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POLYNOMIAL OVERRINGS OF Int($\mathbb{Z}$)

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ABSTRACT. We show that every polynomial overring of the ring Int($\mathbb{Z}$) of polynomials which are integer-valued over $\mathbb{Z}$ may be considered as the ring of polynomials which are integer-valued over some subset of $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$ with respect to the fundamental system of neighbourhoods of 0 consisting of all non-zero ideals of $\mathbb{Z}$.

Introduction. The classical ring of integer-valued polynomials, namely,

$$\mathrm{Int}(\mathbb{Z}) = \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\},$$

is known to be a two-dimensional Prüfer domain (see for instance **[1,** subsection VI.1]). Thus, all the overrings of Int($\mathbb{Z}$), that is, rings between Int($\mathbb{Z}$) and its quotient field $\mathbb{Q}(X)$, are well known *a priori*: they are intersections of localizations of Int($\mathbb{Z}$) at its prime ideals, which are themselves well-known valuation domains. However, the spectrum of Int($\mathbb{Z}$) turns out to be uncountable, so that these intersections of localizations are not so easy to characterize. The aim of this paper is to classify the 'polynomial overrings' of Int($\mathbb{Z}$), that is, rings lying between Int($\mathbb{Z}$) and $\mathbb{Q}[X]$. We first describe them as particular intersections of some families of valuation domains. Furthermore, we will see that the polynomial overrings of Int($\mathbb{Z}$) may be characterized as rings of polynomials which are integer-valued over some subset of $\mathbb{Z}$ or, more generally, of $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$ with respect to the fundamental system of neighborhoods of 0 consisting of all non-zero ideals of $\mathbb{Z}$.

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1. Prime spectrum of $\mathrm{Int}(\mathbb{Z})$ **and localizations.** We first recall the structure of the spectrum of $\mathrm{Int}(\mathbb{Z})$ [**1,** Proposition V.2.7]. A nonzero prime ideal $\mathfrak{P}$ of $\mathrm{Int}(\mathbb{Z})$ lies over a prime ideal of $\mathbb{Z}$, and hence, there are two cases:

- $\mathfrak{P} \cap \mathbb{Z} = (0)$. Then $\mathfrak{P}$ is of the form

$$\mathfrak{P} = \mathfrak{P}_q = q(X)\mathbb{Q}[X] \cap \mathrm{Int}(\mathbb{Z}), \text{ where } q \in \mathbb{Z}[X] \text{ is irreducible.}$$

These ideals $\mathfrak{P}_q$ have height 1 and the polynomial q is uniquely determined.

- $\mathfrak{P} \cap \mathbb{Z} = p\mathbb{Z}$, where $p \in \mathbb{P}$ (we denote by $\mathbb{P}$ the set of prime numbers). Then $\mathfrak{P}$ is of the form

$$\mathfrak{P} = \mathfrak{M}_{p,\alpha} = \{f \in \mathrm{Int}(\mathbb{Z}) \mid f(\alpha) \in p\mathbb{Z}_p\}, \text{ where } \alpha \in \mathbb{Z}_p.$$

These ideals $\mathfrak{M}_{p,\alpha}$ are maximal ideals and the residue field of $\mathfrak{M}_{p,\alpha}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. More precisely,

$$\mathbb{Z}_p \ni \alpha \longmapsto \mathfrak{M}_{p,\alpha} \in \mathrm{Max}\,(\mathrm{Int}(\mathbb{Z}))$$

is a one-to-one correspondence between $\mathbb{Z}_p$ and the set of prime ideals of $\mathrm{Int}(\mathbb{Z})$ lying over p. (Recall that $\mathbb{Z}_p$, the ring of p-adic integers, is uncountable.)

Moreover, given q irreducible in $\mathbb{Z}[X]$, $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, the following holds [**1,** Proposition V.2.5]:

$$\mathfrak{P}_q \subset \mathfrak{M}_{p,\alpha} \Longleftrightarrow q(\alpha) = 0. \tag{1.1}$$

Consequently, given an irreducible polynomial $q \in \mathbb{Z}[X]$, for a fixed prime p, there are at most finitely many ideals $\mathfrak{M}_{p,\alpha}$ containing $\mathfrak{P}_q$; on the other hand, it is known that there exist infinitely many primes p such that $q(X)$ has a root α in $\mathbb{Z}_p$, that is, $\mathfrak{P}_q$ is contained in infinitely many $\mathfrak{M}_{p,\alpha}$'s [**1,** Proposition V.2.8]. In particular, the prime ideals $\mathfrak{P}_q$ are not maximal. From equivalence (1.1), it follows also that the height of $\mathfrak{M}_{p,\alpha}$ is one if and only if α is transcendental over $\mathbb{Q}$, it is two otherwise.

We now describe the localizations of $\mathrm{Int}(\mathbb{Z})$ with respect to these prime ideals (see for example, [**1,** Proposition VI.1.9]). They are the following valuation domains of the field $\mathbb{Q}(X)$:

- $\mathrm{Int}(\mathbb{Z})_{\mathfrak{P}_q} = \mathbb{Q}[X]_{(q)} = \{f(X)/g(X) \in \mathbb{Q}(X) \mid q \nmid g\}$.

- $\mathrm{Int}(\mathbb{Z})_{\mathfrak{M}_{p,\alpha}} = V_{p,\alpha} = \{\varphi \in \mathbb{Q}(X) \mid \varphi(\alpha) \in \mathbb{Z}_p\}$.

Consequently, Int($\mathbb{Z}$) is a Prüfer domain. Moreover,

$$V_{p,\alpha} \subset \mathbb{Q}[X]_{(q)} \Longleftrightarrow \mathfrak{P}_q \subset \mathfrak{M}_{p,\alpha} \Longleftrightarrow q(\alpha) = 0. \tag{1.2}$$

We are interested in the representation of Int($\mathbb{Z}$) as an intersection of valuation overrings. For this purpose, we have to make some choices. First, we may represent Int($\mathbb{Z}$) as the intersection of all of its valuation overrings:

$$\mathrm{Int}(\mathbb{Z}) = \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}(\mathbb{Z})} \mathbb{Q}[X]_{(q)} \cap \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in \mathbb{Z}_p} V_{p,\alpha},$$

where $\mathcal{P}_{\mathrm{irr}}(\mathbb{Z})$ denotes the set of irreducible polynomials of $\mathbb{Z}[X]$. We may look for a more optimal representation of Int($\mathbb{Z}$). To begin with, we may discard from the above representation the valuation domains which are not minimal valuation overrings of Int($\mathbb{Z}$), or, equivalently, the valuation domains which does not correspond to maximal ideals of Int($\mathbb{Z}$) because Int($\mathbb{Z}$) is a Prüfer domain:

$$\mathrm{Int}(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in \mathbb{Z}_p} V_{p,\alpha}. \tag{1.3}$$

The above intersection in (1.3) is uncountable and it is far from being irredundant. Recall that, given a domain D with quotient field K, and a family of valuation overrings $\Lambda = \{V_\lambda\}$ of D (that is, $D \subseteq V_\lambda \subset K$) such that $D = \bigcap_\lambda V_\lambda$, the representation $D = \bigcap_\lambda V_\lambda$ is said *irredundant* if no V_λ is superfluous, that is, for each λ, D is strictly contained in the intersection of the member of Λ distinct from V_λ (**[6]**). For the domain Int($\mathbb{Z}$), there are suitable countable intersections as shown, for instance, by the following equality:

$$\mathrm{Int}(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in \mathbb{Z}} V_{p,\alpha}. \tag{1.4}$$

The fact that every rational function on the right-hand side of equality (1.4), that is, that every $\varphi \in \mathbb{Q}(X)$ such that $\varphi(\mathbb{Z}) \subseteq \mathbb{Z}$ is a polynomial, follows from the observation that a rational function which takes integral values on infinitely many integers is a polynomial (see [**12,** VIII.2 (93)] or [**1,** Proposition X.1.1]).

So, every $V_{p,\alpha}$, $\alpha \in \mathbb{Z}_p \setminus \mathbb{Z}$, $p \in \mathbb{P}$, in the representation (1.3) is superfluous; actually, we will show that, for each $p \in \mathbb{P}$ and $\alpha \in \mathbb{Z}_p$, every $V_{p,\alpha}$ in the above representation is superfluous (Corollary 4.4). However, there is no irredundant representation of $\mathrm{Int}(\mathbb{Z})$ as an intersection of valuation overrings because there is no subset of $\mathbb{Z}$ which is minimal among the subsets of $\mathbb{Z}$ which are dense in $\mathbb{Z}$ for every p-adic topology (see Corollary 3.5 and Remark 5.2). Thus, in the sequel, the only representations that we will consider as 'canonical' will be the intersections of all the minimal valuation overrings as in (1.3).

After some generalities about the overrings of $\mathrm{Int}(\mathbb{Z})$ in Section 2, we consider the representations of the overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$, where p is a fixed prime number and $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ at $p\mathbb{Z}$ in Section 3, as intersections of valuation domains (Proposition 3.3) and then, as rings of integer-valued polynomials on a subset of $\mathbb{Z}_p$ (Theorem 3.11); in particular, we show that there is a one-to-one correspondence between the set of polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$ and the closed subsets of $\mathbb{Z}_p$. In order to globalize these results, we study in Section 4 the valuation overrings of an intersection of valuation domains, characterizing those which are superfluous (Corollary 4.4 and Theorem 4.6). Finally, the polynomial overrings of $\mathrm{Int}(\mathbb{Z})$ are described in Section 5 by their representations as intersection of valuation overrings (Proposition 5.1), and in Section 6 with an interpretation as integer-valued polynomials on a subset of the ring $\widehat{\mathbb{Z}}$ (Theorem 6.2).

2. Generalities about overrings of $\mathrm{Int}(\mathbb{Z})$.

We are interested in rings R which are overrings of $\mathrm{Int}(\mathbb{Z})$, that is,

$$\mathrm{Int}(\mathbb{Z}) \subseteq R \subseteq \mathbb{Q}(X), \tag{2.1}$$

and, in particular, by the *polynomial overrings* of $\mathrm{Int}(\mathbb{Z})$, that is, the rings R which are contained in $\mathbb{Q}[X]$.

Since $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain, we first recall the following fundamental result of [**4**] (see also [**3,** Theorem 26.1]) concerning overrings D' of a Prüfer domain D, that is, rings D' such that $D \subseteq D' \subseteq K$ where K denotes the quotient field of D.

