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# MODULES SATISFYING THE PRIME AND MAXIMAL RADICAL CONDITIONS

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ABSTRACT. In this paper, we introduce and study $\mathbb{P}$-radical and $\mathbb{M}$-radical modules over commutative rings. We say that an $R$-module $M$ is *$\mathbb{P}$-radical* whenever $M$ satisfies the equality $(\sqrt[p]{\mathcal{P}M} : M) = \sqrt{\mathcal{P}}$ for every prime ideal $\mathcal{P} \supseteq Ann(\mathcal{P}M)$, where $\sqrt[p]{\mathcal{P}M}$ is the intersection of all prime submodules of $M$ containing $\mathcal{P}M$. Among other results, we show that the class of $\mathbb{P}$-radical modules is wider than the class of primeful modules (introduced by Lu **[19]**). Also, we prove that any projective module over a Noetherian ring is $\mathbb{P}$-radical. This also holds for any arbitrary module over an Artinian ring. Furthermore, we call an $R$-module $M$ by *$\mathbb{M}$-radical* if $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$, for every maximal ideal $\mathcal{M}$ containing $\mathrm{Ann}\,(M)$. We show that the conditions $\mathbb{P}$-radical and $\mathbb{M}$-radical are equivalent for all $R$-modules if and only if $R$ is a Hilbert ring. Also, two conditions primeful and $\mathbb{M}$-radical are equivalent for all $R$-modules if and only if $\dim(R) = 0$. Finally, we remark that the results of this paper will be applied in a subsequent work of the authors to construct a structure sheaf on the spectrum of $\mathbb{P}$-radical modules in the point of algebraic geometry view.

**1. Some preliminaries.** For an arbitrary commutative ring $R$, the associated *spectrum* $\mathrm{Spec}\,(R)$ of $R$ is the family of its prime ideals. Let $V(I) = \{\mathcal{P} \in \mathrm{Spec}\,(R) : I \subseteq \mathcal{P}\}$ for each ideal $I$ of $R$. One can equip $\mathrm{Spec}\,(R)$ with the so-called *Zariski topology* by considering the sets of the form $V(I)$ as the closed subsets of this topology. Moreover, it is proved that the collection $\{D(f),\ f \in R\}$ is a basis for this

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topology where $D(f) = \{\mathcal{P} \in \mathrm{Spec}\,(R) : f \notin \mathcal{P}\} = \mathrm{Spec}\,(R) \setminus V(f)$ and $V(f) = V(Rf)$ (see [**2, 15**]).

All rings in this article are commutative with identity and modules unital. For a ring $R$, we denote by $\dim(R)$ the *classical Krull dimension* of $R$, and for a submodule $N$ of an $R$-module $M$ we denote the annihilator of the factor module $M/N$ by $(N : M)$, i.e., $(N : M) = \{r \in R \mid rM \subseteq N\}$. $M$ is called *faithful* if $(0 : M) = 0$. A non-zero $R$-module $M$ is called *prime* if the equality $rm = 0$ for $m \in M$; $r \in R$ implies that $m = 0$ or $rM = (0)$ (i.e., $r \in \mathrm{Ann}\,(M)$). We recall that a proper submodule $P$ of $M$ is a *prime submodule* if $M/P$ is a prime module (i.e., for every $r \in R$ and $m \in M$, if $rm \in P$, then $m \in P$ or $r \in (P : M)$). In this case, $\mathcal{P} = (P : M)$ is a prime ideal of $R$ and $M/P$ is a torsion free $R/\mathcal{P}$-module. This motivates one to call the prime submodule $P$ by $\mathcal{P}$-*prime submodule*.

This notion of prime submodule was firstly introduced and systematically studied by Dauns [**9**] and Feller and Swokowski [**12**] and recently has received a good deal of attention from several authors (see for instance, [**4, 6, 7, 16, 20, 21, 22, 24**]). Motivated by algebraic geometry, the set of all prime submodules of $M$ is called the *spectrum* of $M$ and is denoted by $\mathrm{Spec}\,(M)$. Similar to the case of commutative rings, the sets of the form:

$$V(N) = \{P \in \mathrm{Spec}\,(M) \mid (N : M) \subseteq (P : M)\},$$

for any arbitrary submodule $N$ of $M$ allow one to associate the *Zariski topology* on the collection $\mathrm{Spec}\,(M)$, in which each of the sets $V(N)$ is a closed subset in this topology (see for instance, [**17, 18, 23, 26**]).

For an $R$-module $M$, consider the so-called *natural map*:

$$\begin{aligned} \psi : \mathrm{Spec}\,(M) &\longrightarrow \mathrm{Spec}\,(R/\mathrm{Ann}\,(M)) \\ P &\longmapsto (P : M)/\mathrm{Ann}\,(M). \end{aligned}$$

An $R$-module $M$ is called *primeful* if either $M = (0)$ or $M \neq (0)$ and the associated natural map $\psi$ is surjective. This notion of primeful module has been introduced for the first time and extensively studied by Lu in [**19**]. She found out some worthwhile properties of this type of module. This motivated us to extend the class of primeful modules to *wider classes* (which we call $\mathbb{P}$-radical modules and $\mathbb{M}$-radical modules) in which the above-mentioned worthwhile properties are to be preserved.

In fact, in the subsequent results (see **[1]**), we will show that this class of $\mathbb{P}$-radical module is an appropriate one for considering in an algebraic geometry view.

In Section 2, we introduce and study $\mathbb{P}$-radical modules and compare them with primeful modules. In particular, we show that the class of $\mathbb{P}$-radical modules is wider than that of primefuls (Proposition 2.3). Even more, we prove that any projective module over a Noetherian ring is also a $\mathbb{P}$-radical (Theorem 2.5). This also holds for any arbitrary module over an Artinian ring (Theorem 2.13). Furthermore, we call an $R$-module $M$ by $\mathbb{M}$-*radical* if $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$, for every maximal ideal $\mathcal{M}$ containing $\mathrm{Ann}\,(M)$, where $\sqrt[p]{\mathcal{M}M}$ is the intersection of all prime submodules of $M$ containing $\mathcal{M}M$. We see the following chart of implications for $M$:

| $M$ is finitely generated $\Rightarrow$ $M$ is primeful $\Rightarrow$ $M$ is $\mathbb{P}$-radical $\Rightarrow$ $M$ is $\mathbb{M}$-radical |
|---|

We show that the two conditions $\mathbb{P}$-radical and $\mathbb{M}$-radical are equivalent for all $R$-modules if and only if $R$ is a *Hilbert ring* (Theorem 2.11). Also we see that the two conditions primeful and $\mathbb{M}$-radical are equivalent for all $R$-modules if and only if $\dim\,(R) = 0$ (Theorem 2.12). Recall that an $R$-module $M$ is called a multiplication module if, for every submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ (*see* **[3, 11]** for more details). We prove that for a multiplication module $M$ the four conditions of the above table are equivalent (Proposition 2.18). Moreover, we give an analogue of Nakayama's lemma for $\mathbb{P}$-radical modules at the end of Section 2. In Section 3, semisimple primeful modules, semisimple $\mathbb{P}$-radical modules and semisimple $\mathbb{M}$-radical modules are fully investigated. For instance, in Proposition 3.5, it is shown that a semisimple $R$-module $M$ is $\mathbb{M}$-radical *if and only if* there exists a submodule $N$ of $M$ such that:

$$N \cong \bigoplus_{\mathrm{Ann}\,(M)\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} R/\mathcal{M}.$$

Also, a semisimple $R$-module $M$ is $\mathbb{P}$-radical *if and only if* $M$ is a $\mathbb{M}$-radical and $R/\mathrm{Ann}\,(M)$ is a Hilbert ring (Proposition 3.6).

