JOURNAL OF COMMUTATIVE ALGEBRA
Volume 7, Number 3, Fall 2015

MORE PROPERTIES OF ALMOST COHEN-MACAULAY RINGS

CRISTODOR IONESCU

ABSTRACT. Some interesting properties of almost Cohen-Macaulay rings are investigated, and a Serre type property connected with this class of rings is studied.

1. Introduction. A flaw in the first edition of [**5**] in the chapter dedicated to Cohen-Macaulay rings was corrected in the second edition. This led to the study of the so-called almost Cohen Macaulay rings, first by Han [**1**] and later by Kang [**2, 3**]. Since the first of these papers is written in Chinese, the others two are the main references for the subject.

Remark 1.1. Let A be a commutative Noetherian ring, $P \in \operatorname{Spec}(A)$ and $M \neq 0$ a finitely generated A-module. Then $\operatorname{depth}_P(M) \leq \operatorname{depth}_{PA_P} M_P$.

Definition 1.2. (cf. [**1, 2**]). Let A be a commutative Noetherian ring. A finitely generated A-module $M \neq 0$ is called *almost Cohen-Macaulay* if $\operatorname{depth}_P M = \operatorname{depth}_{PA_P} M_P$, for any $P \in \operatorname{Supp}(M)$. A is called an *almost Cohen-Macaulay ring* if it is an almost Cohen-Macaulay A-module, that is, if for any $P \in \operatorname{Spec}(A)$, $\operatorname{depth}_P A = \operatorname{depth}_{PA_P} A_P$.

Several properties of almost Cohen-Macaulay rings are proved in [**2**], and several interesting examples are given in [**3**]. In the following, we are trying to complete the results in [**2**] and to introduce a Serre-type condition that we call (C_k), for any $k \in \mathbb{N}$ condition that is to be to almost Cohen-Macaulay rings what the classical Serre condition (S_k) is to Cohen-Macaulay rings.

2010 AMS *Mathematics subject classification.* Primary 13C14, 13H10.
Keywords and phrases. almost Cohen-Macaulay modules, Serre property.
Received by the editors on March 3, 2014, and in revised form on May 16, 2014.
DOI:10.1216/JCA-2015-7-3-363

2. Properties of almost Cohen-Macaulay rings. All rings considered will be commutative and with unit. We start by reminding the reader about some basic properties of almost Cohen-Macaulay rings.

Remark 2.1. Let A be a Noetherian ring. Then:

(a) A is almost Cohen-Macaulay if and only if $\mathrm{ht}\,(P) \leq 1 + \mathrm{depth}_P A$, for all $P \in \mathrm{Spec}\,(A)$ ([**2,** 1.5]);
(b) A is almost Cohen-Macaulay if and only if A_P is almost Cohen-Macaulay for any $P \in \mathrm{Spec}\,(A)$ if and only if A_Q is almost Cohen-Macaulay for any $Q \in \mathrm{Max}\,(A)$ if and only if $\mathrm{ht}\,(Q) \leq 1 + \mathrm{depth} A_Q$ for any $Q \in \mathrm{Max}\,(A)$ ([**2,** 2.6]);
(c) If A is local, it follows from b) that A is almost Cohen-Macaulay if and only if $\dim(A) \leq 1 + \mathrm{depth}(A)$.

Our first result is a stronger formulation of [**2,** 2.10] and deals with the behavior of almost Cohen-Macaulay rings with respect to flat morphisms.

Proposition 2.2. *Let $u : (A, m) \to (B, n)$ be a local flat morphism of Noetherian local rings.*

(a) *If B is almost Cohen-Macaulay, then A and B/mB are almost Cohen-Macaulay.*
(b) *If A and B/mB are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then B is almost Cohen-Macaulay.*

Proof.