Proposition 2.1. *Let D' be an overring of a Prüfer domain D, and let $\mathcal{S}_{D'}$ be the set of prime ideals $\mathfrak{p}$ of D such that $\mathfrak{p}D' \subsetneq D'$. Then*

(i) *If $\mathfrak{p}'$ is a prime ideal of D' and $\mathfrak{p} = \mathfrak{p}' \cap D$, then $D_{\mathfrak{p}} = D'_{\mathfrak{p}'}$ and $\mathfrak{p}' = \mathfrak{p}D_{\mathfrak{p}} \cap D'$. Therefore D' is Prüfer.*

(ii) *If* $\mathfrak{p}$ *is a non-zero prime ideal of* D*, then* $\mathfrak{p}$ *is in* $\mathcal{S}_{D'}$ *if and only if* $D_{\mathfrak{p}} \supseteq D'$*. Moreover,* $D' = \bigcap_{\mathfrak{p}\in\mathcal{S}_{D'}} D_{\mathfrak{p}}$*.*
(iii) *Every ideal* $\mathfrak{I}'$ *of* D' *is an extended ideal, that is,* $\mathfrak{I}' = (\mathfrak{I}' \cap D)\,D'$*.*
(iv) *The spectrum of* D' *is* $\{\mathfrak{p}D' \mid \mathfrak{p} \in \mathcal{S}_{D'}\}$*.*

In view of the previous proposition, we will use the following terminology: a prime ideal $\mathfrak{p}$ of D is said to *survive* in D' if its extension $\mathfrak{p}D'$ in D' is a proper ideal (that is, $\mathfrak{p}D' \subsetneq D'$, in which case $\mathfrak{p}D'$ is a prime ideal of D' by the above result) and $\mathfrak{p}$ is said to be *lost* in D' otherwise (that is, if $\mathfrak{p}D' = D'$). In particular, every overring D' of a Prüfer domain D is equal to the intersection of the localizations of D at those prime ideals $\mathfrak{p}$ of D which survive in D'.

Example 2.2. Clearly,

$$\mathbb{Q}[X] = \bigcap_{q\in\mathcal{P}_{\text{irr}}} \text{Int}(\mathbb{Z})_{\mathfrak{P}_q} = \bigcap_{q\in\mathcal{P}_{\text{irr}}} \mathbb{Q}[X]_{(q)},$$

where $\mathcal{P}_{\text{irr}} = \mathcal{P}_{\text{irr}}(\mathbb{Z})$ is the set of irreducible polynomials in $\mathbb{Z}[X]$. By **[6,** Remark 1.12], this representation of $\mathbb{Q}[X]$ is irredundant, since $\mathbb{Q}[X]$ is a Dedekind domain and the set of maximal ideals of $\mathbb{Q}[X]$ is in one-to-one correspondence with $\mathcal{P}_{\text{irr}}$, namely $\mathcal{P}_{\text{irr}} \ni q \mapsto q(X)\mathbb{Q}[X]$.

Consequently, for a polynomial overring R, each prime ideal $\mathfrak{P}_q$ of Int($\mathbb{Z}$) must survive in R since it survives in $\mathbb{Q}[X]$, and we have

$$\mathfrak{P}_q\, R = q(X)\mathbb{Q}[X] \cap R.$$

Since we want to describe explicitly R in terms of those prime ideals of the spectrum of Int($\mathbb{Z}$) which survive in R, we are mostly interested in the other prime ideals, those lying over a prime. They are called *unitary* prime ideals because they contain nonzero constants.

The following result of Gilmer and Heinzer is of fundamental importance in order to decide whether an ideal $\mathfrak{p}$ of Int($\mathbb{Z}$) survives or not in some intersection of valuation overrings of Int($\mathbb{Z}$).

Proposition 2.3. (**[6,** Proposition 1.4]). *Let* D *be a Prüfer domain, and let* $\{\mathfrak{p}\} \cup \{\mathfrak{p}_\alpha\}_{\alpha\in\Lambda}$ *be a family of prime ideals of* D*. Then* $D_{\mathfrak{p}} \supseteq \bigcap_{\alpha\in\Lambda} D_{\mathfrak{p}_\alpha}$ *if and only if, for every finitely generated ideal* $\mathfrak{I} \subseteq \mathfrak{p}$*, there exists an* $\alpha \in \Lambda$ *such that* $\mathfrak{I} \subseteq \mathfrak{p}_\alpha$*.*

Corollary 2.4. *If $D_{\mathfrak{p}}$ is not a minimal valuation overring of the Prüfer domain D, then $D_{\mathfrak{p}}$ is superfluous in each representation of D as an intersection of valuation overings in which $D_{\mathfrak{p}}$ appears.*

Proof. (See also [**6,** Lemma 1.6].) Let $\bigcap_{\alpha\in\Lambda} D_{\mathfrak{p}_\alpha}$ be any representation of D, let $\alpha_0 \in \Lambda$, and assume that $D_{\mathfrak{p}_{\alpha_0}}$ is not a superfluous element in this representation. By Proposition 2.3, there exists a finitely generated ideal $\mathfrak{I} \subseteq \mathfrak{p}_{\alpha_0}$ such that $\mathfrak{I} \not\subseteq \mathfrak{p}_\alpha$ for every $\alpha \in \Lambda \setminus \{\alpha_0\}$. Let $\mathfrak{m}$ be a maximal ideal of D containing $\mathfrak{p}_{\alpha_0}$, and let x be any element of $\mathfrak{m}$. Since $D = \bigcap_{\alpha\in\Lambda} D_{\mathfrak{p}_\alpha}$ and $\mathfrak{I} + (x) \not\subseteq \mathfrak{p}_\alpha$ for $\alpha \neq \alpha_0$, necessarily $\mathfrak{I} + (x) \subseteq \mathfrak{p}_{\alpha_0}$. Finally, $\mathfrak{p}_{\alpha_0} = \mathfrak{m}$ is maximal, which is equivalent to the fact that $D_{\mathfrak{p}_{\alpha_0}}$ is a minimal valuation overring of D. □

Remarks 2.5.

(i) The converse of the previous corollary may be false: there are minimal valuation overrings which may be superfluous (cf., Example 4.7 below).

(ii) We have to take care that there is another notion of minimality which depends on the representation that we consider: a valuation domain which is minimal with respect to the elements of some representation of D is not necessarily minimal with respect to another representation (and in particular, with respect to all the valuation overrings of D). For instance, let $p \in \mathbb{P}$, $\alpha_n \in \mathbb{Z}$ $(n \geq 0)$ and $q \in \mathcal{P}_{\mathrm{irr}}(\mathbb{Z})$ be such that $q(\alpha_0) = 0$ and $\lim_{n\to+\infty} v_p(\alpha_n - \alpha_0) = +\infty$. Let $V_q = \mathbb{Q}[X]_{(q)}$. Then, we have:

$$D \doteqdot \left(\bigcap_{n\geq 0} V_{p,\alpha_n}\right) \bigcap V_q = \bigcap_{n\geq 0} V_{p,\alpha_n} = \bigcap_{n>0} V_{p,\alpha_n} = \left(\bigcap_{n>0} V_{p,\alpha_n}\right) \bigcap V_q. \tag{2.2}$$

The first equality follows from the fact that $V_q \supset V_{p,\alpha_0}$ and the second equality from the fact that $\alpha_0 = \lim_{n\to\infty} \alpha_n$ in $\mathbb{Z}_p$ (see Lemma 4.1). The valuation domain V_q is not minimal with respect to the elements of the first representation, while it is for the last one.

(iii) Obviously, a valuation domain which is not minimal with respect to some representation is superfluous for this representation, but Corollary 2.4 says something stronger since a minimal valuation overring of D which appears in some representation of D is a fortiori minimal for this representation. In the last representation of D given in (2.2), V_q is superfluous although it is minimal for this representation, but we could be sure that it is superfluous because it is not a minimal overring of D as shown by the first representation.

Thus, we emphasize that when we speak of a minimal valuation overring of D it is always a valuation domain which is minimal with respect to the family of *all* the valuation overrings of D.

Another important example is the localization of Int($\mathbb{Z}$) with respect to a prime $p \in \mathbb{Z}$.

Example 2.6. For every fixed prime p, we have

$$\operatorname{Int}\left(\mathbb{Z}_{(p)}\right) = \operatorname{Int}(\mathbb{Z})_{(p)},$$

where $\operatorname{Int}(\mathbb{Z})_{(p)}$ is the localization of the $\mathbb{Z}$-module Int($\mathbb{Z}$) at $p\mathbb{Z}$, namely, $\operatorname{Int}(\mathbb{Z})_{(p)} = \{(1/s)f(X) \mid f \in \operatorname{Int}(\mathbb{Z}), s \in \mathbb{Z} \setminus p\mathbb{Z}\}$ (see [**1,** Theorem I.2.3]). Consequently, the prime ideals of Int($\mathbb{Z}$) which survive in $\operatorname{Int}(\mathbb{Z}_{(p)})$ are the non-unitary ideals $\mathfrak{P}_q$ and the unitary ideals $\mathfrak{M}_{p,\alpha}$ lying over the prime p. By a slight abuse of notation, we still denote the corresponding extended ideals in $\operatorname{Int}(\mathbb{Z}_{(p)})$ by $\mathfrak{P}_q$ and $\mathfrak{M}_{p,\alpha}$, respectively. Then we have:

$$\begin{aligned}\operatorname{Int}(\mathbb{Z}_{(p)}) &= \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha \in \mathbb{Z}_p} V_{p,\alpha} \\ &= \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha \in \mathbb{Z}} V_{p,\alpha}.\end{aligned}$$

But, in this local case, an ideal $\mathfrak{P}_q$ may be maximal in $\operatorname{Int}(\mathbb{Z}_{(p)})$: $\mathfrak{P}_q$ is maximal if and only if $q(X)$ has no root in $\mathbb{Z}_p$ ([**1,** Proposition V.2.5]). Therefore, if $\mathcal{P}_{\mathrm{irr}}^{\mathbb{Z}_p}$ denotes the set of irreducible polynomials over $\mathbb{Z}$ which have no roots in $\mathbb{Z}_p$, we have the following representation of $\mathrm{Int}(\mathbb{Z}_{(p)})$ as the intersection of all its minimal valuation overrings (which correspond to the maximal ideals of $\mathrm{Int}(\mathbb{Z}_{(p)})$):

$$\mathrm{Int}(\mathbb{Z}_{(p)}) = \bigcap_{q\in\mathcal{P}_{\mathrm{irr}}^{\mathbb{Z}_p}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha\in\mathbb{Z}_p} V_{p,\alpha}. \tag{2.3}$$

Remark 2.7. It is not difficult to see that $\mathcal{P}_{\mathrm{irr}}^{\mathbb{Z}_p}$ is non-empty: let $g \in \mathbb{Z}_p[X]$ be a monic irreducible polynomial of degree $d \geq 2$. By a corollary of Krasner's lemma (see, for instance, [**9,** Chapter V, Proposition 5.9]), every monic polynomial $q \in \mathbb{Z}_p[X]$ of degree d which is sufficiently close to $g(X)$ with respect to the p-adic valuation is also irreducible over $\mathbb{Z}_p[X]$. Clearly, we may choose such a polynomial $q(X)$ with coefficients in $\mathbb{Z}$. Then, in particular, $q(X)$ is irreducible in $\mathbb{Z}[X]$ and has no roots in $\mathbb{Z}_p$.