In **[1]**, we will employ the results achieved in this paper to study the spectrum of modules from the point of view of algebraic geometry. In particular, we will construct a structure sheaf on the spectrum of the modules which belong to the wide class of $\mathbb{P}$-radicals (see also **[26]**).

**2. $\mathbb{P}$-radical and $\mathbb{M}$-radical modules.** Contrary to the rings with identity, one should notice that not every $R$-module contains a prime submodule. For example, $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module does not contain any prime submodule (see [**4, 12**]). For a proper submodule $N$ of an $R$-module $M$, the *prime radical* $\sqrt[p]{N}$ is the intersection of all prime submodules of $M$ containing $N$. We put $\sqrt[p]{N} = M$ in the case that there is no such prime submodule. Clearly $V(N) = V(\sqrt[p]{N})$. We note that, for each ideal $I$ of $R$, $\sqrt[p]{I} = \sqrt{I}$ (the intersection of all prime ideals of $R$ containing $I$). The prime radicals of submodules are studied by several authors (see for instance, [**5, 6, 20**]).

In [**20**], it is probed whether or not the equality $\sqrt[p]{IM} = \sqrt{I}M$ is satisfied for every ideal $I$ containing $\mathrm{Ann}\,(M)$ in the case of finitely generated $R$-modules $M$. Also, in [**19**], the author extended the investigation to primeful flat content modules (e.g., free modules), and primeful flat modules over rings with Noetherian spectrum.

In this article, we introduce a slight differentiation of the above equality, that is:

$$(\sqrt[p]{IM} : M) = \sqrt{I},$$

for every ideal $I$ containing $\mathrm{Ann}\,(M)$. This radical condition on modules plays a key role in our work to build a desired structure sheaf of modules in [**1**]. The following proposition offers several characterizations of $R$-modules $M$ which satisfy the above condition.

**Proposition 2.1.** *For an $R$-module $M$, the following four statements are equivalent*:

(1) $(\sqrt[p]{IM} : M) = \sqrt{I}$ *for every ideal* $I \supseteq \mathrm{Ann}\,(M)$.
(2) $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$ *for every prime ideal* $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$.
(3) $\sqrt{I} = \bigcap_{P \in V(IM)} (P : M)$, *for every ideal* $I \supseteq \mathrm{Ann}\,(M)$.
(4) $\mathcal{P} = \bigcap_{P \in V(\mathcal{P}M)} (P : M)$, *for every prime ideal* $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$.

*Proof.* (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) are trivial.

For (2) $\Rightarrow$ (1), let $I \supseteq \mathrm{Ann}\,(M)$. We have $I \subseteq (IM : M) \subseteq (\sqrt[p]{IM} : M)$ and $(\sqrt[p]{IM} : M)$ an intersection of all prime ideals. This implies that $\sqrt{I} \subseteq (\sqrt[p]{IM} : M)$. On the other hand, we have:

$$(\sqrt[p]{IM} : M) \subseteq \bigcap_{\mathcal{P} \in V(I)} (\sqrt[p]{\mathcal{P}M} : M) = \bigcap_{\mathcal{P} \in V(I)} \mathcal{P} = \sqrt{I},$$

which immediately gives the desired equality $(\sqrt[p]{IM} : M) = \sqrt{I}$.

For (1) $\Rightarrow$ (3), one checks that, for each $I \supseteq \mathrm{Ann}\,(M)$, we have:

$$\begin{aligned}\sqrt{I} = (\sqrt[p]{IM} : M) &= \Bigg(\Bigg(\bigcap_{P \in V(IM)} P\Bigg) : M\Bigg) \\ &= \bigcap_{P \in V(IM)} (P : M),\end{aligned}$$

as desired.

For (4) $\Rightarrow$ (2), suppose that $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$ is a prime ideal of $R$. Substituting $\bigcap_{P \in V(\mathcal{P}M)} P$ with $\sqrt[p]{\mathcal{P}M}$ in the equality $\bigcap_{P \in V(\mathcal{P}M)}(P : M) = (\bigcap_{P \in V(\mathcal{P}M)} P : M)$ immediately gives the desired equality $\mathcal{P} = (\sqrt[p]{\mathcal{P}M} : M)$. □

**Definition 2.2.** Let $M$ be an $R$-module $M$. We call that $M$ is $\mathbb{P}$-*radical* whenever it satisfies one of the equivalent conditions listed in the above proposition.

The following proposition together with Example 2.6, below, asserts that the class of $\mathbb{P}$-radical is wider than that of primeful modules introduced in [**19**].

**Proposition 2.3.** *Any primeful $R$-module $M$ is $\mathbb{P}$-radical.*

*Proof.* Let $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$ be a prime ideal. By the primeful assumption of $M$, there is some $P \in \mathrm{Spec}\,(M)$ with $(P : M) = \mathcal{P}$. It follows that $\sqrt[p]{\mathcal{P}M} \subseteq P$ and $\mathcal{P} = \sqrt{\mathcal{P}} \subseteq (\sqrt[p]{\mathcal{P}M} : M)$, and consequently we have:

$$\mathcal{P} = \sqrt{\mathcal{P}} \subseteq (\sqrt[p]{\mathcal{P}M} : M) \subseteq (P : M) = \mathcal{P},$$

which immediately implies that $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$. Thus, $M$ is a $\mathbb{P}$-radical module. □

For an arbitrary ring $R$, it is easy to check that every free $R$-module $M$ is primeful. Moreover, it is shown that every finitely generated $R$-module $M$ and also every projective module over a domain is primeful (see [**17,** Theorem 2.2, Corollary 2.6]). Thus, we immediately obtain the following corollary.

**Corollary 2.4.** *Let $R$ be a ring.*

(i) *Every free $R$-module is a $\mathbb{P}$-radical.*
(ii) *Every finitely generated $R$-module is a $\mathbb{P}$-radical.*
(iii) *If $R$ is a domain, then every projective $R$-module is a $\mathbb{P}$-radical.*

We show that Corollary 2.4 (iii) is also true when we replace the phrase "$R$ is a domain" with "$R$ is a Noetherian ring."

