But $\mathfrak{P}$ is prime, so $F - u_\ell \in \mathfrak{P}$ for some $1 \leq \ell \leq t$, implying that $u_\ell \in \mathfrak{P}$. Consequently, $\mathfrak{P} = \mathrm{Int}\,(E, V)$, which is a contradiction. $\square$

Theorem 3.8. $\mathfrak{M}_\infty$ *is a maximal ideal of* $\mathrm{Int}\,(E, V)$, *and the unitary maximal ideals of* $\mathrm{Int}\,(E, V)$ *are exactly* $\mathfrak{M}_\infty$ *and* $\mathfrak{M}_i$, *for* $i \in \mathbb{N}$.

Proof. Let $\mathfrak{M}$ be a maximal ideal of $\mathrm{Int}\,(E, V)$ containing $\mathfrak{M}_\infty$. Then, $\mathfrak{M}$ is unitary and, by Lemma 3.3, $\mathfrak{M} \neq \mathfrak{M}_i$ for all $i \in \mathbb{N}$. By Proposition 3.7, we must have $\mathfrak{M} = \mathfrak{M}_\infty$, so $\mathfrak{M}_\infty$ is maximal. As Proposition 3.7 precludes the existence of unitary maximal ideals other than $\mathfrak{M}_\infty$ and the $\mathfrak{M}_i$, the theorem is proved. $\square$

We now have a complete description of the maximal spectrum of $\mathrm{Int}\,(E, V)$.

Corollary 3.9.

(i) *The non-unitary maximal ideals of* $\mathrm{Int}\,(E,V)$ *all have the form* $q(x)K[x] \cap \mathrm{Int}\,(E,V)$ *for some monic irreducible* $q \in K[x]$.
(ii) *The unitary maximal ideals of* $\mathrm{Int}\,(E,V)$ *are precisely* $\mathfrak{M}_\infty$ *and* $\mathfrak{M}_i$, *for* $i \in \mathbb{N}$.

Having classified all the maximal ideals of $\mathrm{Int}\,(E,V)$, we next determine when the localization of $\mathrm{Int}\,(E,V)$ at a maximal ideal is a valuation domain. By Theorem 3.1, $\mathrm{Int}\,(E,V)_{\mathfrak{M}}$ is a valuation domain for any non-unitary maximal ideal $\mathfrak{M}$. In Sections 4 and 5, we will consider localizations at $\mathfrak{M}_i$ and $\mathfrak{M}_\infty$. As we shall see (Corollary 4.4), $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$ is always a valuation domain. Thus, the determining factor in whether $\mathrm{Int}\,(E,V)$ is Prüfer comes from the maximal ideal $\mathfrak{M}_\infty$.

4. Localizations at $\mathfrak{M}_i$. Our goal in this section is to prove that $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$ is a valuation domain for each $i \in \mathbb{N}$. In fact, we will prove that $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$ equals the valuation domain given in the following definition.

Definition 4.1. For each $i \in \mathbb{N}$, define

$$V_i = \{\phi \in K(x) \mid \phi(\alpha_i) \in V\} = \{\phi \mid v_i(\phi) \geq 0\}.$$

Lemma 4.2. *For each* $i \in \mathbb{N}$, V_i *is a valuation domain, and* $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i} \subseteq V_i$.

Proof. The set V_i is clearly a subring of $K(x)$ and, for each $\phi \in K(x)$, either $v_i(\phi) \geq 0$ or $v_i(1/\phi) \geq 0$. Thus, for each $\phi \in K(x)$, either $\phi \in V_i$ or $\phi^{-1} \in V_i$. By [**6,** Theorem 16.3], V_i is a valuation domain. Also, $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i} \subseteq V_i$ because, if $f \in \mathrm{Int}\,(E,V)$ and $g \notin \mathfrak{M}_i$, then

$$v_i(f/g) = v_i(f) - v_i(g) = v_i(f) - 0 \geq 0. \qquad \square$$

To show that $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$ equals V_i, it will suffice to demonstrate that

$$V_i \subseteq \mathrm{Int}\,(E,V)_{\mathfrak{M}_i}.$$

We prove this in the next theorem by utilizing the polynomials H_i defined in Lemma 3.3.