If we localize each ring of (2.1) at p (that is, with respect to the multiplicative set $\mathbb{Z}\backslash p\mathbb{Z}$), since $\mathrm{Int}(\mathbb{Z})$ is well behaved under localization as seen in Example 2.6, we get

$$\mathrm{Int}(\mathbb{Z}_{(p)}) \subseteq R_{(p)} \subseteq \mathbb{Q}(X), \tag{2.4}$$

where $R_{(p)} = \{(1/n)f \mid f \in R, n \in \mathbb{Z} \setminus p\mathbb{Z}\}$. If R is a polynomial overring of $\mathrm{Int}(\mathbb{Z})$, then $R_{(p)}$ is a polynomial overring of $\mathrm{Int}(\mathbb{Z}_{(p)})$, that is, $R_{(p)} \subseteq \mathbb{Q}[X]$. Clearly, we have

$$R = \bigcap_{p\in\mathbb{P}} R_{(p)}. \tag{2.5}$$

Hence, in order to make our work easier, we fix a prime p, and we continue our discussion for an overring R of $\mathrm{Int}(\mathbb{Z}_{(p)})$.

3. Polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$. In this section, p denotes a fixed prime number, and we consider overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$, that is, rings R such that

$$\mathrm{Int}\left(\mathbb{Z}_{(p)}\right) \subseteq R \subseteq \mathbb{Q}(X).$$

Notation. For every overring R of $\mathrm{Int}(\mathbb{Z}_{(p)})$, we consider the following subsets:

(i) A subset of the ring $\mathbb{Z}_p$ of p-adic integers

$$Z_p(R) \doteqdot \{\alpha \in \mathbb{Z}_p \mid \mathfrak{M}_{p,\alpha} R \subsetneq R\}. \tag{3.1}$$

(ii) For every $\alpha \in \mathbb{Z}_p$ which is not the pole of some element of R, the following subring of the field $\mathbb{Q}_p$ of p-adic numbers

$$R(\alpha) \doteqdot \{f(\alpha) \mid f \in R\} \subseteq \mathbb{Q}_p.$$

Note that $Z_p(R)$ indexes the set of maximal unitary ideals of $\mathrm{Int}(\mathbb{Z}_{(p)})$ which survive in R under extension, and that $R(\alpha)$ is always defined for polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$. For instance, if $R = \mathrm{Int}(\mathbb{Z}_{(p)})$, then $Z_p(R) = \mathbb{Z}_p$ and, for every $\alpha \in Z_p(R) \cap \mathbb{Q}$, $R(\alpha) = \mathbb{Z}_{(p)}$, since $\mathbb{Z}_{(p)}[X] \subset R_{(p)}$ and $R(\alpha) \subseteq \mathbb{Z}_p \cap \mathbb{Q}$.

The following easy proposition characterizes the set $Z_p(R)$ for any overring R.

Proposition 3.1. *Let R be an overring of* $\mathrm{Int}(\mathbb{Z}_{(p)})$ *and $\alpha \in \mathbb{Z}_p$. Then*

$$\alpha \in Z_p(R) \Longleftrightarrow R \subseteq V_{p,\alpha} \Longleftrightarrow R(\alpha) \subseteq \mathbb{Z}_p. \tag{3.2}$$

Moreover, the subset $Z_p(R)$ is closed in $\mathbb{Z}_p$ for the p-adic topology.

Proof. The first equivalence follows from Proposition 2.1. The second equivalence is straightforward from the definitions of $V_{p,\alpha}$ and $R(\alpha)$. Concerning the last assertion, note that, for each $f \in R$, by continuity of f, the subset $\{\alpha \in \mathbb{Z}_p \mid f(\alpha) \in \mathbb{Z}_p\}$ is closed in $\mathbb{Z}_p$. Then, we just have to remark that:

$$Z_p(R) = \bigcap_{f \in R} \{\alpha \in \mathbb{Z}_p \mid f(\alpha) \in \mathbb{Z}_p\}. \qquad \square$$

Corollary 3.2. *Under extension, a prime ideal $\mathfrak{P}_q$ of* $\mathrm{Int}(\mathbb{Z}_{(p)})$ *is maximal in R if and only if $q(X)$ has no roots in $Z_p(R)$.*

Proof. If $\mathfrak{P}_q$ does not become maximal in R under extension, then $\mathfrak{P}_q R$ is strictly contained is some prime ideal $\mathfrak{Q}$ of R. By Proposition 2.1, $\mathfrak{Q}$ must be equal to the extension of some prime ideal of $\mathrm{Int}(\mathbb{Z}_{(p)})$, which must be a maximal ideal $\mathfrak{M}_{p,\alpha}$ containing $\mathfrak{P}_q$, or equivalently, $V_{p,\alpha} \subset \mathbb{Q}[X]_{(q)}$. In particular, $\alpha \in Z_p(R)$ and $q(\alpha) = 0$, by (1.2). The converse is clear. $\square$

3.1. Polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$ as intersections of valuation domains. Now we consider different representations of a polynomial overring R as intersections of valuation overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$.

Proposition 3.3. *Let p be a prime, and let R be any polynomial overring of $\mathrm{Int}(\mathbb{Z}_{(p)})$. We have the following representations of R as an intersection of valuation overrings.*

(i) *The intersection of all the valuation overrings:*

$$R = \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha \in Z_p(R)} V_{p,\alpha}, \tag{3.3}$$

where $\mathcal{P}_{\mathrm{irr}}$ denotes the set of irreducible polynomials of $\mathbb{Z}[X]$, and $Z_p(R)$ is defined by $Z_p(R) \doteqdot \{\alpha \in \mathbb{Z}_p \mid \mathfrak{M}_{p,\alpha} R \subsetneq R\}$.

(ii) *The intersection of all the minimal valuation overrings:*

$$R = \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}^{Z_p(R)}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha \in Z_p(R)} V_{p,\alpha}, \tag{3.4}$$

where $\mathcal{P}_{\mathrm{irr}}^{Z_p(R)}$ denotes the subset of $\mathcal{P}_{irr}$ formed by those polynomials which have no roots in $Z_p(R)$.

(iii) *For every $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{irr}}$ and every $E \subseteq Z_p(R)$, the following intersection of valuation overrings of R:*

$$R_{\mathcal{P},E} = \bigcap_{q \in \mathcal{P}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha \in E} V_{p,\alpha} \tag{3.5}$$

is equal to R if and only if $\mathcal{P} \supseteq \mathcal{P}_{\mathrm{irr}}^{Z_p(R)}$ and E is p-adically dense in $Z_p(R)$.

Proof. Example 2.2 and equivalences (3.2) show clearly that the valuation overrings of R are exactly those which appear in the right-hand side of equality (3.3). The equality follows from the fact that R is an overring of a Prüfer domain, and hence, it is a Prüfer domain, equal to the intersection of all its valuation overrings. Thus, (i) is proved.

The minimal valuation overrings of R correspond to the valuation overrings whose center is a maximal ideal of R. Assertion (ii) is then a consequence of Corollary 3.2.

By equality (3.3), R is contained in any ring of the form $R_{\mathcal{P},E}$. By continuity of the rational functions, if $\beta \in \mathbb{Z}_p$ is the limit of a sequence $\{\alpha_n\}_{n\geq 0}$ of elements of E, then $V_{p,\beta} \supset \bigcap_{n\in\mathbb{N}} V_{p,\alpha_n} \supset \bigcap_{\alpha\in E} V_{p,\alpha}$. As a consequence, if E is dense in $Z_p(R)$, then $\bigcap_{\alpha\in E} V_{p,\alpha} = \bigcap_{\alpha\in Z_p(R)} V_{p,\alpha}$, and hence, once more by equality (3.4), $R_{\mathcal{P},E} = R$.

Let us now prove the converse assertion of (iii). Assume first that $\mathcal{P} \not\supset \mathcal{P}_{\text{irr}}^{Z_p(R)}$. Then, there exists $r \in \mathcal{P}_{\text{irr}} \setminus \mathcal{P}$ without any root in $Z_p(R)$. Let $m = \sup\{v_p(r(\alpha)) \mid \alpha \in Z_p(R)\}$. Since $Z_p(R)$ is closed, m is finite since otherwise there would exist a sequence $\{\alpha_n\}_{n\geq 0}$ of elements of $Z_p(R)$ such that $v_p(r(\alpha_n)) \geq n$, and, by compactness of $Z_p(R)$, there would exist a subsequence which converges to an element β, which then would be a root of $r(X)$ in $Z_p(R)$. Consider now the rational function $\varphi(X) = p^m/r(X)$. For every $\alpha \in Z_p(R)$, $v_p(r(\alpha)) \leq m$, and hence, $\varphi \in V_{p,\alpha}$. Consequently, $\varphi \in \bigcap_{q\in\mathcal{P}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{\alpha\in Z_p(R)} V_{p,\alpha}$, while clearly $\varphi \notin \mathbb{Q}[X]_{(r)}$. Thus, $R \subsetneq R_{\mathcal{P},E}$.