**Theorem 2.5.** *Every projective $R$-module is $\mathbb{P}$-radical whenever $R$ is a Noetherian ring.*

*Proof.* Suppose that $M$ is a projective $R$-module and $\mathcal{P} \supseteq \operatorname{Ann}(M)$. We claim that $\mathcal{P}M \neq M$. Indeed, if $\mathcal{P}M = M$, then the collection $\mathcal{A} = \{I \trianglerighteq R \mid \ IM \neq M \text{ and } \operatorname{Ann}(M) \subseteq I \subseteq \mathcal{P}\}$ is not empty since $\operatorname{Ann}(M) \in \mathcal{A}$. The Noetherian assumption of $R$ implies that $\mathcal{A}$ has a maximal element, say $\mathcal{P}_0$. If $\mathcal{P}_0$ is not a prime ideal of $R$, then there exist $a, b \in R \setminus \mathcal{P}_0$ such that $ab \in \mathcal{P}_0$. It follows that $(\mathcal{P}_0 + Ra)M = (\mathcal{P}_0+Rb)M = M$, and so $M = (\mathcal{P}_0+Ra)(\mathcal{P}_0+Rb)M \subseteq \mathcal{P}_0M$, which is a contradiction. Since for each projective $R$-module $M$ and each ideal $I$ of $R$, the factor module $M/IM$ is also projective as an $R/I$-module, thus $\overline{M} := M/\mathcal{P}_0M$ is projective as an $\overline{R} := R/\mathcal{P}_0$-module. Furthermore, $\overline{R}$ is a domain, and hence $\overline{M}$ is a $\mathbb{P}$-radical $\overline{R}$-module according to Corollary 2.4 (iii). Now, if $r \in R \setminus \mathcal{P}_0$, then $(Rr + \mathcal{P}_0)M = M$, and it follows that $r+\mathcal{P}_0 \notin \operatorname{Ann}_{\overline{R}}(\overline{M})$, i.e., $\operatorname{Ann}_{\overline{R}}(\overline{M}) = (0)$. Since $\overline{\mathcal{P}} := \mathcal{P}/\mathcal{P}_0$ is a prime ideal of $\overline{R}$, we must have $(\sqrt[p]{\overline{\mathcal{P}M}} : \overline{M}) = \overline{\mathcal{P}}$. But, the equality $\mathcal{P}M = M$ implies that $\overline{\mathcal{P}M} = \overline{M}$, and so $(\sqrt[p]{\overline{\mathcal{P}M}} : \overline{M}) = R$, which is a contradiction. Thus, $\mathcal{P}M \neq M$ and so $\mathcal{P}M$ is a proper submodule of $M$. Suppose $F = M \oplus L$ where $F$ is a free $R$-module and $L$ is a submodule of $F$. Clearly, $\mathcal{P}F$ is a prime submodule of $F$, namely, $F/\mathcal{P}F$ is a prime $R$-module and $\operatorname{Ann}(F/\mathcal{P}F) = \mathcal{P}$. Since $F/\mathcal{P}F \cong M/\mathcal{P}M \oplus L/\mathcal{P}L$, we conclude that $M/\mathcal{P}M$ is also a prime $R$-module with $\operatorname{Ann}(M/\mathcal{P}M) = \operatorname{Ann}(F/\mathcal{P}F) = \mathcal{P}$, and hence $\mathcal{P}M$ is a prime submodule of $M$ with $(\mathcal{P}M : M) = \mathcal{P}$. Thus, $(\sqrt[p]{\mathcal{P}M} : M) = (\mathcal{P}M : M) = \mathcal{P}$, which implies satisfaction of the $\mathbb{P}$-radical condition on the $R$-module $M$. $\square$

The following example shows that the converse of Proposition 2.3 is not true in general and, consequently, the class of $\mathbb{P}$-radical modules contains the class of primeful $R$-modules, properly. Moreover, the next proposition helps us to better recognize the relationship between these two types of modules.

**Example 2.6.** (see also [**19,** page 136, Example 1]). Consider the $\mathbb{Z}$-module $M := \bigoplus_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ for the set of prime integers $\Omega$. One easily convinces oneself that $\operatorname{Ann}(M) = 0$ and according to [**19**], for any non-zero prime ideal $(p)$ of $\mathbb{Z}$, $pM$ is a $(p)$-prime submodule of $M$, while it doesn't have any (0)-prime submodule. Consequently, $M$ is not a primeful $\mathbb{Z}$-module. The zero submodule (0) of $M$ is an intersection of maximal submodules and hence $\sqrt[p]{(0)} = (0)$. Thus:

$$(\sqrt[p]{(0)M} : M) = ((0) : M) = \operatorname{Ann}(M) = (0).$$

Furthermore, if $(q)$ is a non-zero prime (maximal) ideal of $\mathbb{Z}$, then $(q)M = \bigoplus_{q\neq p\in\Omega} \mathbb{Z}/p\mathbb{Z} \neq M$. It follows that $(q) = (\sqrt[p]{(q)M} : M)$, and hence $M$ is $\mathbb{P}$-radical.

**Proposition 2.7.** *For every ring $R$, there is a non-primeful $\mathbb{P}$-radical $R$-module $M$ if and only if there exist some prime ideals $\mathcal{P}$ and $\{\mathcal{P}_i\}_{i\in I}$ of $R$ such that $\mathcal{P} \subset \mathcal{P}_i$ and $\mathcal{P} = \bigcap_{i\in I} \mathcal{P}_i$.*

*Proof.* Let $M$ be a non-primeful $\mathbb{P}$-radical $R$-module and $\mathcal{P}$ a prime ideal of $R$ such that $M$ does not have any $\mathcal{P}$-prime submodules. Then, for any prime submodule $N$ of $M$ with $\mathcal{P}M \subseteq N$, we should have $\mathcal{P} \subset \mathcal{P}_N := (N : M)$. On the other hand, the $\mathbb{P}$-radical property of $M$ implies that:

$$\begin{aligned}\mathcal{P} = (\sqrt[p]{\mathcal{P}M} : M) &= \left(\bigcap_{N\in X} N : M\right)\\ &= \bigcap_{N\in X} (N : M) = \bigcap_{N\in X} \mathcal{P}_N,\end{aligned}$$

where $X = \{N \mid N \in \operatorname{Spec}(M) \text{ with } \mathcal{P}M \subseteq N\}$. For the converse, there is no loss of generality in assuming that $\mathcal{P} = \bigcap_{i\in I}\mathcal{P}_i$ and, for each prime ideal $\mathcal{Q} \supset \mathcal{P}$ of $R$, there exists $i \in I$ such that $\mathcal{Q} = \mathcal{P}_i$. Consider the $R$-module $M = \bigoplus_{i\in I} R/\mathcal{P}_i$. Then, $\operatorname{Ann}(M) = \bigcap_{i\in I}\mathcal{P}_i = \mathcal{P}$ and, for each $j \in I$, $N_j = \bigoplus_{j\neq i\in I} R/\mathcal{P}_i$ is a $\mathcal{P}_j$-prime submodule of $M$ with $\mathcal{P}M \subseteq \mathcal{P}_jM \subseteq N_j$. Thus, for each prime ideal $\mathcal{P}_j \supset \operatorname{Ann}(M) = \mathcal{P}$, we have $(\sqrt[p]{\mathcal{P}_jM} : M) = \mathcal{P}_j$ (since $\mathcal{P}_j \subseteq (\sqrt[p]{\mathcal{P}_jM} : M) \subseteq (N_j : M) = \mathcal{P}_j$). On the other hand,