(a) We have

$$\begin{aligned}\dim(A) &= \dim(B) - \dim(B/mB) \leq 1 + \mathrm{depth} B - \dim(B/mB)\\ &\leq 1 + \mathrm{depth} B - \mathrm{depth}(B/mB) = 1 + \mathrm{depth} A.\end{aligned}$$

We also have

$$\begin{aligned}\dim(B/mB) - \mathrm{depth}(B/mB) &= (\dim(B) - \mathrm{depth} B)\\ &\quad - (\dim(A) - \mathrm{depth} A)\\ &\leq 1 - (\dim(A) - \mathrm{depth} A) \leq 1.\end{aligned}$$

(b) Since u is flat, we have

$$\begin{aligned}\dim(B) &= \dim(A) + \dim(B/mB) \leq 1 + \operatorname{depth}(A) + \operatorname{depth}(B/mB)\\ &= 1 + \operatorname{depth}(B).\end{aligned}$$ □

Question 2.3. *We do not know of any example of a local flat morphism of Noetherian local rings $u : (A, m) \to (B, n)$ such that A and B/mB are almost Cohen-Macaulay and B is not almost Cohen-Macaulay.*[1]

Corollary 2.4. *Let A be a Noetherian local ring, $I \neq A$ an ideal contained in the Jacobson radical of A and $\widehat{A}$ the completion of A in the I-adic topology. Then A is almost Cohen-Macaulay if and only if $\widehat{A}$ is almost Cohen-Macaulay.*

Proof. Since I is contained in the Jacobson radical of A, the canonical morphism $A \to \widehat{A}$ is faithfully flat and $\operatorname{Max}(A) \cong \operatorname{Max}(\widehat{A})$. Moreover, if $m \in \operatorname{Max}(A)$ and $\widehat{m}$ is the corresponding maximal ideal of $\widehat{A}$, the closed fiber of the morphism $A_m \to \widehat{A}_{\widehat{m}}$ is a field. Now apply Proposition 2.2. □

Corollary 2.5. (see [**2,** 1.6]). *Let A be a Noetherian ring and $n \in \mathbb{N}$. Then A is almost Cohen-Macaulay if and only if $A[[X_1, \ldots, X_n]]$ is almost Cohen-Macaulay.*

Proof. Suppose that A is almost Cohen-Macaulay. We may clearly assume that A is local and $n = 1$. By [**2,** 1.3], we get that $A[X]_{(X)}$ is almost Cohen-Macaulay. Now apply Corollary 2.4. The converse is clear. □

For the next corollary, we need some notation.

Notation 2.6. If $\mathbf{P}$ is a property of Noetherian local rings, we denote by $\mathbf{P}(A) := \{Q \in \operatorname{Spec}(A) \mid A_Q \text{ has the property } \mathbf{P}\}$ and by $\mathbf{NP}(A) := \{Q \in \operatorname{Spec}(A) \mid A_Q \text{ does not have the property } \mathbf{P}\} = \operatorname{Spec}(A) \setminus \mathbf{P}(A)$.

Definition 2.7. Let A be a Noetherian ring. According to Notation 2.6, the set

$$\mathbf{aCM}\,(A) := \{P \in \operatorname{Spec}(A) \mid A_P \text{ is almost Cohen-Macaulay}\}$$

is called the *almost Cohen-Macaulay locus* of A.

Corollary 2.8. *Let $u : A \to B$ be a morphism of Noetherian local rings and $\varphi : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ the induced morphism on the spectra. If the fibers of u are Cohen-Macaulay, then $\varphi^{-1}(\mathbf{aCM}\,(A)) = \mathbf{aCM}\,(B)$.*

Proof. Obvious from Proposition 2.2. □

In Cohen-Macaulay rings, chains of prime ideals behave very well, in the sense that Cohen-Macaulay rings are universally catenary (see [**5**]). This is no longer the case for almost Cohen-Macaulay rings.

Example 2.9. There exists a local almost Cohen-Macaulay ring which is not catenary.

Proof. Indeed, by [**2,** Example 2], any Noetherian normal integral domain of dimension 3 is almost Cohen-Macaulay. In [**6**], such a ring which is not catenary is constructed. □

The next result shows that some of the formal fibers of almost Cohen-Macaulay rings are almost Cohen-Macaulay. A stronger fact will be proved in Proposition 2.13.

Proposition 2.10. *Let A be a Noetherian local almost Cohen-Macaulay ring, $P \in \operatorname{Spec}(A), Q \in \operatorname{Ass}(\widehat{A}/P\widehat{A})$. Then $\widehat{A}_Q/P\widehat{A}_Q$ is almost Cohen-Macaulay.*

Proof. We have

$$\begin{aligned}\dim(\widehat{A}_Q/P\widehat{A}_Q) &= \dim \widehat{A}_Q - \dim A_P\\ &\leq \operatorname{depth}\widehat{A}_Q + 1 - \dim A_P\\ &\leq \operatorname{depth}\widehat{A}_P + 1 - \dim A_P\\ &= \operatorname{depth}(\widehat{A}_Q/P\widehat{A}_Q) + 1.\end{aligned}$$

□

The following result shows that the almost Cohen-Macaulay property is preserved by tensor products and finite field extensions.