Assume now that E is not p-adically dense in $Z_p(R)$. It remains to prove that again we have a strict containment: $R \subsetneq R_{\mathcal{P},E}$. For this, it is enough to prove that:

$$R \subsetneq \left(\bigcap_{q\in\mathcal{P}_{\text{irr}}} \mathbb{Q}[X]_{(q)}\right) \cap \left(\bigcap_{\alpha\in E} V_{p,\alpha}\right) = \{f(X) \in \mathbb{Q}[X] \mid f(E) \subseteq \mathbb{Z}_p\}.$$

This strict containment is a clear consequence of Proposition 3.10 below. $\square$

Remark 3.4. By Remark 2.7 and, by the fact that $\mathcal{P}_{\text{irr}}^{\mathbb{Z}_p} \subseteq \mathcal{P}_{\text{irr}}^{Z_p(R)}$ for each overring R of Int$(\mathbb{Z}_{(p)})$, it follows that $\mathcal{P}_{\text{irr}}^{Z_p(R)}$ is always non-empty. Note though, that the complement of $\mathcal{P}_{\text{irr}}^{Z_p(R)}$ may be empty, for example, if $Z_p(R)$ is formed by elements of $\mathbb{Z}_p$ which are transcendental over $\mathbb{Q}$.

As for Int$(\mathbb{Z})$ or for Int$(\mathbb{Z}_{(p)})$, an overring R does not have in general an irredundant representation as intersection of valuation overrings. There does exist an irredundant representation in some particular cases, as the next result shows.

Corollary 3.5. *A polynomial overring R of* $\mathrm{Int}(\mathbb{Z}_{(p)})$ *admits an irredundant representation if and only if $Z_p(R)$ contains a p-adically dense subset E formed by isolated points.*

Proof. Assume that $R_{\mathcal{P},E}$ is an irredundant representation of R. By Proposition 3.3 (iii), $\mathcal{P} = \mathcal{P}_{\mathrm{irr}}^{Z_p(R)}$ and E is dense in $Z_p(R)$. Moreover, for each $\alpha_0 \in E$, $R = R_{\mathcal{P},E} \subsetneq R_{\mathcal{P},E\setminus\{\alpha_0\}}$; thus, the topological closure of $E \setminus \{\alpha_0\}$ is strictly contained in that of E, which means that α_0 is isolated in E. The reverse implication is obvious still by Proposition 3.3 (iii). $\square$

For instance, we can consider E to be equal to the set of distinct elements of a convergent sequence $\{\alpha_n\}_{n\geq 0}$ with limit α, so that $Z_p(R) = E \cup \{\alpha\}$.

3.2. Polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$ as integer-valued polynomials rings. Contrary to equality (3.4), equality (3.3) shows that a polynomial overring R depends only on $Z_p(R)$. In order to describe how a polynomial overring R of $\mathrm{Int}(\mathbb{Z}_{(p)})$ is characterized by its associated set $Z_p(R)$, we recall the following definition (for example, see [**10, 11**]).

Definition 3.6. For every subset E of $\mathbb{Z}_p$, the ring formed by the polynomials of $\mathbb{Q}[X]$ whose values on E are p-integers is denoted by:

$$\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p) \doteqdot \{f \in \mathbb{Q}[X] \mid f(E) \subset \mathbb{Z}_p\}.$$

In particular, for $E = \mathbb{Z}_p$, we set $\mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_p) \doteqdot \mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_p,\mathbb{Z}_p)$.

By definition (or by convention) we set $\mathrm{Int}_{\mathbb{Q}}(\emptyset,\mathbb{Z}_p) = \mathbb{Q}[X]$ (after all, any polynomial is integer-valued over the empty-set). Note also that $\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p) = \mathbb{Q}[X] \cap \mathrm{Int}(E,\mathbb{Z}_p)$, where $\mathrm{Int}(E,\mathbb{Z}_p) = \{f \in \mathbb{Q}_p[X] \mid f(E) \subseteq \mathbb{Z}_p\}$. The following equality follows from a continuity-density argument:

$$\mathrm{Int}(\mathbb{Z}_{(p)}) = \mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_p). \tag{3.6}$$

Proposition 3.7. *Let R be a polynomial overring of* $\mathrm{Int}(\mathbb{Z}_{(p)})$*, and let* $Z_p(R) = \{\alpha \in \mathbb{Z}_p \mid \mathfrak{M}_{p,\alpha} R \subsetneq R\}$*. Then*

$$R = \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p).$$

Proof. The containment $R \subseteq \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$ follows from Proposition 3.1: $\alpha \in Z_p(R)$ if and only $R(\alpha) \subseteq \mathbb{Z}_p$. Thus, we have the chain of inclusions

$$\mathrm{Int}(\mathbb{Z}_{(p)}) \subseteq R \subseteq \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p) \subseteq \mathbb{Q}[X].$$

In order to prove the converse containment, it is sufficient to show that each prime ideal of $\mathrm{Int}(\mathbb{Z}_{(p)})$ which survives in R also survives in $\mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$. In fact, since we are dealing with Prüfer domains, if a prime ideal $\mathfrak{P}$ of $\mathrm{Int}(\mathbb{Z}_{(p)})$ is such that $\mathfrak{P}R \subsetneq R$, then $\mathfrak{P}R$ is a prime ideal of R and these extensions comprise the whole spectrum of R by Proposition 2.1 (iv). We then use the well-known fact that an integral domain is equal to the intersection of the localizations at its own prime ideals.

For what we have already said, all the prime non-unitary ideals survive in both rings since they survive in $\mathbb{Q}[X]$. Let $\mathfrak{M}_{p,\alpha}$ be a maximal unitary ideal which survives in R. By definition of $Z_p(R)$, $\alpha \in Z_p(R)$. Now, $\mathfrak{M}_{p,\alpha}$ survives in $\mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$ if and only if $\mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$ is contained in $V_{p,\alpha}$, that is, each polynomial of $\mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$ is integer-valued on α. Since $\alpha \in Z_p(R)$, the conclusion follows. $\square$

In particular, from Proposition 3.7, we have a complete characterization of the family $\mathcal{R}_p$ of polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$:

Corollary 3.8. *If $\mathcal{F}(\mathbb{Z}_p)$ denotes the family of closed subsets of $\mathbb{Z}_p$, then*

$$\mathcal{R}_p = \{\mathrm{Int}_{\mathbb{Q}}(F, \mathbb{Z}_p) \mid F \in \mathcal{F}(\mathbb{Z}_p)\}.$$

Proposition 3.7 says how R is characterized by the closed subset $Z_p(R) \subseteq \mathbb{Z}_p$. In order to prove that, for different closed subsets of $\mathbb{Z}_p$ we get different polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$, we recall the notion of polynomial closure introduced by Gilmer [**5**] and McQuillan [**8**].

Definition 3.9. For any subset $E \subseteq \mathbb{Z}_p$ (E is not necessarily closed), the *p-polynomial closure* of E is the largest subset $\overline{E}$ of $\mathbb{Z}_p$ (containing

E) such that

$$\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p)=\mathrm{Int}_{\mathbb{Q}}(\overline{E},\mathbb{Z}_p).$$

Equivalently,

$$\begin{aligned}\overline{E}&=\{\alpha\in\mathbb{Z}_p\mid \mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p)(\alpha)\subset\mathbb{Z}_p\}\\&=\{\alpha\in\mathbb{Z}_p\mid \mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p)\subset V_{p,\alpha}\}\\&=Z_p(\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p)),\end{aligned}$$

where the last equality follows by Proposition 3.1.

Proposition 3.10. *For any subset $E\subseteq\mathbb{Z}_p$, the following subsets are equal*:

(i) *the p-polynomial closure of E,*
(ii) *the p-adic topological closure of E,*
(iii) $Z_p(\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p))$.

For equivalence between the polynomial closure and the topological closure, see for instance, [**1,** Theorem IV.1.15].

The next theorem shows that the closed subsets of $\mathbb{Z}_p$ are in one-to-one correspondence with the polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$.

Theorem 3.11. *Let $\mathcal{R}_p$ be the set of polynomial overrings of* $\mathrm{Int}(\mathbb{Z}_{(p)})$, *and let $\mathcal{F}(\mathbb{Z}_p)$ be the family of closed subsets of $\mathbb{Z}_p$. The following maps which reverse the containments are inverse to each other*:

$$\varphi_p:\mathcal{R}_p\ni R\longmapsto Z_p(R)\in\mathcal{F}(\mathbb{Z}_p)$$

and

$$\psi_p:\mathcal{F}(\mathbb{Z}_p)\ni F\longmapsto \mathrm{Int}_{\mathbb{Q}}(F,\mathbb{Z}_p)\in\mathcal{R}_p.$$

Proof. By Proposition 3.7, $\psi_p\circ\varphi_p=id_{\mathcal{R}_p}$. Now, considering $\varphi_p\circ\psi_p$: for every $F\in\mathcal{F}(\mathbb{Z}_p)$, one has $\varphi_p(\psi_p(F))=Z_p(\mathrm{Int}(F,\mathbb{Z}_p))=\{\alpha\in\mathbb{Z}_p\mid$ for all $f\in\mathrm{Int}(F,\mathbb{Z}_p)$ $f(\alpha)\in\mathbb{Z}_p\}=F$ by Proposition 3.10 since F is assumed to be closed. Consequently, $\varphi_p\circ\psi_p=id_{\mathcal{F}(\mathbb{Z}_p)}$. □

We end this section with the characterization of minimal ring extensions of the family $\mathcal{R}_p$. Recall that $R_1 \subsetneq R_2 \in \mathcal{R}_p$ forms a minimal ring extension if there is no ring in between R_1 and R_2.

Proposition 3.12. *Let $R = \mathrm{Int}_{\mathbb{Q}}(F, \mathbb{Z}_p)$ where $F = Z_p(R)$ is a closed subset of $\mathbb{Z}_p$. There is a bijection between the minimal ring extensions of R in $\mathcal{R}_p$ and the subset F_0 formed by the isolated points of F, which is given by:*

$$F_0 \ni \alpha \longmapsto \mathrm{Int}_{\mathbb{Q}}(F \setminus \{\alpha\}, \mathbb{Z}_p).$$

We stress that we are interested only in *polynomial* ring extensions of R, that is, elements of the family $\mathcal{R}_p$. Note that the proposition says that R has no minimal ring extension in $\mathcal{R}_p$ if and only if $Z_p(R)$ has no isolated points.

Proof. Let $S \in \mathcal{R}_p$ be a proper extension of R. Then, by Theorem 3.11, $S = \mathrm{Int}_{\mathbb{Q}}(E, \mathbb{Z}_p)$ where $E = Z_p(S)$ is a closed subset strictly contained in F. For every $\alpha \in F \setminus E$, the subset $E \cup \{\alpha\}$ is closed and the ring $T = \mathrm{Int}_{\mathbb{Q}}(E \cup \{\alpha\}, \mathbb{Z}_p)$ satisfies $R \subseteq T \subsetneq S$ since $E \subsetneq E \cup \{\alpha\} \subseteq F$.