$$(\sqrt[p]{\mathcal{P}M} : M) \subseteq \left(\bigcap_{i\in I} N_i : M\right) = \bigcap_{i\in I}(N_i : M) = \bigcap_{i\in I}\mathcal{P}_i = \mathcal{P},$$

and hence $M$ is a $\mathbb{P}$-radical module. We claim that $M$ doesn't have any $\mathcal{P}$-prime submodule. Otherwise, let $N$ be a $\mathcal{P}$-prime submodule of $M$ with $(N : M) = \mathcal{P}$. Since $N \neq M$, there exists $j \in I$ such that $(\cdots, 0, 1 + \mathcal{P}_j, 0, \cdots) \notin N$. Since $\mathcal{P}_j(\cdots, 0, 1 + \mathcal{P}_j, 0, \cdots) \in N$, we should have $\mathcal{P}_jM \subseteq N$, namely, $\mathcal{P}_j \subseteq \mathcal{P}$, which is a contradiction. Thus, $M$ is not primeful. □

According to Proposition 2.1, an $R$-module $M$ is $\mathbb{P}$-radical if and only if $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$ for each prime ideal $\mathcal{P} \supseteq \operatorname{Ann}(M)$. Now let us generalize it by the notion of $\mathbb{M}$-radical modules.

**Definition 2.8.** An $R$-module $M$ is called $\mathbb{M}$-*radical* (or *Maxful*) whenever $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$ for each maximal ideal $\mathcal{M} \supseteq \operatorname{Ann}(M)$.

The following evident proposition offers several other characterizations of $\mathbb{M}$-radical modules.

**Proposition 2.9.** *The following statements are equivalent for a nonzero R-module M:*

(1) *$M$ is an $\mathbb{M}$-radical module.*
(2) *$\mathcal{M}M \neq M$ for every maximal ideal $\mathcal{M} \supseteq \operatorname{Ann}(M)$.*
(3) *$\mathcal{P}M \neq M$ for every prime ideal $\mathcal{P} \supseteq \operatorname{Ann}(M)$.*
(4) *There is a maximal submodule $P$ of $M$ such that $(P : M) = \mathcal{M}$ for every maximal ideal $\mathcal{M} \supseteq \operatorname{Ann}(M)$.*
(5) *There is a prime submodule $P$ of $M$ such that $(P : M) = \mathcal{M}$ for every maximal ideal $\mathcal{M} \supseteq \operatorname{Ann}(M)$.*

*Proof.* (1) $\Rightarrow$ (2) can be immediately obtained from the direct definition of the $\mathbb{M}$-radical modules.

For (2) $\Rightarrow$ (1), suppose that $\mathcal{M}M \neq M$ for every maximal ideal $\mathcal{M} \supseteq \mathrm{Ann}\,(M)$. Then, for each maximal ideal $\mathcal{M} \supseteq \mathrm{Ann}\,(M)$, $\mathcal{M}M$ is a prime submodule of $M$, and so $\sqrt[p]{\mathcal{M}M} = \mathcal{M}M$. It follows that $\mathcal{M} \subseteq (\sqrt[p]{\mathcal{M}M} : M) = (\mathcal{M}M : M) = \mathcal{M}$, and hence $(\sqrt[p]{\mathcal{M}M} : M) = \mathcal{M}$, as desired.

The case (2) $\Leftrightarrow$ (3) is obvious.

For (2) $\Rightarrow$ (5), suppose that $\mathcal{M} \supseteq \mathrm{Ann}\,(M)$ is a maximal ideal. Then the non-equality $\mathcal{M}M \neq M$ implies that $\mathcal{M}M$ is a prime submodule with $(\mathcal{M}M : M) = \mathcal{M}$.

Two cases, (5) $\Rightarrow$ (2) and (4) $\Rightarrow$ (5), are immediate and for (5) $\Rightarrow$ (4), suppose that $(P : M) = \mathcal{M}$ where $P$ is a prime submodule of $M$ and $\mathcal{M} \supseteq \mathrm{Ann}\,(M)$ is a maximal. Then $M/P$ is an $R/\mathcal{M}$-vector space, and hence $M/P$ has a maximal $R/\mathcal{M}$-subspace such as $K/P$. Then, $K \subseteq M$ is a maximal $R$-submodule and $(K : M) = \mathcal{M}$, as desired. □

A commutative ring $R$ is called a *Hilbert ring*–or *Jacobson* or *Jacobson-Hilbert* ring–if every prime ideal of $R$ is an intersection of its maximal ideals. The class of commutative Hilbert rings is closed under the finite polynomial rings forming. On the other hand, we have already observed some modules $M$ having no prime submodule (for example $\mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module). We call such modules *primeless*. We recall that a module $M$ over a domain $R$ is called *divisible* if $rM = M$ for each $0 \neq r \in R$ and is called *torsion* if $\mathrm{Ann}\,(m) \neq 0$ for each $m \in M$. We shall be interested in seeing, under which conditions, two notions of $\mathbb{P}$-radical and $\mathbb{M}$-radical are equivalent. We characterize this equivalency using the notion of Hilbert rings. But, at first, we need the following lemma.

**Lemma 2.10.** (see [**23,** Lemma 1.3 (i)]). *Let $R$ be a domain. Then every torsion divisible $R$-module is primeless.*

**Theorem 2.11.** *For an arbitrary ring $R$, every $\mathbb{M}$-radical $R$-module is $\mathbb{P}$-radical if and only if $R$ is a Hilbert ring.*

*Proof.* Let every $\mathbb{M}$-radical $R$-module be $\mathbb{P}$-radical. To obtain a contradiction, suppose that $\mathcal{P}$ is a prime ideal of $R$ which is not an intersection of the maximal ideals of $R$. One convinces oneself that every $\mathbb{M}$-radical $R/\mathcal{P}$-module is also a $\mathbb{P}$-radical $R/\mathcal{P}$-module. Let $S = R/\mathcal{P}$ and $Q$ be the field of fraction of $S$. Since $\mathcal{P}$ is not a maximal ideal, $S$ is not a field and so $S \neq Q$. For a non-zero proper $S$-submodule $K$ of $Q$ the quotient $L := Q/K$ is a torsion divisible $S$-module and so, by the above lemma, $L$ is a primeless $S$-module. Now consider the $S$-module:

$$M = \bigoplus_{\mathcal{M} \in \operatorname{Max}(S)} S/\mathcal{M} \oplus L.$$