Theorem 4.3. *Let $i \in \mathbb{N}$. Then, $V_i \subseteq \mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$.*

Proof. Let $\phi \in V_i$, and write $\phi = f/g$, where $f, g \in V[x]$ with no common factors. Then, $f, g \in \mathrm{Int}\,(E,V)$. If $v_i(g) = 0$, then $g \notin \mathfrak{M}_i$, and hence, $\phi \in \mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$. So, assume that $v_i(g) > 0$.

Let $\beta = g(\alpha_i)$. Then, $\beta \neq 0$ because $\phi \in V_i$. Let H_i be as in Lemma 3.3. By construction, there exists $\rho > 0$ such that

$$v_j(H_i) \geq \rho \quad \text{for } j \neq i.$$

Since the value group of K has rank one, we can choose $n \in \mathbb{N}$ such that $v_j(H_i^n) > v(\beta)$ for all $j \neq i$. Decompose ϕ as follows:

$$\phi = ((fH_i^n)/\beta) \Big/ ((gH_i^n)/\beta).$$

To show that $\phi \in \mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$, it suffices to show that $fH_i^n/\beta \in \mathrm{Int}\,(E,V)$ and $gH_i^n/\beta \in \mathrm{Int}\,(E,V) \setminus \mathfrak{M}_i$.

When $j \neq i$,

$$v_j(fH_i^n/\beta) = v_j(f) + v_j(H_i^n) - v(\beta),$$

and this is non-negative because $f \in V[x]$ and

$$v_j(H_i^n) > v(\beta).$$

Furthermore,

$$v_i(fH_i^n/\beta) = v_i(\phi) \geq 0,$$

because $v_i(H_i) = 0$ and $\phi \in V_i$. So, fH_i^n/β is a polynomial in $K[x]$ and $v_j(fH_i^n/\beta) \geq 0$ for all $j \in \mathbb{N}$. Hence,

$$fH_i^n/\beta \in \mathrm{Int}\,(E,V).$$

Applying a similar argument to gH_i^n/β shows that $v_j(gH_i^n/\beta) \geq 0$ for $j \neq i$ and $v_i(gH_i^n/\beta) = 0$. Thus,

$$gH_i^n/\beta \in \mathrm{Int}\,(E,V) \setminus \mathfrak{M}_i. \qquad \square$$

Corollary 4.4. *For each $i \in \mathbb{N}$, $\mathrm{Int}\,(E,V)_{\mathfrak{M}_i}$ is a valuation domain.*

Given Theorem 3.1 and Corollary 4.4, we see that $\mathrm{Int}\,(E,V)$ being Prüfer depends entirely on the localization at $\mathfrak{M}_\infty$.

Corollary 4.5. $\operatorname{Int}(E,V)$ *is a Prüfer domain if and only if the localization* $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ *is a valuation domain.*

The examination of $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ is the topic of the next section.

5. Localization at $\mathfrak{M}_\infty$. In contrast to the situation with $\mathfrak{M}_i$, $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ is not always a valuation domain; it depends on E. We borrow the following definitions from Kaplansky [**7**].

Definition 5.1. The pseudo-convergent sequence $E = (\alpha_i)_{i\in\mathbb{N}}$ is said to be of *transcendental type* if $(v_j(f))_{j\in\mathbb{N}}$ eventually stabilizes for every $f \in K[x]$. If $(v_j(f))_{j\in\mathbb{N}}$ is eventually strictly increasing for at least one $f \in K[x]$, then we say that E is of *algebraic type*.