Therefore, the extension $R \subsetneq S$ is minimal if and only if there is no closed subset G such that $E \subsetneq G \subsetneq F$. Consequently, if the extension $R \subsetneq S$ is minimal, then necessarily $F = E \cup \{\alpha\}$. The fact that E is closed in $E \cup \{\alpha\} = F$ implies that α is isolated in F. Conversely, if $\alpha \in F$ is isolated in F, then $F \setminus \{\alpha\}$ is closed in F, and clearly there is no closed subset G properly lying between $F \setminus \{\alpha\}$ and F. Thus, we may conclude that S is a minimal extension of R if and only if $S = \mathrm{Int}_{\mathbb{Q}}(F \setminus \{\alpha\}, \mathbb{Z}_p)$ where $\alpha \in F$ is an isolated point. If $\alpha \neq \alpha'$ are two distinct isolated points of F, then by Theorem 3.11, the corresponding minimal ring extensions of R are distinct because $F \setminus \{\alpha\} \neq F \setminus \{\alpha'\}$. □

4. Valuation overrings of an intersection of valuation domains. The aim of this section is to characterize whether a valuation overring of Int($\mathbb{Z}$) as described in Section 1 contains a given intersection of valuation overrings of Int($\mathbb{Z}$). We will apply the obtained results to describe the representations of every polynomial overring of Int($\mathbb{Z}$) as intersections of valuation domains. In order to do this, we will extensively use Proposition 2.3. To ease the notation, we set $V_q = \mathbb{Q}[X]_{(q)}$, for $q \in \mathcal{P}_{\rm irr}$. Moreover, since now we are going to consider arbitrary intersections of unitary valuation domains for different $p \in \mathbb{P}$, we generalize the notation $R_{\mathcal{P},E_p}$ used in formula (3.5) in the following way: if $\mathcal{P} \subseteq \mathcal{P}_{\rm irr}$ and if, for each $p \in \mathbb{P}$, $E_p \subseteq \mathbb{Z}_p$, then we set

$$R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}} = \bigcap_{q\in\mathcal{P}} V_q \cap \bigcap_{p\in\mathbb{P}} \bigcap_{\alpha\in E_p} V_{p,\alpha}.$$

If the subset E_p of $\mathbb{Z}_p$ is empty for some $p \in \mathbb{P}$, then the corresponding intersection $\bigcap_{\alpha\in E_p} V_{p,\alpha}$ is set to be equal to $\mathbb{Q}(X)$. We consider a similar convention for the set of non-unitary valuation overrings V_q if $\mathcal{P} = \emptyset$. In particular, if E_p is empty for all $p \in \mathbb{P}$ except p_0, then the intersection corresponds to the ring $R_{\mathcal{P},E_{p_0}}$.

We want to determine which are the valuation overrings of a ring $R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$ as above. We distinguish the case of a unitary valuation overring $V_{p,\alpha}$ (whose center is a unitary prime ideal of $\mathrm{Int}(\mathbb{Z})$) from a non-unitary valuation overring V_q (whose center is non-unitary).

4.1. Unitary valuation overrings. We begin to determine unitary valuation overrings of an arbitrary intersection of $V_{p,\alpha}$ for a fixed prime p, and possibly some non-unitary valuation domains V_q's. We remark first that, given a subset E of $\mathbb{Z}_p$, if V_{p_0,α_0} is an overring of $\cap_{\alpha\in E}V_{p,\alpha}$, where $p_0 \in \mathbb{P}$ and $\alpha_0 \in \mathbb{Z}_{p_0}$, then $p_0 = p$. In fact, if that were not true, then $1/p_0$, which is in $\cap_{\alpha\in E}V_{p,\alpha}$ would also belong to V_{p_0,α_0}, which is a contradiction. Therefore, we can just consider valuation overrings which lie above the same prime p.

The next result is an obvious consequence of Proposition 3.10 (see also [**7**, Lemma 26]; although [**7**, Section 5] is entitled *Overrings of* $\mathrm{Int}(\mathbb{Z})$, the author's point of view is quite different from ours).

Lemma 4.1. *Let $p \in \mathbb{P}$, $E \subseteq \mathbb{Z}_p$, $\mathcal{P} \subseteq \mathcal{P}_{\rm irr}$ and $\alpha_0 \in \mathbb{Z}_p$. The following assertions are equivalent*:

(i) $R_{\mathcal{P},E} \subseteq V_{p,\alpha_0}$,
(ii) $\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p) \subseteq V_{p,\alpha_0}$,
(iii) α_0 *belongs to the topological closure* $\overline{E}$ *of* E *in* $\mathbb{Z}_p$.

In particular, $R_{\mathcal{P},E} = R_{\mathcal{P},\overline{E}}$ *and* $Z_p(R_{\mathcal{P},E}) = \overline{E}$.

Proof. (i) $\to$ (ii). $\mathrm{Int}_{\mathbb{Q}}(E,\mathbb{Z}_p)$ is contained in $R_{\mathcal{P},E}$.

(ii) $\leftrightarrow$ (iii). (ii) means that, for every $f \in \mathrm{Int}_{\mathbb{Q}}(E, \mathbb{Z}_p)$, $f(\alpha_0) \in \mathbb{Z}_p$, that is, α_0 belongs to the p-polynomial closure of E; thus, we may conclude with Proposition 3.10.

(ii) $\to$ (i). Assume that V_{p,α_0} is an overring of $R = \mathrm{Int}_{\mathbb{Q}}(E, \mathbb{Z}_p)$. We use Proposition 2.3 to get the claim. Let $I \subset R$ be a finitely generated ideal contained in $\mathfrak{M}_{p,\alpha_0}$ and let $J = I + (p)$. Since J is not contained in any non-unitary prime ideal of R, then it follows from (3.4) that J is contained in some unitary prime ideal $\mathfrak{M}_{p,\alpha}$ of R where $\alpha \in E$. In particular, I is contained in this ideal $\mathfrak{M}_{p,\alpha}$ and we conclude that $V_{p,\alpha_0} \supseteq \bigcap_{\alpha \in E} V_{p,\alpha} \supseteq R_{\mathcal{P},E}$.

The last claims follow immediately. $\square$

Lemma 4.2. *For each $p \in \mathbb{P}$, let $E_p \subseteq \mathbb{Z}_p$. Let $p_0 \in \mathbb{P}$ and $\alpha_0 \in \mathbb{Z}_{p_0}$. Then*

$$\bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in E_p} V_{p,\alpha} \subset V_{p_0,\alpha_0} \Longleftrightarrow \bigcap_{\alpha \in E_{p_0}} V_{p_0,\alpha} \subset V_{p_0,\alpha_0}$$

Proof. One implication is obvious. Conversely, assume that V_{p_0,α_0} is an overring of the intersection $\bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in E_p} V_{p,\alpha}$. Let I be a finitely generated ideal contained in $\mathfrak{M}_{p_0,\alpha_0}$, and let $J = I + (p_0)$. Since for all $p \neq p_0$ and, for all $\alpha \in E_p$, we have $J \not\subset \mathfrak{M}_{p,\alpha}$, it follows that $J \subseteq \mathfrak{M}_{p_0,\alpha}$ for some $\alpha \in E_{p_0}$. In particular, $I \subseteq \mathfrak{M}_{p_0,\alpha}$. By Proposition 2.3, we may conclude. $\square$

Both previous lemmas lead to the following proposition.

Proposition 4.3. *For each $p \in \mathbb{P}$, let $E_p \subseteq \mathbb{Z}_p$. Let $p_0 \in \mathbb{P}$, let $\alpha_0 \in \mathbb{Z}_p$, and let $\mathcal{P}$ be any subset of $\mathcal{P}_{\mathrm{irr}}$. Then the following conditions are equivalent*:

(i) $\bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in E_p} V_{p,\alpha} \subset V_{p_0,\alpha_0}$.
(i′) $R_{\mathcal{P},(E_p)_{p \in \mathbb{P}}} \subset V_{p_0,\alpha_0}$.
(ii) $\bigcap_{\alpha \in E_{p_0}} V_{p_0,\alpha} \subset V_{p_0,\alpha_0}$.
(ii′) $R_{\mathcal{P},E_{p_0}} \subset V_{p_0,\alpha_0}$.
(iii) α_0 *is in the topological closure of* E_{p_0} *in* $\mathbb{Z}_{p_0}$.

Corollary 4.4. *Let $\mathcal{P} \subseteq \mathcal{P}_{\rm irr}$ and, for each $p \in \mathbb{P}$, let $E_p \subseteq \mathbb{Z}_p$. Then, V_{p_0,α_0} where $p_0 \in \mathbb{P}$ and $\alpha_0 \in E_{p_0}$ is not a superfluous valuation overring of $R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$ if and only if α_0 is an isolated point of E_{p_0}.*

Proof. V_{p_0,α_0} is not a superfluous valuation overring of $R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$ if and only if the intersection of the valuation domains of the family $\{V_q \mid q \in \mathcal{P}\}\cup\{V_{p,\alpha}|\alpha \in E_p, p \in \mathbb{P}\}\setminus\{V_{p_0,\alpha_0}\}$ is not contained in V_{p_0,α_0}. By Proposition 4.3, this condition is equivalent to $\alpha_0 \notin \overline{E_{p_0} \setminus \{\alpha_0\}}$, that is, α_0 is an isolated point of E_{p_0}. □

4.2. Non-unitary valuation overrings. Now we consider the case of a non-unitary valuation domain $V_q = \mathbb{Q}[X]_{(q)}$, $q \in \mathcal{P}_{\rm irr}$, containing an arbitrary intersection of unitary and non-unitary valuation domains.