Since, for each $\mathcal{M}_1 \in \operatorname{Max}(S)$, we have:

$$\mathcal{M}_1 M = \bigoplus_{\mathcal{M}_1 \neq \mathcal{M} \in \operatorname{Max}(S)} S/\mathcal{M} \oplus L$$

is a prime $S$-module with $(\mathcal{M}_1 M : M) = \mathcal{M}_1$, then $M$ is $\mathbb{M}$-radical. We claim that every prime $S$-submodule of $M$ is also of the above form. To see this, let $P$ be a prime $S$-submodule of $M$. We have $\bigoplus_{\mathcal{M} \in \operatorname{Max}(S)} S/\mathcal{M} \not\subseteq P$; otherwise, we must have $M/P \cong L/T$ for some proper submodule $T$ of $L$, which is a contradiction according to the primeless assumption of $L$. Then there is some $\mathcal{M}_1 \in \operatorname{Max}(S)$ such that $(0, \cdots, 0, 1 + \mathcal{M}_1, 0 \cdots) \notin P$. Since $\mathcal{M}_1(0, \cdots, 0, 1 + \mathcal{M}_1, 0 \cdots) \subseteq P$, hence $\mathcal{M}_1 M \subseteq N$, and consequently, we have $\mathcal{M}_1 M = N$. It follows that:

$$\sqrt[p]{(0)} = \bigcap_{\mathcal{M} \in \operatorname{Max}(R)} \mathcal{M} M = L.$$

Now the equality $\operatorname{Ann}({}_S L) = (0)$ implies that $\operatorname{Ann}({}_S M) = (0)$ and hence:

$$\begin{aligned}(\sqrt[p]{(0)M} : M) &= (\sqrt[p]{(0)} : M) = (L : M) = \\ &= \Big((0) : \bigoplus_{\mathcal{M} \in \operatorname{Max}(S)} S/\mathcal{M}\Big) = \bigcap_{\mathcal{M} \in \operatorname{Max}(R)} \mathcal{M}.\end{aligned}$$

But $\bigcap_{\mathcal{M} \in \operatorname{Max}(R)} \mathcal{M} \neq (0)$ since $\mathcal{P} \neq \bigcap_{\mathcal{P} \subseteq \mathcal{M} \in \operatorname{Max}(R)} \mathcal{M}$. Thus, $(\sqrt[p]{(0)M} : M) \neq (0)$, which is impossible (since $M$ is a $\mathbb{P}$-radical module). Thus, $R$ is a Hilbert ring.

For the converse, assume that $M$ is an $\mathbb{M}$-radical module and $\mathcal{P}$ is a prime ideal of $R$ with $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$. Since $R$ is a Hilbert ring, $\mathcal{P} = \bigcap_{\mathcal{P}\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} \mathcal{M}$. The $\mathbb{M}$-radical property of $M$ implies that $(\mathcal{M}M : M) = \mathcal{M}$ for each $\mathcal{P} \subseteq \mathcal{M} \in \mathrm{Max}\,(R)$, and hence we have:

$$\begin{aligned}\left(\sqrt[p]{\mathcal{P}M} : M\right) &\subseteq \Big(\bigcap_{\mathcal{P}\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} \mathcal{M}M : M\Big)\\ &= \bigcap_{\mathcal{P}\subseteq\mathcal{M}\in\mathrm{Max}(R)} (\mathcal{M}M : M)\\ &= \bigcap_{\mathcal{P}\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} \mathcal{M} = \mathcal{P}.\end{aligned}$$

This inequality, together with the fact $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (\sqrt[p]{\mathcal{P}M} : M)$, implies that $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$, as desired. $\square$

According to the results obtained so far, we have the following chart of implications for an arbitrary $R$-module $M$:

| $M$ is primeful $\Rightarrow$ $M$ is $\mathbb{P}$-radical $\Rightarrow$ $M$ is $\mathbb{M}$-radical |
|---|

Moreover, none of the implications is reversible in general. However, for zero-dimensional rings, we have the following result, which expresses the relationship between primeful and $\mathbb{M}$-radical modules.

**Theorem 2.12.** *For an arbitrary ring $R$, every $\mathbb{M}$-radical module is also primeful if and only if* $\dim\,(R) = 0$.

*Proof.* If every $\mathbb{M}$-radical $R$-module is primeful, then, according to Theorem 2.11, $R$ is a Hilbert ring. Suppose, contrary to our claim, that $\dim\,(R) \geq 2$ and $\mathcal{P}$ is a non-maximal prime ideal of $R$. Thus, $\mathcal{P} = \bigcap_{\mathcal{P}\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} \mathcal{M}$, and according to Proposition 2.7 there exists an $R$-module $M$ which is $\mathbb{P}$-radical but not primeful, a contradiction. Thus, $\dim\,(R) = 0$. The converse is clear. $\square$

We shall be interested in seeing whether every $R$-module is primeful (and hence $\mathbb{P}$-radical and $\mathbb{M}$-radical). This question leads us to the concept of *Artinian rings*.

**Theorem 2.13.** *For an Artinian ring $R$, every $R$-module is primeful (and hence $\mathbb{P}$-radical and $\mathbb{M}$-radical).*

*Proof.* We know that the dimension of the Artinian rings is identically zero and hence the three concepts of primeful, $\mathbb{P}$-radical and $\mathbb{M}$-radical are equivalent for all modules defined over such rings. Since $R$ is Artinian, we can express it by:

$$R = R_1 \times \cdots \times R_n, \quad n \in \mathbb{N},$$

where each $R_i$ is an Artinian local ring. In the special case $n = 1$, where $R$ is a local ring with maximal ideal $\mathcal{M}$, consider the non-zero $R$-module $M$. Then $\operatorname{Ann}(M) \neq R$, and so $\operatorname{Ann}(M) \subseteq \mathcal{M}$. By Proposition 2.9, it is sufficient to show that $\mathcal{M}M \neq M$. If $R$ is a domain (field), then $\mathcal{M} = (0)$, and so the proof is complete. Hence, to obtain a contradiction, suppose that $R$ is not a domain and also $\mathcal{M}M = M$. Then there are some non-zero elements $a, b \in R$ such that $ab = 0$. Thus, $(Ra)(Rb)M = (0)$, and consequently we should have either $RaM \neq M$ or $RbM \neq M$. It follows that $\mathcal{A} := \{I \trianglerighteq R \mid IM \neq M \text{ and } I \neq (0)\}$ is a non-empty set of the ideals of $R$. Since $R$ is Noetherian, $\mathcal{A}$ has a maximal element such as $\mathcal{P}$ which is not a prime (maximal) ideal of $R$ according to the inequality $\mathcal{P}M \neq M$. Thus, there exist two ideals $A$ and $B$ of $R$ such that $\mathcal{P} \subset A$, $\mathcal{P} \subset B$ and $AB \subseteq \mathcal{P}$. Now, using the equalities $AM = BM = M$ implies that $M = ABM \subseteq \mathcal{P}M$, a contradiction. Now, assume that $n \geq 2$, and for each $i$ $(1 \leq i \leq n)$ let $\mathcal{M}_i$ be the maximal ideal of the local ring $R_i$. Furthermore, consider the non-zero $R$-module $M$ with $\operatorname{Ann}(M) \subseteq \mathcal{M}$ where $\mathcal{M}$ is a maximal ideal of $R$. Then $\mathcal{M}$ is of the form $R_1 \times \cdots \times R_{i-1} \times \mathcal{M}_i \times R_{i+1} \cdots \times R_n$ for some $1 \leq i \leq n$. Without loss of generality, we can assume that $i = 1$. Again, according to Proposition 2.9, it is sufficient to show that the inequality $\mathcal{M}M = (\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M \neq M$ holds. On the contrary, suppose that $(\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M = M$. Consider two ideals $I = R_1 \times (0) \times \cdots \times (0)$ and $J = (0) \times R_2 \times \cdots \times R_n$ of $R$. We have $J = \operatorname{Ann}(I)$, and hence $R_1 \cong R/J$ which implies that $\overline{M} = IM$ is a unitary $R_1$-module (in fact, $\overline{M} = (R_1 \times 0 \times \cdots \times 0)M$ is a unitary $R_1$-module with $r_1\overline{m}$ defined to be $r_1(1, 0, \cdots, 0)\overline{m}$ for $r_1 \in R_1$ and $\overline{m} \in \overline{M}$). We claim that $\overline{M} \neq (0)$, otherwise, $R_1 \times (0) \times \cdots \times (0) \subseteq \operatorname{Ann}(M) \subseteq \mathcal{M}_1 \times R_2 \times \cdots \times R_n$, which is a contradiction. Thus, $\overline{M}$ is a non-zero $R_1$-module and so, according to the case $n = 1$, we have $\mathcal{M}_1\overline{M} \neq \overline{M}$, i.e., $(\mathcal{M}_1 \times 0 \times \cdots \times 0)M \neq (R_1 \times 0 \times \cdots \times 0)M$.