Proposition 2.11. *Let k be a field and A and B two k-algebras such that $A \otimes_k B$ is a Noetherian ring. If A and B are almost Cohen-Macaulay and one of them is Cohen-Macaulay, then $A \otimes_k B$ is almost Cohen-Macaulay.*

Proof. Let $P \in \operatorname{Spec}(A)$. We have a flat morphism $B \to B \otimes_k k(P)$. Let $Q \in \operatorname{Spec}(B)$. Set $T := A/P \otimes_k B/Q = A \otimes_k B/(P \otimes_k B + A \otimes_k Q)$. Then $k(P) \otimes_k k(Q)$ is a ring of fractions of T, hence Noetherian by assumption. By [**7,** Proposition 5], it follows that $k(P) \otimes_k k(Q)$ is locally a complete intersection. Now let $Q \in \operatorname{Spec}(B)$ and $P = Q \cap A$. By the above, the flat local morphism $A_P \to (B \otimes_k k(P))_Q$ has a complete intersection closed fiber; hence, the ring $(B \otimes_k k(P))_Q$ is almost Cohen-Macaulay by Proposition 2.2. Now consider the flat morphism $A \to A \otimes_k B$ and let $Q \in \operatorname{Spec}(A \otimes_k B)$ and $P = Q \cap A$. Then the flat local morphism $A_P \to (A \otimes_k B)_Q$ has a complete intersection closed fiber, whence $(A \otimes_k B)_Q$ is almost Cohen-Macaulay. $\square$

Corollary 2.12. *Let k be a field, A a Noetherian k-algebra which is almost Cohen-Macaulay and L a finite field extension of k. Then $A \otimes_k L$ is almost Cohen-Macaulay.*

As for the Cohen-Macaulay property, the formal fibers of factorizations of almost Cohen-Macaulay rings are almost Cohen-Macaulay.

Proposition 2.13. *Let B be a local almost Cohen-Macaulay ring, I an ideal of B and $A = B/I$. Then the formal fibers of A are almost Cohen-Macaulay.*

Proof. We have $\widehat{A} = \widehat{B} \otimes_B A = \widehat{B}/I\widehat{B}$; hence, the formal fibers of A are exactly the formal fibers of B in the prime ideals of B containing I. Let P be such a prime ideal, let $S = B \setminus P$ and let $C := S^{-1}(\widehat{B}/I\widehat{B})$. Also let $Q \in \operatorname{Spec}(C)$. There exists $Q' \in \operatorname{Spec}(\widehat{B})$ such that $Q = Q'C$ and $Q' \cap B = P$. Thus, we have a local flat morphism $B_Q \to \widehat{B}_{Q'}$. But B is almost Cohen-Macaulay; hence, $\widehat{B}_{Q'}$

and consequently $C_Q \cong \widehat{B}_{Q'}/P\widehat{B}_{Q'}$ are almost Cohen-Macaulay, by Proposition 2.2. □

3. The property (C_n)**.** Recall that, given a natural number n, a Noetherian ring A is said to have Serre property (S_n) if $\operatorname{depth}(A_P) \geq \min(\operatorname{ht} P, n)$ for any prime ideal $P \in \operatorname{Spec}(A)$. Moreover, A is Cohen-Macaulay if and only if A has the property (S_n) for any $n \in \mathbb{N}$ (see [**5,** (17.I)]). We will try to characterize almost Cohen-Macaulay rings in a similar way.

Definition 3.1. Let $n \in \mathbb{N}$ be a natural number. We say that a Noetherian ring A has the property (C_n) if $\operatorname{depth}(A_P) \geq \min(\operatorname{ht} P, n) - 1$, for all $P \in \operatorname{Spec}(A)$.

Remark 3.2. (a) It is clear that $(C_n) \Rightarrow (C_{n-1})$ and that $(S_n) \Rightarrow (C_n)$, for all $n \in \mathbb{N}$.
(b) It is also clear that if A has (C_n), then A_P has (C_n), for all $P \in \operatorname{Spec}(A)$.