Lemma 4.5. *Let $\mathcal{P} \subset \mathcal{P}_{\rm irr}$ and $q_0 \in \mathcal{P}_{\rm irr}$. Then*

$$\bigcap_{q\in\mathcal{P}} V_q \subseteq V_{q_0} \Longleftrightarrow q_0 \in \mathcal{P}.$$

Proof. One direction is obvious. Conversely, suppose V_{q_0} is an overring of the intersection of the V_q's, $q \in \mathcal{P}$. If $q_0 \notin \mathcal{P}$, we have a contradiction since

$$\frac{1}{q_0} \in \bigcap_{q\in\mathcal{P}} V_q \quad \text{and} \quad \frac{1}{q_0} \notin V_{q_0}.$$

□

Theorem 4.6. *Let $q \in \mathcal{P}_{\rm irr}$ and, for each $p \in \mathbb{P}$, let $F_p \subseteq \mathbb{Z}_p$ be a (possibly empty) closed set of p-adic integers. Then*

$$\bigcap_{p\in\mathbb{P}} \bigcap_{\alpha\in F_p} V_{p,\alpha} \subset V_q \Longleftrightarrow \exists(\alpha_p) \in \prod_{p\in\mathbb{P}} F_p$$
$$\textit{such that } \sum_{p\in\mathbb{P}} v_p(q(\alpha_p)) = +\infty.$$

Note that the latter condition means that: either there exist $p \in \mathbb{P}$ and $\alpha_p \in F_p$ such that $q(\alpha_p) = 0$, or there exist infinitely many primes $p_n \in \mathbb{P}$ and some $\alpha_{p_n} \in F_{p_n}$ such that $v_{p_n}(q(\alpha_{p_n})) \geq 1$. Example 4.7 below shows that the latter condition can really occur.

Proof. Since $\mathfrak{P}_q = \{qf \in \mathrm{Int}(\mathbb{Z}) \mid f \in \mathbb{Q}[X]\}$ and $\mathbb{Q}[X]$ is countable, we may fix a sequence $\{f_n\}_{n\geq 0}$ of polynomials in $\mathbb{Q}[X]$ such that the qf_n's generate $\mathfrak{P}_q$. We also consider the set

$$\mathbb{P}_q \doteqdot \{p \in \mathbb{P} \mid \text{there exists } \alpha_p \in F_p \text{ such that } q \in \mathfrak{M}_{p,\alpha_p}\}.$$

Assume that there exists $(\alpha_p) \in \prod_{p\in\mathbb{P}} F_p$ such that $\sum_{p\in\mathbb{P}} v_p(q(\alpha_p)) = \infty$. If, for some prime p and some $\alpha \in F_p$, $q(\alpha) = 0$, then $V_q \supset V_{p,\alpha}$, and hence, $V_q \supset \bigcap_{p\in\mathbb{P}} \bigcap_{\alpha\in F_p} V_{p,\alpha}$. Suppose that, for each $p \in \mathbb{P}$, $q(X)$ has no roots in F_p. It follows that the set $\mathbb{P}_q$ is infinite. Let $I \subseteq \mathfrak{P}_q$ be any finitely generated ideal. There exists n such that $I \subseteq (qf_1, \dots, qf_n)$. Since, for almost all $p \in \mathbb{P}$, the polynomials $f_1, \dots, f_n$ are in $\mathbb{Z}_{(p)}[X]$, there exists $p \in \mathbb{P}_q$ such that $f_1, \dots, f_n \in \mathbb{Z}_{(p)}[X]$, and hence, for the above $\alpha_p \in F_p$, $v_p(q(\alpha_p) f_j(\alpha_p)) \geq v_p(q(\alpha_p)) > 0$, for $1 \leq j \leq n$. Consequently, $I \subseteq \mathfrak{M}_{p,\alpha_p}$, which shows by Proposition 2.3 that $V_q \supset \bigcap_{p\in\mathbb{P}} \bigcap_{\alpha\in F_p} V_{p,\alpha}$.

Conversely, assume that $\mathbb{P}_q = \{p_1, \dots, p_s\}$ and $m_i = \sup\{v_{p_i}(q(\alpha)) \mid \alpha \in F_{p_i}\} < \infty$ for $i = 1, \dots, s$ ($\Leftrightarrow$ $q(\alpha) \neq 0$, for each $\alpha \in F_{p_i}$, $i = 1, \dots, s$ since F_{p_i} is closed.) Then, consider the rational function:

$$\varphi(X) = \prod_{i=1}^{s} p_i^{m_i} \frac{1}{q(X)}.$$

For every $p_i \in \mathbb{P}_q$ and every $\alpha_{p_i} \in F_{p_i}$, $v_p(q(\alpha_{p_i})) \leq m_i$, and hence, $\varphi \in V_{p_i,\alpha_{p_i}}$. For every $p \in \mathbb{P}\setminus\mathbb{P}_q$ and every $\alpha_p \in F_p$, $v_p(q(\alpha_p)) = 0$, and hence, $\varphi \in V_{p,\alpha_p}$. Consequently, $\varphi \in \bigcap_{p\in\mathbb{P}} \bigcap_{\alpha_p\in F_p} V_{p,\alpha_p}$, while clearly $\varphi \notin V_q$. $\square$

Example 4.7. This example shows that a minimal non-unitary valuation overring of some ring $R_{\mathcal{P},(F_p)_{p\in\mathbb{P}}}$ can be superfluous. Let $q(X) = X$. Suppose $\mathfrak{P}_q = \bigcup_{n\in\mathbb{N}} I_n$, where $\{I_n\}_{n\geq 0}$ is an increasing sequence of ideals, each of them generated by $Xf_1(X), \dots, Xf_n(X)$, for some $f_i \in \mathbb{Q}[X]$. Let p_n be the nth prime. For each $n \in \mathbb{N}$, there exists $a_n \in \mathbb{N}$ large enough such that $I_n \subset \mathfrak{M}_{p_n,p_n^{a_n}}$, exactly by the same argument as the above proof. Then, by Proposition 2.3, V_q is an overring of $\bigcap_{n\in\mathbb{N}} V_{p_n,p_n^{a_n}}$, even though, by (1.2), $V_{p_n,p_n^{a_n}} \not\subset V_q$ for each $n \in \mathbb{N}$. Hence, V_q is a minimal overring of $\bigcap_{n\in\mathbb{N}} V_{p_n,p_n^{a_n}} \cap V_q$ which is superfluous. Or, if we want to consider integer-valued polynomials, let

$E = \cup_{n\in\mathbb{N}}\{p_n^{a_n}\}$. Then we have

$$\mathrm{Int}(E,\mathbb{Z}) = \bigcap_{q\in\mathcal{P}_{\mathrm{irr}}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{p\in\mathbb{P}} \bigcap_{n\in\mathbb{N}} V_{p,p_n^{a_n}},$$

where the minimal valuation overring V_X of $\mathrm{Int}(E,\mathbb{Z})$ is superfluous.

Remark 4.8. Let $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{irr}}, q_0 \in \mathcal{P}_{\mathrm{irr}}$, and, for each $p \in \mathbb{P}$, let F_p be a closed subset of $\mathbb{Z}_p$.

(i) Theorem 4.6 may be generalized to an arbitrary intersection of unitary and non-unitary valuation domains:

$$\begin{aligned}&R_{\mathcal{P},(F_p)_{p\in\mathbb{P}}} \subset V_{q_0} \iff \text{ either } q_0 \in \mathcal{P}\\ &\text{or}\\ &\text{there exists } (\alpha_p) \in \prod_{p\in\mathbb{P}} F_p\\ &\text{such that } \sum_{p\in\mathbb{P}} v_p(q_0(\alpha_p)) = +\infty.\end{aligned}$$

In fact, if V_{q_0} is an overring of $R_{\mathcal{P},(F_p)_{p\in\mathbb{P}}}$ and there is no $(\alpha_p) \in \prod_{p\in\mathbb{P}} F_p$ such that $\sum_{p\in\mathbb{P}} v_p(q_0(\alpha_p)) = +\infty$, then, by Theorem 4.6, V_{q_0} is not an overring of $\bigcap_{p\in\mathbb{P}}\bigcap_{\alpha\in F_p} V_{p,\alpha}$. Hence, by the techniques of Proposition 2.3, V_{q_0} is easily seen to be an overring of $\bigcap_{q\in\mathcal{P}} V_q$, and so, by Lemma 4.5, $q_0 \in \mathcal{P}$, as desired. Conversely, by Theorem 4.6, each condition on the right-hand side implies that V_{q_0} is an overring of $R_{\mathcal{P},(F_p)_{p\in\mathbb{P}}}$.

(ii) If F_p is an empty set for all but finitely many primes $\{p_1,\ldots,p_n\}$ (for example, overrings $R_{\mathcal{P},F_p}$ of $\mathrm{Int}(\mathbb{Z}_{(p)})$), then V_q is an overring of $R_{\mathcal{P},(F_p)_{p\in\mathbb{P}}}$ if and only if $q(X)$ has a root in some F_{p_i}, $i = 1,\ldots,n$, or $q \in \mathcal{P}$. Therefore, a minimal non-unitary valuation overring V_q of $R_{\mathcal{P},(F_{p_i})_{i=1,\ldots,n}}$ is not superfluous.

5. Polynomial overrings of $\mathrm{Int}(\mathbb{Z})$ **as intersections of valuation domains.** We consider now a polynomial overring R of $\mathrm{Int}(\mathbb{Z})$. Analogously to the previous case of overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$, we consider the subset $Z_p(R)$.

Notation. For every ring R such that $\mathrm{Int}(\mathbb{Z}) \subseteq R \subseteq \mathbb{Q}[X]$ and every $p \in \mathbb{P}$, let $Z_p(R)$ be the following subset of $\mathbb{Z}_p$:

$$Z_p(R) \doteqdot \{\alpha \in \mathbb{Z}_p \mid \mathfrak{M}_{p,\alpha} R \subsetneq R\}. \tag{5.1}$$

We already introduced $Z_p(R)$ in (3.1) for polynomial overrings of $\mathrm{Int}(\mathbb{Z}_{(p)})$. Fortunately, both notations agree with each other since, clearly, $Z_p(R) = Z_p(R_{(p)})$:

$$\alpha \in Z_p(R) \Longleftrightarrow R \subseteq V_{p,\alpha} \Longleftrightarrow R_{(p)} \subseteq V_{p,\alpha} \Longleftrightarrow \alpha \in Z_p(R_{(p)}). \tag{5.2}$$

Analogously to Proposition 3.3, we now consider the representations of R as an intersection of valuation overrings.

Proposition 5.1. *Let R be any polynomial overring of* $\mathrm{Int}(\mathbb{Z})$*. We have the following representations of R as an intersection of valuation overrings.*

(i) *The intersection of all the valuation overrings*:

$$R = \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in Z_p(R)} V_{p,\alpha}; \tag{5.3}$$

(ii) *The intersection of all the minimal valuation overrings*:

$$R = \bigcap_{q \in \mathcal{P}_{\mathrm{irr}}^{Z(R)}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in Z_p(R)} V_{p,\alpha}, \tag{5.4}$$

where $\mathcal{P}_{\mathrm{irr}}^{Z(R)}$ denotes the set of irreducible polynomials of $\mathbb{Z}[X]$ which have no roots in $Z_p(R)$ whatever $p \in \mathbb{P}$.