On the other hand, for each $m \in M$, we have $(1, 0, \cdots, 0)m \in M = (\mathcal{M}_1 \times R_2 \times \cdots \times R_n)M$ and, consequently, for each $m \in M$, we have:

$$(1, 0, \ldots, 0)m = \textstyle\sum_{j=1}^{k} (p_{1j}, r_{2j}, \ldots, r_{nj})m_j,$$

where $k \in \mathbb{N}$, $m_j \in M$, $p_{1j} \in \mathcal{M}_1$ and $r_{ij} \in R_i$. Thus:

$$\begin{aligned}(1, 0, \ldots, 0)m &= (1, 0, \ldots, 0)^2 m \\ &= \sum_{j=1}^{k} (p_{1j}, 0, \ldots, 0)m_j,\end{aligned}$$

and hence $(1, 0, \ldots, 0)m \in (\mathcal{M}_1 \times (0) \times \cdots \times (0))M$, for each $m \in M$. It follows that $(R_1 \times 0 \times \cdots \times 0)M \subseteq (\mathcal{M}_1 \times (0) \times \cdots \times (0))M$, i.e., $\mathcal{M}_1\overline{M} = \overline{M}$, which is a contradiction. □

The following example shows that the converse of the above theorem is not true, in general.

**Example 2.14.** Let $K$ be a field, $D := K[\{x_i : i \in \mathbb{N}\}]$ (a unique factorization domain) and $R = K[\{x_i : i \in \mathbb{N}\}]/(\{x_i x_j : i, j \in \mathbb{N}\})$, where $(\{x_i x_j : i, j \in \mathbb{N}\})$ is the ideal of $D$ generated by $\{x_i x_j : i, j \in \mathbb{N}\} \subseteq D$. Furthermore, for each $k \in \mathbb{N}$, let $\overline{x}_k = x_k + (\{x_i x_j : i, j \in \mathbb{N}\})$ and $\mathcal{M} = (\{\overline{x}_k : k \in \mathbb{N}\})$. The ideal $\mathcal{M}$ is simply the image of the maximal ideal $\mathcal{N} = (\{x_k : k \in \mathbb{N}\})$ of $D$. Clearly $\mathcal{M}^2 = (0)$, and so $R$ is a local zero-dimensional ring, but is not Artinian (Noetherian). The equality $\mathcal{M}^2 = (0)$ implies that, for each non-zero $R$-module, we have $M$, $\mathcal{M}M \neq M$, and hence according to Proposition 2.9 and Theorem 2.12, $M$ is a $\mathbb{P}$-radical module. Thus, every $R$-module is $\mathbb{P}$-radical, while $R$ is not Artinian.

A ring $R$ is called a *Max-ring* (or a *Bass ring*) if every non-zero $R$-module has a maximal submodule. Also, a ring $R$ is called a *P-ring* if every non-zero $R$-module has a prime submodule. It is proved that the commutative P-rings coincide with the Max-rings (*see* [**7,** Theorem 3.9]). Moreover, we have the following lemma.

**Lemma 2.15.** (see [**14,** Theorem 2]). *For a commutative ring $R$, the following conditions are equivalent*:

(1) *R is a max ring;*
(2) *$R/J(R)$ is a regular ring and $J(R)$ is a t-nilpotent ideal.*

The following theorem offers several characterizations of Noetherian rings $R$ over which every module is $\mathbb{P}$-radical.

**Theorem 2.16.** *Consider the following statements for a ring $R$:*

(1) *$R$ is an Artinian ring.*
(2) *Every $R$-module is primeful.*
(3) *Every $R$-module is $\mathbb{P}$-radical.*
(4) *Every $R$-module is $\mathbb{M}$-radical.*
(5) *$R$ is a Max-ring.*
(6) *$R$ is a P-ring.*
(7) $\dim(R) = 0$.

*Then* $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Rightarrow (7)$. *Moreover, when $R$ is Noetherian* (*or domain*), *all the seven statements are equivalent.*

*Proof.* $(1) \Rightarrow (2)$ is Theorem 2.13.
$(2) \Rightarrow (3)$ is Proposition 2.3. $(3) \Rightarrow (4)$ is clear.
$(4) \Rightarrow (5)$. Assume that every $R$-module is $\mathbb{M}$-radical. For an arbitrary non-zero $R$-module $M$ we have $\mathrm{Ann}(M) \neq R$ and so there is a maximal ideal $\mathcal{M}$ of $R$ such that $\mathrm{Ann}(M) \subseteq \mathcal{M}$. Since $M$ is maxful, there is, moreover, a prime submodule $P$ of $M$ with $(P : M) = \mathcal{M}$. Thus, $M/P$ is an $R/\mathcal{M}$-module ($R/\mathcal{M}$-vector space), and hence, $M/P$ has a maximal $R/\mathcal{M}$-submodule such as $K/P$. It is easy to see that $K < M$ is a maximal $R$-submodule. Thus, every non-zero $R$-module has a maximal submodule, namely, $R$ is a Max-ring.
$(5) \Leftrightarrow (6)$ is by [**7,** Theorem 3.9].
$(6) \Rightarrow (7)$. Let $R$ be a P-ring and $\mathcal{P}$ a non-maximal prime ideal of $R$. For $R' := R/\mathcal{P}$, consider $K$ as the field of fractions of $R'$. Since $R'$ is not a field, then $R' \neq K$, and $K$ is a divisible $R'$-module. It follows that $K/R'$ is a non-zero torsion divisible $R'$-module. Then, according to Lemma 2.10, $K/R'$ is a primeless $R'$-module. Now, one convinces oneself that $K/R'$ is a primeless $R$-module and consequently $R$ is not a P-ring, which is a contradiction.
$(4) \Rightarrow (2)$. If (4) holds, one concludes from $(4) \Rightarrow (7)$ that $\dim(R) = 0$. Then, according to Theorem 2.12, every $R$-module is primeful.