Theorem 3.3. *A Noetherian ring A is almost Cohen-Macaulay if and only if A has the property (C_n) for every $n \in \mathbb{N}$.*

Proof. Assume that A is almost Cohen-Macaulay, and let $P \in \operatorname{Spec}(A)$. Then A_P is almost Cohen-Macaulay; hence, $\operatorname{depth}(A_P) \geq \operatorname{ht}(P) - 1$. If $n \geq \operatorname{ht}(P)$, then $\min(\operatorname{ht}(P), n) = \operatorname{ht}(P)$. Hence, $\operatorname{depth}(A_P) \geq \min(n, \operatorname{ht}(P)) - 1$. If $n < \operatorname{ht}(P)$, then $\min(n, \operatorname{ht}(P)) = n$, so that $\operatorname{depth}(A_P) \geq \operatorname{ht}(P) - 1 > n - 1 = \min(\operatorname{ht}(P), n) - 1$.

For the converse, let $P \in \operatorname{Spec}(A)$, $\operatorname{ht}(P) = l$. Then

$$\operatorname{depth}(A_P) \geq \min(l, \operatorname{ht}(P)) - 1 = \operatorname{ht}(P) - 1. \qquad \square$$

Proposition 3.4. *Let $k \in \mathbb{N}$. A Noetherian ring A has the property (C_k) if and only if A_P is almost Cohen-Macaulay for any $P \in \operatorname{Spec}(A)$ with* $\operatorname{depth}(A_P) \leq k - 2$.

Proof. Let $P \in \operatorname{Spec}(A)$ be such that $\min(k, \operatorname{ht}(P)) - 1 \leq \operatorname{depth}(A_P) \leq k - 2$. If $\operatorname{ht}(P) \leq k$, then $\operatorname{depth}(A_P) \geq \operatorname{ht}(P) - 1$. And, if $\operatorname{ht}(P) > k$, then it follows that $k - 2 > \operatorname{depth}(A_P) \geq k - 1$. This is a contradiction.

Conversely, let $P \in \operatorname{Spec}(A)$. If $\operatorname{depth}(A_P) \leq k-2$, then A_P is almost Cohen-Macaulay, hence $\operatorname{ht}(P) - 1 \leq \operatorname{depth}(A_P) \leq k-2$. Thus, $\min(\operatorname{ht}(P), k) = \operatorname{ht}(P)$, whence $\operatorname{depth}(A_P) \geq \min(k, \operatorname{ht}(P)$. If $k-2 < \operatorname{depth}(A_P)$, then $\operatorname{ht}(P) > k-2$. Hence, $\operatorname{depth}(A_P) \geq \min(k, \operatorname{ht}(P)) - 1$. □

Proposition 3.5. *Let A be a Noetherian ring, $k \in \mathbb{N}$ and $x \in A$ a non zero divisor. If A/xA has the property (C_k), then A has the property (C_k).*

Proof. Let $Q \in \operatorname{Spec}(A)$ be such that $\operatorname{depth}(A_Q) = n \leq k-2$. If $x \in Q$, then $\operatorname{depth}(A/xA)_Q = n-1 \leq k-3$. Then $\operatorname{ht}(Q/xA) \leq n-1+1 = n$; hence, $\operatorname{ht}(Q) \leq n+1 = \operatorname{depth} A_Q + 1$. If $x \notin Q$, let $P \in \operatorname{Min}(Q+xA)$. Then $(Q+xA)A_P$ is PA_P-primary and $\operatorname{depth}(A_P) \leq \operatorname{depth}(A_Q)+1 = n+1$. Then $\operatorname{depth}(A/xA)_Q = n-1$; hence, $\operatorname{ht}(P/xA) \leq n$. It follows that $\operatorname{ht}(P) \leq n+1 = \operatorname{depth}(A_P)+1$. □

Definition 3.6. We say that a property $\mathbf{P}$ of Noetherian local rings satisfies Nagata's criterion (NC) if the following holds: if A is a Noetherian ring such that, for every $P \in \mathbf{P}(A)$, the set $\mathbf{P}(A/P)$ contains a non-empty open set of $\operatorname{Spec}(A/P)$, then $\mathbf{P}(A)$ is open in $\operatorname{Spec}(A)$.