(iii) *For every $\mathcal{P} \subseteq \mathcal{P}_{\mathrm{irr}}$ and every $E_p \subseteq Z_p(R) (p \in \mathbb{P})$, the following intersection of valuation overrings of R*:

$$R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}} = \bigcap_{q \in \mathcal{P}} \mathbb{Q}[X]_{(q)} \cap \bigcap_{p \in \mathbb{P}} \bigcap_{\alpha \in E_p} V_{p,\alpha} \tag{5.5}$$

is equal to R if and only if

(a) *$\mathcal{P} \supseteq \mathcal{P}_{\mathrm{irr}}^{Z_0(R)}$ where $\mathcal{P}_{\mathrm{irr}}^{Z_0(R)}$ is formed by the irreducible polynomials q of $\mathbb{Z}[X]$ such that, for every $p \in \mathbb{P}$, q has no root in $Z_p(R)$, and there do not exist two infinite sequences $\{p_j\}_{j\in\mathbb{N}}$ and $\{\alpha_j\}_{j\in\mathbb{N}}$ where $p_i \in \mathbb{P}$, $\alpha_j \in Z_{p_j}(R)$, and $v_{p_j}(q(\alpha_j)) > 0$,*

(b) *for every $p \in \mathbb{P}$, E_p is p-adically dense in $Z_p(R)$.*

Proof. Formula (5.3) is clearly a consequence of Proposition 3.3 (i). Analogously to formula (3.4), formula (5.4) follows from the globalization of Corollary 3.2: a prime ideal $\mathfrak{P}_q$ of $\mathrm{Int}(\mathbb{Z})$ is maximal in R if and only if, for each $p \in \mathbb{P}$, $q(X)$ has no roots in $Z_p(R)$.

It remains to prove assertion (iii). If (a) and (b) hold, then, by Proposition 4.3 and Theorem 4.6, the right-hand side of formula (5.5) is equal to the right-hand side of formula (5.3), and hence to R.

Assume now that (a) does not hold. There exists $r \in \mathcal{P}_{\mathrm{irr}}^{Z_0(R)} \setminus \mathcal{P}$. Since $E_p \subseteq Z_p(R)$ for every $p \in \mathbb{P}$ and $r \notin \mathcal{P}$, it follows from Theorem 4.6 that $V_r = \mathbb{Q}[X]_{(r)} \not\supseteq R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$; in particular, $\mathbb{Q}[X] \not\supseteq R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$, and hence, $R \subsetneq R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$.

Finally, assume that (b) does not hold. There is some $p_0 \in \mathbb{P}$ such that E_{p_0} is not p_0-adically dense in $Z_{p_0}(R)$; in other words, there is some $\alpha_0 \in Z_{p_0}(R)$ which is not in the topological closure of E_{p_0}. By Proposition 4.3, $\bigcap_{p\in\mathbb{P}} \bigcap_{\alpha\in E_p} V_{p,\alpha} \cap \mathbb{Q}[X] \not\subseteq V_{p_0,\alpha_0}$, and hence, once more, $R \subsetneq R_{\mathcal{P},(E_p)_{p\in\mathbb{P}}}$. $\square$

Remark 5.2. We can generalize Corollary 3.5 to overrings of $\mathrm{Int}(\mathbb{Z})$ in the following way: a polynomial overring R of $\mathrm{Int}(\mathbb{Z})$ admits an irredundant representation if and only if, for each $p \in \mathbb{P}$, $Z_p(R)$ contains a p-adically dense subset formed by isolated points.

6. Polynomial overrings of $\mathrm{Int}(\mathbb{Z})$ **as integer-valued polynomial rings over subsets of** $\widehat{\mathbb{Z}}$**.** In this section, we give another point of view about polynomial overrings R of $\mathrm{Int}(\mathbb{Z})$, in order to represent them as rings of integer-valued polynomials. We know that $R = \cap_{p\in\mathbb{P}} R_{(p)}$ and that $R_{(p)} = \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$. Consequently, R is equal to an intersection of different integer-valued polynomial rings as p runs through the set of prime numbers:

$$R = \bigcap_{p\in\mathbb{P}} \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p). \tag{6.1}$$

However, it seems to be more convenient to consider all the p-adic completions $\mathbb{Z}_p$ at the same time. Classically, the way to do that is via the ring of *finite adeles* $\mathcal{A}_f(\mathbb{Q})$ ('finite' refers to the fact we forget the Archimedean absolute value). A finite *adele* is an element $\underline{\alpha} = (\alpha_p)_p$ of the product $\prod_{p\in\mathbb{P}}\mathbb{Q}_p$ such that, for all but finitely many p's, α_p belongs to $\mathbb{Z}_p$ (for instance, see [**9,** subsection 6.2, page 286]).

Note that $\mathbb{Q}$ embeds diagonally into $\prod_{p\in\mathbb{P}}\mathbb{Q}_p$ and its image is in $\mathcal{A}_f(\mathbb{Q})$. Actually, $\mathbb{Q}$ embeds into the group of units of $\mathcal{A}_f(\mathbb{Q})$. Recall that this group, denoted by $\mathcal{I}_f(\mathbb{Q})$ and called *finite ideles*, is formed by the elements $\underline{\alpha} = (\alpha_p)_p \in \prod_{p\in\mathbb{P}}\mathbb{Q}_p^*$ such that $v_p(a_p) = 0$, for all but finitely many p. Given $\underline{\alpha} = (\alpha_p)_p \in \mathcal{A}_f(\mathbb{Q})$ and $f \in \mathbb{Q}[X]$, we clearly have

$$f(\underline{\alpha}) = (f(\alpha_p))_p \in \mathcal{A}_f(\mathbb{Q}) \subset \prod_{p\in\mathbb{P}}\mathbb{Q}_p,$$

that is, every polynomial with rational coefficients maps an adele into an adele. For this reason, the ring of integer-valued polynomials over the ring of finite adeles is trivial:

$$\begin{aligned}\mathbb{Q}[X] &= \mathrm{Int}_{\mathbb{Q}}(\mathcal{A}_f(\mathbb{Q}))\\ &= \{f \in \mathbb{Q}[X] \mid f(\underline{\alpha}) \in \mathcal{A}_f(\mathbb{Q}),\ \text{for all } \underline{\alpha} \in \mathcal{A}_f(\mathbb{Q})\}.\end{aligned}$$

However, note that $\mathcal{A}_f(\mathbb{Q})$ contains as a subring the product $\prod_{p\in\mathbb{P}}\mathbb{Z}_p$, which is isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$ with respect to the fundamental system of neighborhoods of 0 consisting of all the non-zero ideals of $\mathbb{Z}$.

Given $f \in \mathbb{Q}[X]$ and $\underline{\alpha} \in \mathcal{A}_f(\mathbb{Q})$, we say that f is *integer-valued* at $\underline{\alpha}$ if $f(\underline{\alpha}) = (f(\alpha_p))_p \in \widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$. Then, analogously to Definition 3.6, we introduce the following:

Definition 6.1. For every subset $\underline{E}$ of $\widehat{\mathbb{Z}}$, the ring of integer-valued polynomials on $\underline{E}$ is

$$\mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}) = \{f \in \mathbb{Q}[X] \mid f(\underline{\alpha}) \in \widehat{\mathbb{Z}},\ \text{for all } \underline{\alpha} \in \underline{E}\}.$$

Notation. For each polynomial overring R of Int($\mathbb{Z}$), we consider the following set of finite adeles:

$$\underline{Z}_R \doteqdot \prod_p Z_p(R) \subseteq \prod_p \mathbb{Z}_p = \widehat{\mathbb{Z}}.$$

Clearly, $\underline{Z}_R = \{(\alpha_p)_p \in \widehat{\mathbb{Z}} \mid \mathfrak{M}_{p,\alpha_p} R \subsetneq R, \text{ for all } p \in \mathbb{P}\}$. With the previous notation, equality (6.1) may then be written:

$$R = \mathrm{Int}_{\mathbb{Q}}(\underline{Z}_R, \widehat{\mathbb{Z}}), \tag{6.2}$$

which means that every polynomial overring R of $\mathrm{Int}(\mathbb{Z})$ may be considered as the ring formed by polynomials which are integer-valued over a subset of $\widehat{\mathbb{Z}}$. Note that since, for each $p \in \mathbb{P}$, $Z_p(R)$ is a closed subset of $\mathbb{Z}_p$, and hence, is compact, the subset $\underline{Z}_R$ is also compact in $\widehat{\mathbb{Z}}$ where $\widehat{\mathbb{Z}} = \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is endowed with the product topology.

The following theorem is the globalized version of Theorem 3.11.

Theorem 6.2. *Let $\mathcal{R}$ be the set of polynomial overrings of* $\mathrm{Int}(\mathbb{Z})$*, and let $\mathcal{F}(\widehat{\mathbb{Z}})$ be the family of compact subsets of $\widehat{\mathbb{Z}}$ of the form $\prod_{p\in\mathbb{P}} F_p$ where F_p is a closed subset of $\mathbb{Z}_p$. The following maps which reverse the containments are inverse to each other*:

$$\varphi : \mathcal{R} \ni R \longmapsto \underline{Z}_R = \prod_{p\in\mathbb{P}} Z_p(R) \in \mathcal{F}(\widehat{\mathbb{Z}})$$

and

$$\psi : \mathcal{F}(\widehat{\mathbb{Z}}) \ni \underline{F} \longmapsto \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}}) \in \mathcal{R}.$$