The *breadth* of E, denoted by $\operatorname{Br}(E)$, is defined to be

$$\operatorname{Br}(E) = \{b \in V \mid v(b) > v(\alpha_{i+1} - \alpha_i) \text{ for all } i \in \mathbb{N}\}.$$

The breadth of E always forms an ideal of V. Given a pseudo-limit α of E in K, all other pseudo-limits of E in K have the form $\alpha + b$ for some $b \in \operatorname{Br}(E)$ [**7,** Lemma 3]. In particular, if a pseudo-limit of E exists and $\operatorname{Br}(E) = (0)$, then the pseudo-limit is unique.

We can use the breadth and the type of E to classify exactly when $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ is a valuation domain.

Theorem 5.2. *Let E be a pseudo-convergent sequence in V. Then,* $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ *is a valuation domain if and only if E is of transcendental type or* $\operatorname{Br}(E) = (0)$*. Consequently,* $\operatorname{Int}(E,V)$ *is a Prüfer domain if and only if E is of transcendental type or* $\operatorname{Br}(E) = (0)$*.*

The proof of Theorem 5.2 is more complicated than our work in earlier sections and relies on some theorems from [**4**]. We will prove the theorem via a number of intermediary results. We begin with the following lemma about the values of rational functions in $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$.

Lemma 5.3.

(i) *If $g \in \operatorname{Int}(E,V) \setminus \mathfrak{M}_\infty$, then $v_j(g) = 0$ for all sufficiently large j.*

(ii) *If* $f/g \in \mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty}$*, then* $v_j(f/g) \geq 0$ *for all sufficiently large* j*.*

Proof.

(i) Since

$$g \in \mathrm{Int}\,(E,V), \quad v_j(g) \geq 0 \text{ for all } j \in \mathbb{N}.$$

But, since $g \notin \mathfrak{M}_\infty$, $v_j(g) = 0$ for infinitely many j. So, $(v_j(g))_{j\in\mathbb{N}}$ eventually stabilizes at 0.

(ii) With $f \in \mathrm{Int}\,(E,V)$ and $g \in \mathrm{Int}\,(E,V) \setminus \mathfrak{M}_\infty$, we have $v_j(f) \geq 0$ for all j, and $v_j(g) = 0$ for sufficiently large j. Hence, $v_j(f/g)$ is eventually non-negative. □

Next, we will show that, if E is of transcendental type, then $\mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty}$ is a valuation domain. Our approach is similar to our work in Section 4.

Definition 5.4. We define

$$V_\infty = \{\phi \in K(x) \mid \phi(\alpha_i) \in V \text{ for all but finitely many } i \in \mathbb{N}\}.$$

It is straightforward to prove that V_∞ is a subring of $K(x)$. Whether it is a valuation domain depends on E.

Proposition 5.5. *Assume that* E *is of transcendental type. Then,* V_∞ *is a valuation domain.*

Proof. Let $\phi \in K(x)$. Since E is of transcendental type, both the numerator and denominator of ϕ are eventually stable; hence, $(v_j(\phi))_{j\in\mathbb{N}}$ also eventually stabilizes, say at ε. If $\varepsilon \geq 0$, then $\phi \in V_\infty$ and, if $\varepsilon < 0$, then $1/\phi \in V_\infty$. □

Theorem 5.6. *Assume that* E *is of transcendental type. Then,*

$$\mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty} = V_\infty.$$

Proof. The containment $\mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty} \subseteq V_\infty$ follows from Lemma 5.3 (ii). So, it suffices to prove that $V_\infty \subseteq \mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty}$. Let $f/g \in V_\infty$, where $f, g \in V[x]$, and find $n \in \mathbb{N}$ such that $v_j(f)$ and $v_j(g)$ are stable for all $j \geq n$. Then, $v_j(f/g)$ is also stable for $j \geq n$. Since $f/g \in V_\infty$, we must have $v_j(f/g) \geq 0$ for such j, and hence $v_j(f) - v_j(g) \geq 0$.