Finally, if $R$ is a Noetherian ring, then $\dim(R) = 0$ if and only if $R$ is Artinian. Thus, (7) $\Rightarrow$ (1) holds whenever $R$ is a Noetherian ring. $\square$

The following is now immediate.

**Corollary 2.17.** *For a domain $R$, the following statements are equivalent*:

(1) *Every $R$-module is primeful.*
(2) *Every $R$-module is $\mathbb{P}$-radical.*
(3) *Every $R$-module is $\mathbb{M}$-radical.*
(4) *$R$ is a field* (*i.e., $R$ is an Artinan domain*).

The following proposition demonstrates the relationship between finitely generated, primeful, $\mathbb{P}$-radical and $\mathbb{M}$-radical modules in the interesting case of multiplication modules.

**Proposition 2.18.** *Consider the following statements for a non-zero $R$-module $M$*:

(1) *$M$ is finitely generated.*
(2) *$M$ is primeful.*
(3) *$M$ is a $\mathbb{P}$-radical module.*
(4) $(\mathcal{P}M : M) = \mathcal{P}$ *for every prime ideal* $\mathcal{P} \supseteq \operatorname{Ann}(M)$.
(5) *$M$ is a $\mathbb{M}$-radical module.*

*Then* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5). *In particular, when $M$ is a multiplication module, then* (5) $\Rightarrow$ (1) *holds, too.*

*Proof.* (1) $\Rightarrow$ (2) holds by [**19,** Proposition 3.8], and (2) $\Rightarrow$ (3) holds by Proposition 2.3. Moreover, (3) $\Rightarrow$ (4) is straightforward by the fact $\mathcal{P} \subseteq (\mathcal{P}M : M) \subseteq (\sqrt[p]{\mathcal{P}M} : M)$ for every prime ideal $\mathcal{P} \supseteq \operatorname{Ann}(M)$. To prove (4) $\Rightarrow$ (5), we know that the equality $(\mathcal{P}M : M) = \mathcal{P}$ implies that $\mathcal{P}M \neq M$ for every prime ideal $\mathcal{P} \supseteq \operatorname{Ann}(M)$. Thus, $M$ is an $\mathbb{M}$-radical module. When $M$ is a multiplication module, then (5) $\Rightarrow$ (1) holds according to [**19,** Proposition 3.8]. This completes the proof. $\square$

We conclude this section with the next analogue of Nakayama's lemma.

**Proposition 2.19.** *Let $M$ be an $\mathbb{M}$-radical $R$-module. Then $M$ satisfies the following assertion* (NAK): *If $I$ is an ideal of $R$ contained in the Jacobson radical $J(R)$ with $IM = M$, then $M = (0)$.*

*Proof.* If $M \neq (0)$ then, $\mathrm{Ann}\,(M) \neq R$. Hence, for a maximal ideal $\mathcal{M}$ of $R$ containing $\mathrm{Ann}\,(M)$, we have $I \subseteq \mathcal{M}$ and $IM = M = \mathcal{M}M$, which is a contradiction. □

**3. Characterization of semisimple $\mathbb{P}$-radical modules.** Recall that, for an $R$-module module $M$, the *socle* of $M$ (denoted by $\mathrm{soc}\,(M)$) is the sum of all simple (minimal) submodules of $M$. If there are no minimal submodules in $M$, we put $\mathrm{soc}\,(M) = (0)$. Thus, $M$ is a semisimple module whenever $\mathrm{soc}\,(M) = M$. Furthermore, a semisimple module $M$ is called *homogeneous* if any two simple submodules of $M$ are isomorphic. One checks that an $R$-module $M$ is homogeneous semisimple if and only if $\mathrm{Ann}\,(M)$ is a maximal ideal. In this section, we aim to characterize semisimple $\mathbb{P}$-radical modules. First we need the following definition.

**Definition 3.1.** Let $R$ be a ring. A semisimple $R$-module $M$ is called *full semisimple* if, for each maximal ideal $\mathcal{M} \supseteq \mathrm{Ann}\,(M)$ the simple $R$-module $R/\mathcal{M}$ can be embedded in $M$, namely, there exists a submodule $N$ of $M$ such that $N \cong \bigoplus_{\mathrm{Ann}\,(M)\subseteq\mathcal{M}\in\mathrm{Max}\,(R)} R/\mathcal{M}$.

**Example 3.2.** Consider the $\mathbb{Z}$-module $M = \bigoplus_{p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ where $\Omega$ is the set of prime integers. Obviously, $M$ is a full semisimple $\mathbb{Z}$-module while the semisimple $\mathbb{Z}$-module $M_1 = \bigoplus_{2\neq p\in\Omega} \mathbb{Z}/p\mathbb{Z}$ is not full semisimple since $\mathrm{Ann}\,(M_1) = (0) \subseteq 2\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ is not a submodule of $M_1$.

The proof of the following result is straightforward and left to the reader.

**Proposition 3.3.** *Let $M$ be a semisimple $R$-module such that* $\mathrm{Ann}\,(M)$ *is a finite intersection of maximal ideals. Then $M$ is full semisimple. In particular, all finitely generated semisimple modules are full semisimple as well as all homogenous semisimple modules.*

**Lemma 3.4.** *Let $M$ be an $R$-module with the non-zero socle. Then $M$ is a prime module if and only if $M$ is a homogeneous semisimple module.*

*Proof.* Let $M$ be a prime module with non-zero socle, and let $Rm$ be a simple submodule of $M$ for some $m \in M$. Then $\operatorname{Ann}(m) = \operatorname{Ann}(M) = P$, and hence $P$ is a maximal ideal of $R$. Since $\operatorname{Ann}(m) = \operatorname{Ann}(m')$ for each $0 \neq m' \in M$, then $M$ is a homogeneous semisimple $R$-module. The converse is evident. □

We are now in a position to show that the two concepts of $\mathbb{M}$-radical and full semisimple are equivalent in the case of semisimple modules.