An interesting study of the Nagata criterion is performed in [**4**].

Theorem 3.7. *Let $k \in \mathbb{N}$. Property (C_k) satisfies* (NC).

Proof. Let $Q \in C_k(A)$. Then $\operatorname{depth}(A_Q) \geq \min(k, \operatorname{ht}(Q)) - 1$.

Case a). $\operatorname{ht}(Q) \leq k$. Then $\min(k, \operatorname{ht}(Q)) = \operatorname{ht}(Q)$; hence, $\operatorname{depth}(A_Q) + 1 \geq \operatorname{ht}(Q)$ and A_Q is almost Cohen-Macaulay. Let $f \in A \setminus Q$ be such that

$$\dim(A_P) = \dim(A_Q) + \dim(A_P/QA_P)$$

and

$$\operatorname{depth}(A_P) = \operatorname{depth}(A_Q) + \operatorname{depth}(A_P/QA_P)$$

for any $P \in D(f) \cap V(Q) \cap NT_k(A)$. Then

$$\operatorname{depth}(A_P) \ngeq \min(k, \operatorname{ht}(P)) - 1.$$

Case a1). $\operatorname{ht}(P) \leq k$. Then $\min(k, \operatorname{ht}(P)) = \operatorname{ht}(P)$; hence, $\operatorname{depth}(A_P) + 1 < \operatorname{ht}(P)$. Then

$$\begin{aligned}\operatorname{depth}(A_P/QA_P) + 1 &= \operatorname{depth}(A_P) - \operatorname{depth}(A_Q) + 1 \\ &< \operatorname{ht}(P) - \operatorname{depth}(A_Q) \leq \operatorname{ht}(P) - \operatorname{ht}(Q) + 1.\end{aligned}$$

Then $\operatorname{depth}(A_P/QA_P) < \dim(A_P/QA_P) = \dim(A_P) - \dim(A_Q)$, and it follows that A_P/QA_P is not (C_k).

Case a2). $\operatorname{ht}(P) > k$. Then $\min(k, \operatorname{ht}(P)) = k$; hence, $\operatorname{depth}(A_P) < k - 1$. It follows that

$$\begin{aligned}\operatorname{depth}(A_P/QA_P) &= \operatorname{depth}(A_P) - \operatorname{depth}(A_Q) \\ &< k - 1 + 1 - \operatorname{ht}(Q) = k - \operatorname{ht}(Q).\end{aligned}$$

This implies that A_P/QA_P is not (C_k).

Case b). $\operatorname{ht}(Q) > k$. Then $\min(k, \operatorname{ht}(Q)) = k$ and $\operatorname{depth}(A_Q) + 1 \geq k$. Since $\operatorname{ht}(P) > k$, it follows that $\min(k, \operatorname{ht}(P)) = k$ and $\operatorname{depth}(A_P) + 1 < k$. Let $x_1, \ldots, x_r$ be an A_Q-regular sequence. Then there exists $f \in A \setminus Q$ such that $x_1, \ldots, x_r$ is A_f-regular. If $P \in D(f) \cap V(Q)$, it follows that A_P is (C_k). □

Corollary 3.8. *The property almost Cohen-Macaulay satisfies* (NC).

Theorem 3.9. *Let A be a quasi-excellent ring and $k \in \mathbb{N}$. Then $C_k(A)$ and* $\mathbf{aCM}(A)$ *are open in the Zariski topology of* $\operatorname{Spec}(A)$.

Proof. Let $P \in \operatorname{Spec}(A)$. Then $\mathbf{aCM}(A/P)$ and $C_k(A/P)$ contain the non-empty open set $\mathbf{Reg}(A/P) = \{P \in \operatorname{Spec}(A) \mid A_P \text{ is regular}\}$. Now apply Theorems 3.7 and 3.8. □

Corollary 3.10. *Let A be a complete semilocal ring and $k \in \mathbb{N}$. Then $C_k(A)$ and $aCM(A)$ are open in the Zariski topology of* $\operatorname{Spec}(A)$.