Let

$$H(x) = \prod_{0 \leq \ell \leq n} (x - \alpha_\ell),$$

and let

$$\beta = g(\alpha_{n+1})H(\alpha_{n+1});$$

note that $v(\beta) = v_{n+1}(g) + v_{n+1}(H)$. Decompose f/g as

$$f/g = (fH/\beta) \Big/ (gH/\beta).$$

We claim that

$$fH/\beta \in \mathrm{Int}\,(E, V)$$

and

$$gH/\beta \in \mathrm{Int}\,(E, V) \setminus \mathfrak{M}_\infty.$$

By Lemma 2.1 (ii),

$$v_j(H) = v_{n+1}(H) \text{ for all } j \geq n+1.$$

From this, we get that

$$v_j(gH/\beta) = 0 \quad \text{ for all } j \geq n+1,$$

and clearly, $v_j(gH/\beta) = \infty$ when $0 \leq j \leq n$. So,

$$gH/\beta \in \mathrm{Int}\,(E, V) \setminus \mathfrak{M}_\infty.$$

Finally, for fH/β, we have

$$v_j(fH/\beta) = \infty$$

when $0 \leq j \leq n$ and

$$v_j(fH/\beta) = v_j(f) - v_j(g) \geq 0$$

when $j \geq n$. So, $fH/\beta \in \mathrm{Int}\,(E, V)$, completing the proof. □

Corollary 5.7. *If E is of transcendental type, then* $\mathrm{Int}\,(E, V)$ *is a Prüfer domain.*

This handles the situation when E is of transcendental type. When E is of algebraic type, we rely on the next theorem. For an (eventually) pseudo-convergent sequence $(a_i)_{i\in\mathbb{N}}$, we use the notation $(v(a_i))_{i\in\mathbb{N}} \to \infty$ to mean that the pseudo-convergent sequence $(a_i)_{i\in\mathbb{N}}$ is eventually increasing and the values $v(a_i)$ are unbounded.

Theorem 5.8. *Consider the following four conditions.*

(i) *There exists* $q \in K[x]$ *such that* $(v_j(q))_{j\in\mathbb{N}} \to \infty$.
(ii) $\mathrm{Br}(E) = (0)$.
(iii) $\mathrm{Int}(E, V)_{\mathfrak{M}_\infty}$ *is a valuation domain.*
(iv) *If* $q \in K[x]$ *and* $(v_j(q))_{j\in\mathbb{N}}$ *is eventually increasing, then* $(v_j(q))_{j\in\mathbb{N}} \to \infty$.

For any pseudo-convergent E*, we have* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *When* E *is of algebraic type,* (iv) $\Rightarrow$ (i)*, and hence all four conditions are equivalent.*

When E is of algebraic type, it is clear that (iv) $\Rightarrow$ (i). We prove the other implications without any assumption on the type of E.

(i) $\Rightarrow$ (ii). Let $q \in K[x]$ be such that $(v_j(q))_{j\in\mathbb{N}} \to \infty$. Following [**7,** Theorem 3], we may assume q is irreducible. Indeed, if $q = q_1q_2$ for some $q_1, q_2 \in K[x]$, then either $(v_j(q_1)) \to \infty$ or $(v_j(q_2)) \to \infty$. So, without loss of generality, assume that q is irreducible.

Let L be the splitting field of q over K, and let w be an extension of v to L. Factor q as

$$q(x) = (x - \beta_1)(x - \beta_2)\cdots(x - \beta_t)$$

for some (not necessarily distinct) $\beta_\ell \in L$. Then, for at least one ℓ, $(w(\alpha_j - \beta_\ell)) \to \infty$ as $j \to \infty$. Thus, for sufficiently large j,

$$\begin{aligned} v(\alpha_{j+1} - \alpha_j) &= w(\alpha_{j+1} - \alpha_j) \\ &= w((\alpha_{j+1} - \beta_\ell) + (\beta_\ell - \alpha_j)) \\ &= w(\beta_\ell - \alpha_j), \end{aligned}$$

so $(v(\alpha_{j+1} - \alpha_j)) \to \infty$. Hence, $\mathrm{Br}(E) = (0)$. □

Before proving (ii) $\Rightarrow$ (iii), we recall a topological definition. A topological space X is *precompact* when its completion is compact [**1,** Section 4, Definition 2]. Now, [**4,** Theorem 4.1] asserts that $\operatorname{Int}(E,V)$ is a Prüfer domain when E is precompact with respect to the topology on K induced by v. So, we will prove that, if $\operatorname{Br}(E)=(0)$, then E is precompact.