**Proposition 3.5.** *Let $M$ be a semisimple $R$-module. Then $M$ is $\mathbb{M}$-radical if and only if $M$ is full semisimple.*

*Proof.* Since $M$ is a semisimple $R$-module, we can assume that $M = \bigoplus_{i\in I} R/\mathcal{M}_i$ where $I$ is an index set and each $\mathcal{M}_i$ is a maximal ideal of $R$. Let $M$ be an $\mathbb{M}$-radical module. Then $\operatorname{Ann}(M) = \bigcap_{i\in I} \mathcal{M}_i$. Suppose that $\mathcal{M} \supseteq \operatorname{Ann}(M)$ is a maximal ideal of $R$. If $\mathcal{M} \neq \mathcal{M}_i$, then, $\mathcal{M}(R/\mathcal{M}_i) = R/\mathcal{M}_i$ for each $i \in I$. It follows that $\mathcal{M}M = M$, which is in contrary to Proposition 2.9 (2). Thus, $\mathcal{M} = \mathcal{M}_i$ for some $i \in I$ and hence $R/\mathcal{M}$ can be embedded in $M$. To prove the converse, assume that $\mathcal{M} \supseteq \operatorname{Ann}(M)$ is a maximal ideal of $R$. Since $M$ is full semisimple, then $\mathcal{M} = \mathcal{M}_i$ for each $i \in I$ and so $\mathcal{M}M \neq M$. Thus $M$ is an $\mathbb{M}$-radical module, according to Proposition 2.9. □

**Proposition 3.6.** *For a semisimple $R$-module $M$, the following statements are equivalent*:

(1) *$M$ is a $\mathbb{P}$-radical module.*
(2) *$M$ is an $\mathbb{M}$-radical module and $R/\operatorname{Ann}(M)$ is a Hilbert ring.*
(3) *$M$ is full semisimple and $R/\operatorname{Ann}(M)$ is a Hilbert ring.*

*Proof.* Since $M$ is a semisimple $R$-module, we can assume that $M = \bigoplus_{i\in I} R/\mathcal{M}_i$ where $I$ is an index set and each $\mathcal{M}_i$ is a maximal ideal of $R$. To prove (1) ⇒ (2), one should notice that since every $\mathbb{P}$-radical module is $\mathbb{M}$-radical, then it is sufficient to show that $R/\operatorname{Ann}(M)$ is a Hilbert ring. Suppose $\mathcal{P} \supseteq \operatorname{Ann}(M)$ is a prime ideal of $R$. Thus, we have $(\sqrt[p]{\mathcal{P}M} : M) = \mathcal{P}$ and hence $\sqrt[p]{\mathcal{P}M} \neq M$. We can assume that $\sqrt[p]{\mathcal{P}M} = \bigcap_{\lambda\in\Lambda} P_\lambda$, where $\Lambda$ is an index set and each $P_\lambda$ is a prime submodule of $M$ containing $\mathcal{P}M$. Hence, the factor module $M/P_\lambda$ is a prime semisimple module for each $P_\lambda$ and, consequently, $M/P_\lambda$ is a homogenous semisimple $R$-module, according to by Lemma 3.4. More precisely, $\mathcal{M}_\lambda := (P_\lambda : M)$ is a maximal ideal of $R$. Thus:

$$\mathcal{P} = (\sqrt[p]{\mathcal{P}M} : M) = \left(\bigcap_{\lambda\in\Lambda} P_\lambda : M\right) = \bigcap_{\lambda\in\Lambda} (P_\lambda : M) = \bigcap_{\lambda\in\Lambda} \mathcal{M}_\lambda,$$

and hence every prime ideal $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$ is an intersection of maximal ideals of $R$, namely, $R/\mathrm{Ann}\,(M)$ is a Hilbert ring. (2) $\Leftrightarrow$ (3) and (2) $\Rightarrow$ (1) hold by Proposition 3.5 and Theorem 2.11, respectively. $\square$

**Corollary 3.7.** *For a semisimple $R$-module $M$ where $R$ is either a Hilbert ring or a domain of dimension one, the following statements are equivalent:*

(1) *$M$ is $\mathbb{P}$-radical.*
(2) *$M$ is $\mathbb{M}$-radical.*
(3) *$M$ is full semisimple.*

*Proof.* If $R$ is a Hilbert ring then by Proposition 3.5 and Theorem 2.11, the proof is complete. Thus, we can assume that $R$ is a domain of dimension one. Since $M$ is a semisimple $R$-module, we can assume that $M = \bigoplus_{i\in I} R/\mathcal{M}_i$ where $I$ is an index set and each $\mathcal{M}_i$ is a maximal ideal of $R$. Now (1) $\Rightarrow$ (2) is straightforward and (2) $\Rightarrow$ (3) is Proposition 3.5. To prove (3) $\Rightarrow$ (1), assume that $\mathcal{P} \supseteq \mathrm{Ann}\,(M)$ is a prime ideal of $R$, and let $\mathcal{M} \supseteq \mathcal{P}$ be a maximal ideal. Since $R$ is a domain with $\dim(R) = 1$, either $\mathcal{P} = (0)$ or $\mathcal{M} = \mathcal{P}$. If $\mathcal{M} = \mathcal{P}$, then $\mathcal{P}$ is one of the maximal ideals $\mathcal{M}_i$ in the direct summand of $M$ and so $\mathcal{P}M = \mathcal{M}_iM \neq M$. Clearly, $\mathcal{P}M$ is a prime submodule of $M$ with $(\mathcal{P}M : M) = \mathcal{P}$. It follows that $(\sqrt[p]{\mathcal{P}M} : M) = (\mathcal{P}M : M) = \mathcal{P}$. Otherwise, if $\mathcal{P} = (0)$ then, $\mathrm{Ann}\,(M) = (0) = \bigcap_{i\in I} \mathcal{M}_i$. We know that every proper submodule of a semisimple module is an intersection of maximal submodules and furthermore each maximal submodule is a prime submodule. Hence, $\sqrt[p]{\mathcal{P}M} = \sqrt[p]{(0)} = (0)$, which implies that $(\sqrt[p]{(0)M} : M) = ((0) : M) = \mathrm{Ann}\,(M) = (0) = \sqrt{(0)}$. Thus, $M$ is a $\mathbb{P}$-radical module. $\square$

We conclude this paper with the following result that offers several characterizations for semisimple primeful modules.

**Corollary 3.8.** *The following statements are equivalent for an arbitrary semisimple $R$-module $M$:*

(1) *$M$ is primeful.*
(2) *$M$ is $\mathbb{P}$-radical and* $\dim(R/\mathrm{Ann}(M)) = 0$.
(3) *$M$ is $\mathbb{M}$-radical and* $\dim(R/\mathrm{Ann}(M)) = 0$.
(4) *$M$ is full semisimple and* $\dim(R/\mathrm{Ann}(M)) = 0$.

*Proof.* To prove (1) $\Rightarrow$ (2), it is sufficient to show that $\dim(R/\mathrm{Ann}(M)) = 0$. For a prime ideal $\mathcal{P} \supseteq \mathrm{Ann}(M)$ of $R$, there is a prime submodule $P$ of $M$ such that $(P : M) = \mathcal{P}$. Since $M/P$ is a prime semisimple $R$-module, then $\mathcal{P}$ is a maximal ideal according to Lemma 3.4. Thus, $\dim(R/\mathrm{Ann}(M)) = 0$. The assertion (2) $\Rightarrow$ (3) is a consequence of Theorem 2.12. Furthermore, (3) $\Rightarrow$ (4) is Proposition 3.5. $\square$

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