Corollary 3.11. *Let A be a Noetherian local ring with Cohen-Macaulay formal fibers. Then* $\mathbf{aCM}(A)$ *is open.*

Proof. Follows from Corollaries 2.8 and 3.10. □

Proposition 3.12. *Let $u : A \to B$ be a flat morphism of Noetherian rings and $k \in \mathbb{N}$. If B has (C_k), then A has (C_k).*

Proof. We may assume that A and B are local rings and that u is local. Let $P \in \mathrm{Spec}\,(A)$ and $Q \in \mathrm{Min}\,(PB)$. Then $\dim(B_Q/PB_Q) = 0$; hence,

$$\begin{aligned}\mathrm{depth}(A_P) &= \mathrm{depth}(B_Q) \geq \min(k, \dim(B_Q)) - 1 \\ &= \min(k, \dim(A_P)) - 1.\end{aligned}$$ □

Proposition 3.13. *Let $u : A \to B$ be a flat morphism of Noetherian rings and $k \in \mathbb{N}$.*

a) *If A has (C_k) and all the fibers of u have (S_k), then B has (C_k).*
b) *If A has (S_k) and all the fibers of u have (C_k), then B has (C_k).*

Proof. a) Let $Q \in \mathrm{Spec}\,(B)$, $P = Q \cap A$. Then, by flatness, we have

$$\dim(B_Q) = \dim(A_P) + \dim(B_Q/PB_Q),$$
$$\mathrm{depth}(B_Q) = \mathrm{depth}(A_P) + \mathrm{depth}(B_Q/PB_Q).$$

By assumption, we have

$$\mathrm{depth}(A_P) \geq \min(k, \mathrm{ht}\,(P)) - 1,$$
$$\mathrm{depth}(B_Q/PB_Q) \geq \min(k, \dim(B_Q/PB_Q)).$$

Hence, we have

$$\begin{aligned}\mathrm{depth}(B_Q) &= \mathrm{depth}(A_P) + \mathrm{depth}(B_Q/PB_Q) \\ &\geq \min(k, \mathrm{ht}\,(P)) - 1 \\ &\quad + \min(k, \dim(B_Q/PB_Q)) \\ &\quad = \min(k, \mathrm{ht}\,(B_Q)) - 1.\end{aligned}$$

b) The proof is the same. □

As a corollary we get a new proof of a previous result.

Corollary 3.14. *Let $u : A \to B$ be a flat morphism of Noetherian rings.*

a) *If B is almost Cohen-Macaulay, then A is almost Cohen-Macaulay.*

b) *If A is almost Cohen-Macaulay and the fibers of u are Cohen-Macaulay, then B is almost Cohen-Macaulay.*

Example 3.15. Let k be a field, and let X_0, X_1, X_2, Y_1, Y_2 be indeterminates. Set $B = k[[X_0, X_1, X_2]]/(X_0) \cap (X_0, X_1)^2 \cap (X_0, X_1, X_2)^3$ and $A := B[[Y_1, Y_2]]$. It is easy to see that A is a Noetherian local ring with $\dim(A) = 5$, $\operatorname{depth}(A) = 2$. It is also not difficult to see that A has property (C_3) and not property (C_4). Other similar examples can easily be constructed.

Example 3.16. Let k be a field, X and Y indeterminates and consider the ring $A = k[[X, Y]]/(X^2, XY)$. Then A has (C_2) and not (S_2).

ENDNOTES

1. An example was given by M. Tabaâ, *Sur le produit tensoriel d'algèbres*, preprint, `arXiv:1304.5395`.

REFERENCES

1. Y. Han, D-*rings*, Acta Math. Sinica **4** (1998), 1047–1052.

2. M.C. Kang, *Almost Cohen-Macaulay modules*, Comm. Alg. **29** (2001), 781–787.

3. ______, *Addendum to "Almost Cohen-Macaulay modules,"* Comm. Alg. **30** (2002), 1049–1052.

4. C. Massaza, *Su alcuni aspetti assiomatici del criterio di Nagata*, Rend. Circ. Mat. Palermo **30** (1981), 365–377.

5. H. Matsumura, *Commutative algebra*, 2nd edition, Benjamin Cummings, 1980.

6. T. Ogoma, *Non-catenary pseudo-geometric normal rings*, Japan J. Math. **6** (1980), 147–163.

7. M. Tousi and S. Yassemi, *Tensor products of some special rings*, J. Algebra **268** (2003), 672–675.

Institute of Mathematics, Simion Stoilow of the Romanian Academy, P.O. Box 1-764, RO 014700 Bucharest, Romania

Email address: cristodor.ionescu@imar.ro