(ii) $\Rightarrow$ (iii). Assume that $\operatorname{Br}(E)=(0)$, which implies that $(v(\alpha_{i+1}-\alpha_i))\to\infty$. Also, from [**1,** Section 3, Proposition 1], it follows that K is metrizable. Let $\widehat{K}$ be the completion of K with respect to v, and let $\widehat{E}$ be the corresponding completion of E (that is, the topological closure of E in $\widehat{K}$).

The condition $(v(\alpha_{i+1}-\alpha_i))\to\infty$ implies that $(\alpha_i)_{i\in\mathbb{N}}$ is a Cauchy sequence in K, so E comprises a convergent sequence in $\widehat{K}$. Hence, every sequence in $\widehat{E}$ has a subsequence converging to a limit point in $\widehat{E}$. Thus, $\widehat{E}$ is compact, $\operatorname{Int}(E,V)$ is a Prüfer domain by [**4,** Theorem 4.1], and so $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$ is a valuation domain. $\square$

(iii) $\Rightarrow$ (iv). Here, we prove the contrapositive. Assume $q\in K[x]$ is such that $(v_j(q))_{j\in\mathbb{N}}$ is eventually increasing, but is bounded above. Let $\beta\in V$ be such that $v_j(q)<v(\beta)$ for all $j\in\mathbb{N}$. Let $\phi=\beta/q$. We claim that neither ϕ nor $1/\phi$ is an element of $\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$.

By construction, $v_j(1/\phi)<0$ for all j. This violates the conclusion of Lemma 5.3 (ii), so

$$\frac{1}{\phi}\notin\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}.$$

Suppose now that $\phi\in\operatorname{Int}(E,V)_{\mathfrak{M}_\infty}$, and write $\phi=f/g$, where $f\in\operatorname{Int}(E,V)$ and

$$g\in\operatorname{Int}(E,V)\setminus\mathfrak{M}_\infty.$$

By Lemma 5.3, $(v_j(g))_{j\in\mathbb{N}}$ eventually stabilizes at 0. Hence, for sufficiently large j, we have

$$v_j(f)=v_j(\phi)=v(\beta)-v_j(q).$$

But, $(v_j(q))_{j\in\mathbb{N}}$ is increasing, so $(v_j(f))_{j\in\mathbb{N}}$ is decreasing. This contradicts Lemma 2.1 (iii). Thus,

$$\phi \notin \mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty},$$

and so $\mathrm{Int}\,(E,V)_{\mathfrak{M}_\infty}$ is not a valuation domain. □

This completes the proof of Theorem 5.8. The equivalence of (ii) and (iii) when E is algebraic gives us:

Corollary 5.9. *Assume that E is of algebraic type. Then,* $\mathrm{Int}\,(E,V)$ *is Prüfer if and only if* $\mathrm{Br}\,(E)=(0)$.

At this point, we have a complete proof of Theorem 5.2. To summarize the argument: if E is of transcendental type, then $\mathrm{Int}\,(E,V)$ is Prüfer by Theorem 5.6 and Corollary 5.7. If $\mathrm{Br}\,(E)=(0)$, then $\mathrm{Int}\,(E,V)$ is Prüfer by Theorem 5.8. Finally, if E is not of transcendental type and $\mathrm{Br}\,(E)\neq(0)$, then E must be of algebraic type, and we see that $\mathrm{Int}\,(E,V)$ is not Prüfer by Corollary 5.9.

It remains to demonstrate that the two conditions in Theorem 5.2, E being of transcendental type and $\mathrm{Br}\,(E)=(0)$, are, in general, independent of one another. We give two examples to illustrate this.