Proof. By equality (6.2), $\psi \circ \varphi = id_{\mathcal{R}}$. Now consider $\varphi \circ \psi$. For every $\underline{F} = \prod_p F_p \in \mathcal{F}(\mathbb{Z}_p)$, one has

$$\begin{aligned}
\varphi(\psi(\underline{F})) &= \underline{Z}_{\mathrm{Int}_{\mathbb{Q}}(\underline{F},\mathbb{Z}_p)} \\
&= \Big\{(\alpha_p)_p \in \prod_p \mathbb{Z}_p \mid \text{ for all } f \in \mathrm{Int}(\underline{F}, \widehat{\mathbb{Z}}), \\
&\qquad\qquad\qquad \text{for all } p \in \mathbb{P}, f(\alpha_p) \in \mathbb{Z}_p\Big\} \\
&= \Big\{(\alpha_p)_p \in \prod_p \mathbb{Z}_p \mid \mathrm{Int}(\underline{F}, \widehat{\mathbb{Z}}) \subseteq V_{p,\alpha_p}, \text{ for all } p \in \mathbb{P}\Big\},
\end{aligned}$$

which is equal to $\underline{F}$ by Proposition 4.3. ∎

Remark 6.3. Let $\underline{F}$ be a generic compact subset of $\widehat{\mathbb{Z}}$, and consider the following ring of integer-valued polynomials:

$$R = \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}}).$$

For each $p \in \mathbb{P}$, let $\pi_p : \widehat{\mathbb{Z}} \to \mathbb{Z}_p$ be the canonical projection. Then, for each $f \in \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}})$, and for each $\alpha_p \in \pi_p(\underline{F})$, $p \in \mathbb{P}$, we have $f(\alpha_p) \in \mathbb{Z}_p$. Consequently, $f \in \mathrm{Int}_{\mathbb{Q}}(\pi_p(\underline{F}), \mathbb{Z}_p)$. Therefore,

$$R \subseteq \bigcap_p \mathrm{Int}_{\mathbb{Q}}(\pi_p(\underline{F}), \mathbb{Z}_p) = \mathrm{Int}_{\mathbb{Q}}\left(\prod_p \pi_p(\underline{F}), \prod_p \mathbb{Z}_p\right) \subseteq \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}}) = R,$$

since $\underline{F} \subseteq \prod_p \pi_p(\underline{F})$. Finally,

$$\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}}) = \mathrm{Int}_{\mathbb{Q}}\left(\prod_p \pi_p(\underline{F}), \widehat{\mathbb{Z}}\right).$$

Since the projections π_p are closed maps, each $\pi_p(\underline{F})$ is a closed subset of $\mathbb{Z}_p$. Therefore, by Theorem 6.2, we have proved that $\underline{Z}_{\mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}})} = \prod_p \pi_p(\underline{F})$, which is an element of $\mathcal{F}(\widehat{\mathbb{Z}})$. In other words, $\prod_p \pi_p(\underline{F})$ is precisely equal to the subset of $\widehat{\mathbb{Z}}$ of those $\underline{\alpha}$ such that $f(\underline{\alpha}) \in \widehat{\mathbb{Z}}$, for each $f \in \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}})$. Generalizing the terminology of subsection 3.2, one could say that the *polynomial closure* of $\underline{F} \subseteq \widehat{\mathbb{Z}}$ is the compact subset $\prod_p \pi_p(\underline{F})$.

Remark 6.4. Let $E \subseteq \mathbb{Z}$ be an infinite subset. We denote by $\widehat{E}$ the topological closure of E in $\widehat{\mathbb{Z}} = \prod_{p \in \mathbb{P}} \mathbb{Z}_p$, by E_p the topological closure of E in $\mathbb{Z}_p$, for each prime p and by $\underline{E}$ the direct product $\prod_{p \in \mathbb{P}} E_p \subseteq \widehat{\mathbb{Z}}$. By Remark 6.3, $\underline{E}$ is the polynomial closure of E in $\widehat{\mathbb{Z}}$. It is easy to see that

$$\widehat{E} \subseteq \underline{E},$$

since the canonical embedding of E into $\widehat{\mathbb{Z}}$ is contained in $\prod_{p \in \mathbb{P}} E$ (in fact, strictly contained if $\mathrm{Card}\,(E) \neq 1$) whose topological closure is $\underline{E}$. Moreover, for each prime p, $\pi_p(\widehat{E}) = \pi_p(\underline{E}) = E_p$. In particular,

$$\mathrm{Int}(E, \mathbb{Z}) = \mathrm{Int}_{\mathbb{Q}}(\widehat{E}, \widehat{\mathbb{Z}}) = \mathrm{Int}_{\mathbb{Q}}(\underline{E}, \widehat{\mathbb{Z}}).$$

We know that, locally, for a subset $E \subseteq \mathbb{Z}_{(p)}$, the polynomial closure of E in $\mathbb{Z}_p$ coincides with its topological closure in $\mathbb{Z}_p$ (Proposition 3.10 and Theorem 3.11). The global situation can be different: as the next example shows, in general, $\widehat{E}$ can be strictly contained in $\underline{E}$. By definition, an element $\underline{\alpha} \in \underline{E}$ has the property that, for each finite set of primes $\{p_1, \ldots, p_k\}$ and finite set of non-negative integers $\{k_1, \ldots, k_s\}$, there exists $a_i \in E$ such that $a_i \equiv \alpha_{p_i} \pmod{p_i^{k_i}}$, for $i = 1, \ldots, s$. In order for $\underline{\alpha}$ to belong to $\widehat{E}$, there should exist $a \in E$ which is a simultaneous solution of all the previous congruences.

Example 6.5. Let $E = \mathbb{Z} \setminus \{-7 + 8 \cdot 9k \mid k \in \mathbb{Z}\}$. It is easy to see that E is dense in $\mathbb{Z}_p$ for each prime p, so the polynomial closure of E in $\widehat{\mathbb{Z}}$ is equal to $\widehat{\mathbb{Z}}$. However, since there is no $a \in E$ such that the following congruences are satisfied:

$$a \equiv 1 \pmod{8}, \qquad a \equiv 2 \pmod{9},$$

it follows that $\widehat{E} \subsetneq \underline{E}$.

Example 6.6. Let us consider the ring $\mathrm{Int}(\mathbb{Z})$. Since

$$\mathrm{Int}(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} \mathrm{Int}(\mathbb{Z}_{(p)}),$$

by (3.6), we have

$$\mathrm{Int}(\mathbb{Z}) = \bigcap_{p \in \mathbb{P}} \mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_p) = \mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}, \widehat{\mathbb{Z}}) \doteqdot \mathrm{Int}_{\mathbb{Q}}(\widehat{\mathbb{Z}}).$$

Note the analogy of the previous equation with (3.6).

Corollary 6.7. *For each $p \in \mathbb{P}$, let $R(p)$ be a polynomial overring of $\mathrm{Int}_{\mathbb{Q}}(\mathbb{Z}_{(p)})$. Then, there exists a polynomial overring R of $\mathrm{Int}(\mathbb{Z})$ such that $R_{(p)} = R(p)$ for each $p \in \mathbb{P}$.*

Proof. By Theorem 3.11, the choice for each $p \in \mathbb{P}$ of an overring $R(p)$ of $\mathrm{Int}(\mathbb{Z}_{(p)})$ corresponds to a closed subset $F_p = Z_p(R(p)) \subseteq \mathbb{Z}_p$. Moreover, $R(p) = \mathrm{Int}_{\mathbb{Q}}(F_p, \mathbb{Z}_p)$. Let $\underline{F} = \prod_{p \in \mathbb{P}} F_p \subseteq \widehat{\mathbb{Z}}$, and let

$$R = \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}}) = \bigcap_{p \in \mathbb{P}} R(p).$$

We claim that R is the desired polynomial overring of Int($\mathbb{Z}$). Since $R_{(p)}$ and $R(p)$ are elements of the family $\mathcal{R}_p$, by Theorem 3.11, it is sufficient to show that $Z_p(R_{(p)}) = F_p$. Proposition 4.3 and (5.2) allow us to conclude. $\square$

Remark 6.8. With our interpretation in terms of finite adeles, we may formulate Theorem 4.6 in another way. Let R be a polynomial overring of Int($\mathbb{Z}$). Let $R_{\mathcal{P},\underline{Z}_R}$ be a representation of R as an intersection of valuation overrings (Proposition 5.1). Theorem 4.6 says that, for every $q \in \mathcal{P}$, V_q is superfluous if and only if there exists $\underline{\alpha} \in \underline{Z}_R$ such that $q(\underline{\alpha})$ is not invertible in $\mathcal{A}_f(\mathbb{Q})$. The valuation domain V_q is surperfluous in all representations of R if and only if $q(\underline{Z}_R) \not\subseteq \mathcal{I}_f(\mathbb{Q})$.

To end our study, we now show under which conditions a polynomial overring R of Int($\mathbb{Z}$) is of the simple form Int$(E, \mathbb{Z})$ where E is a subset of $\mathbb{Z}$.

Corollary 6.9. *A polynomial overring R of* Int($\mathbb{Z}$) *is a ring of integer-valued polynomials on a subset of $\mathbb{Z}$ if and only if, for each prime p, the subset $E = \cap_p(Z_p(R) \cap \mathbb{Z})$ is dense in $Z_p(R)$ for the p-adic topology. If this condition holds, then $R = \mathrm{Int}(E, \mathbb{Z})$.*

Proof. Clearly, $E = \{a \in \mathbb{Z} \mid R(a) \subseteq \mathbb{Z}\}$ and, if R is a ring of integer-valued polynomials on a subset of $\mathbb{Z}$, the subset E is convenient. Moreover, the equality $R = \mathrm{Int}(E, \mathbb{Z})$ holds if and only if both rings have the same localizations at each prime p. For every p, $\mathrm{Int}(E, \mathbb{Z})_{(p)} = \mathrm{Int}_{\mathbb{Q}}(E, \mathbb{Z}_p)$ and $R_{(p)} = \mathrm{Int}_{\mathbb{Q}}(Z_p(R), \mathbb{Z}_p)$ by Proposition 3.7. Thus, by Proposition 3.10, both localizations are equal if and only if E is dense in $Z_p(R)$. $\square$

Example 6.10. For each p, let us consider the following closed subset of $\mathbb{Z}_p$: $F_p = \{p\} \cup (\mathbb{Z}_p \setminus p\mathbb{Z}_p)$. Let $\underline{F} = \prod_p F_p \subseteq \widehat{\mathbb{Z}}$ and $R = \mathrm{Int}_{\mathbb{Q}}(\underline{F}, \widehat{\mathbb{Z}})$. Does there exist an $E \subseteq \mathbb{Z}$ such that $R = \mathrm{Int}(E, \mathbb{Z})$? Yes, $R = \mathrm{Int}(\mathbb{P}, \mathbb{Z})$ since, for each p, the topological closure of $\mathbb{P}$ in $\mathbb{Z}_p$ is F_p. Actually, the subset E suggested in Corollary 6.9 is $\mathbb{P} \cup \{\pm 1\}$, namely, the polynomial closure of $\mathbb{P}$ in $\mathbb{Z}$ (about $\mathrm{Int}(\mathbb{P}, \mathbb{Z})$ see **[2]**).

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