Example 5.10. Let $\mathbb{Q}^+$ denote the positive rational numbers. Let y be an indeterminate, let $R=\mathbb{Q}[\{y^e\}_{e\in\mathbb{Q}^+}]$, let $\mathfrak{m}$ be the maximal ideal of R generated by $\{y^e\}_{e\in\mathbb{Q}^+}$, and let $V=R_{\mathfrak{m}}$. The fraction field K of V is then a valued field with valuation group isomorphic to the additive group of the rational numbers, and V is a non discrete one-dimensional valuation domain.

Let $E=\{y,y^2,y^3,\ldots\}$. Then, E is pseudo-convergent, the breadth of E is (0) and E is of algebraic type because $(v_j(x))\to\infty$ as $j\to\infty$. Note that an alternate example, where the values of E stabilize instead of increasing, is given by

$$E=\{y+y^2,y+y^3,y+y^4,\ldots\}.$$

For this latter choice of E, the breadth is still (0), and we have $(v_j(x-y))\to\infty$.

Example 5.11. Let V and K be as in Example 5.10. We demonstrate the existence of pseudo-convergent sequences of transcendental type and nonzero breadth. (The existence of these sequences was not in doubt prior to this paper, but neither an example nor a proof of existence was found in the available literature.)

It follows from Kaplansky's work in **[7]** that a pseudo-convergent sequence with a transcendental pseudo-limit is of transcendental type (hence the terminology). So, it suffices to show that there exists a pseudo-convergent sequence in V with a transcendental pseudo-limit and nonzero breadth.

Consider a real number $d = 0.d_1d_2d_3\ldots$, where each d_ℓ is either 1 or 2. Given such a real number, for each $\ell > 0$ let $e_\ell = 0.d_1d_2\ldots d_\ell$. For each $i > 0$, let

$$\alpha_i = \sum_{\ell=1}^{i} y^{e_\ell}.$$

Take $E_d = \{\alpha_1, \alpha_2, \ldots\}$. Then,

$$v(\alpha_{i+1} - \alpha_i) = e_{i+1},$$

and the sequence $(e_i)_{i>0}$ is increasing, so E_d is pseudo-convergent. Also, $(e_i)_{i>0}$ is bounded above, so $\mathrm{Br}\,(E_d) \neq (0)$. A pseudo-limit of E_d in $\widehat{K}$ is given by

$$L_d := \sum_{\ell=1}^{\infty} y^{e_\ell}.$$

Now, for real numbers d and d' of the above form, $L_{d'}$ is a pseudo-limit of E_d if and only if $d = d'$. Indeed, if $d \neq d'$, then $(v(L_{d'} - \alpha_i))_{i>0}$ will stabilize as soon as the decimal expansions of d and d' are different. Since there are uncountably many such d, there are uncountably many L_d. However, K and its algebraic closure are both countable. Hence, there exists a d such that L_d is transcendental over K, and the corresponding pseudo-convergent sequence E_d provides the needed example.

We close with an example, mentioned in the introduction, of an infinite subset E of V that is not precompact, but for which $\mathrm{Int}\,(E, V)$ is Prüfer.

Example 5.12. Let V and K be as in the previous two examples. Then, K is metrizable, and we let $\widehat{K}$ be the completion of K with respect to v.

Let the E_d be as in Example 5.11. Choose d such that E_d is of transcendental type; then, $\text{Int}\,(E_d, V)$ is Prüfer. Note that, for $i > j$, we have

$$v(\alpha_i - \alpha_j) = e_{j+1},$$

and the sequence $(e_j)_{j>0}$ is bounded above. Because of this, the only Cauchy sequences in E_d are those which are eventually constant; hence, $\widehat{E_d} = E_d$. Moreover, the sequence $(\alpha_1, \alpha_2, \ldots)$ has no convergent subsequence, so $\widehat{E_d}$ is not compact, and thus E_d is not precompact.

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