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WHEN IS THE COMPLEMENT OF THE ZERO-DIVISOR GRAPH OF A COMMUTATIVE RING PLANAR?

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ABSTRACT. Let R be a commutative ring with identity admitting at least two distinct zero-divisors a, b with $ab \neq 0$. In this article, necessary and sufficient conditions are determined in order that $(\Gamma(R))^c$ (that is, the complement of the zero-divisor graph of R) is planar. It is noted that, if $(\Gamma(R))^c$ is planar, then the number of maximal N-primes of (0) in R is at most three. Firstly, we consider rings R admitting exactly three maximal N-primes of (0) and present a characterization of such rings in order that the complement of their zero-divisor graphs be planar. Secondly, we consider rings R admitting exactly two maximal N-primes of (0) and investigate the problem of when the complement of their zero-divisor graphs is planar. Thirdly, we consider rings R admitting only one maximal N-prime of (0) and determine necessary and sufficient conditions in order that the complement of their zero-divisor graphs be planar.

1. Introduction. All rings considered in this article are nonzero commutative rings with identity. Unless otherwise specified, we consider rings which admit at least two nonzero zero-divisors. For any ring R, and for any R-module M, *the set of zero-divisors of M as an R-module* denoted by $Z_R(M)$ is defined as $Z_R(M) = \{r \in R \mid rm = 0 \text{ for some } m \in M \setminus \{0\}\}$. In the special case when $M = R$, $Z_R(R)$ is simply denoted by $Z(R)$. We denote by $Z(R)^*$ the set of all nonzero zero-divisors of R.

Let R be a ring which is not an integral domain. Recall from [**5**] that *the zero-divisor graph* of R, denoted by $\Gamma(R)$, is defined as the graph whose vertex set is $Z(R)^*$ and distinct $x, y \in Z(R)^*$ are joined by an edge in this graph if and only if $xy = 0$. Many interesting and inspiring theorems are known about zero-divisor graphs. Several researchers investigated in the area of zero-divisor graphs, and the theorems proved

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in this topic either characterize rings R such that $\Gamma(R)$ has a specific graph theoretic property or describe the relationship between the ring theoretic properties of R and the graph theoretic properties of $\Gamma(R)$. Among many interesting research papers that appeared in this area, we mention the following research papers which mainly motivate the present work [**1, 4, 5, 8, 9, 15, 16**].

First, it is useful to recall the following definitions from the theory of graphs. Let $G = (V, E)$ be a graph. Recall from [**7**, Definition 8.1.1] that G is said to be *planar* if G can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G.

Let $n \in \mathbf{N}$. A complete graph on n vertices is denoted by K_n. For any set A, we denote by $|A|$, the cardinality of A. A graph G is said to be *bipartite* if its vertex set can be partitioned into two nonempty subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. A bipartite graph with vertex partition V_1 and V_2 is said to be *complete* if each element of V_1 is adjacent to every element of V_2. Let $m, n \in \mathbf{N}$. Let $G = (V, E)$ be a complete bipartite graph with $V = V_1 \cup V_2$. If $|V_1| = m$ and $|V_2| = n$, then G is denoted by $K_{m,n}$ [**7**, Definitions 1.1.12].

Recall that two adjacent edges of a graph G are said to be *in series* if their common end vertex is of degree two [**11**, page 9]. Two graphs are said to be *homeomorphic* if one graph can be obtained from the other by insertion of vertices of degree two or by the merger of edges in series [**11**, page 100]. It is useful to note from [**11**, page 93] that the graph K_5 is referred to as *Kuratowski's first graph* and the graph $K_{3,3}$ is referred to as *Kuratowski's second graph*. The celebrated theorem of Kuratowski states that a graph G is planar if and only if G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [**11**, Theorem 5.9].

Let $G = (V, E)$ be a simple graph. Recall from [**7**, Definition 1.1.13] that the *complement* of G denoted by G^c is defined by taking $V(G^c) = V$ and making two vertices u and v adjacent in G^c if and only if they are not adjacent in G.

Let R be a commutative ring with identity, and let $|Z(R)^*| \geq 2$. Suppose that $(\Gamma(R))^c$ contains at least one edge. Motivated by the research work done on planar zero-divisor graphs which is presented in the research papers [**1, 4, 9, 15, 16**], in this article, we make an attempt to determine all rings R such that $(\Gamma(R))^c$ is planar. For an excellent survey of zero-divisor graphs in commutative rings and a clear history on the problem of planar zero-divisor graphs, the reader is referred to [**3**].

Let $G = (V, E)$ be a graph. By a *clique* of G we mean a complete subgraph of G [**7**, Definition 1.2.2]. Let $G = (V, E)$ be a simple graph. By the *clique number* of G, denoted by $\omega(G)$, we mean the largest positive integer n such that G contains a clique on n vertices [**7**, page 185, Definition]. If G contains a clique on n vertices for all $n \geq 1$, then we set $\omega(G) = \infty$.

We now proceed to describe the results that are proved in this article. First, it is convenient to name the following conditions satisfied by a graph $G = (V, E)$ so that it can be used throughout this work.

Let $G = (V, E)$ be a graph.

(i) We say that G satisfies (C_1) if G does not contain K_5 as a subgraph (that is, equivalently, if $\omega(G) \leq 4$).

(ii) We say that G satisfies (C_2) if G does not contain $K_{3,3}$ as a subgraph.

Next, we recall the following definition from commutative ring theory. Let I be an ideal of a ring R, $I \neq R$. A prime ideal P of R is said to be a *maximal N-prime* of I if P is maximal with respect to the property of being contained in $Z_R(R/I)$ [**13**]. Note that maximal N-primes always exist by theorems of Zorn and Krull [**14**, Section 1-1]. Thus, a prime ideal P of R is a maximal N-prime of (0) if P is maximal with respect to the property of being contained in $Z(R)$. Let $\{P_\alpha\}_{\alpha \in \Lambda}$ be the set of all maximal N-primes of (0) in R. It is well known that $Z(R) = \cup_{\alpha \in \Lambda} P_\alpha$ [**14**, Theorem 2].

This article consists of five sections. Let R be a commutative ring with identity. Initially it is observed in Section 2 (see Proposition 2.1) that if R has more than three maximal N-primes of (0), then $(\Gamma(R))^c$ does not satisfy (C_1) and it does not satisfy (C_2). The main result of this section is Proposition 2.4 which determines necessary and sufficient conditions in order that $(\Gamma(R))^c$ be planar where R is a ring with exactly three maximal N-primes of (0).

In Section 3, we consider commutative rings R with identity such that R admits exactly two maximal N-primes of (0) and focus our study on the problem of characterizing when $(\Gamma(R))^c$ is planar. With the help of Lemma 3.1, Remark 3.2, Lemmas 3.3 and 3.4 and Proposition 3.5, the main result of this section is deduced in Remark 3.6, which provides, up to isomorphism of rings, a complete list of rings R such that $(\Gamma(R))^c$ is planar.

In Section 4, we consider commutative rings R with identity such that $|Z(R)^*| \geq 2$, R has exactly one maximal N -prime of (0), and determine necessary and sufficient conditions in order that $(\Gamma(R))^c$ be planar under the assumption that $(\Gamma(R))^c$ contains at least one edge. The main results proved in this section are Propositions 4.15 and 4.19 and Corollary 4.20. We prove in Proposition 4.15 that $(\Gamma(R))^c$ satisfies (C_2) implies that $(\Gamma(R))^c$ satisfies (C_1). It is shown in Proposition 4.19 that $(\Gamma(R))^c$ satisfies (C_2) if and only if $|R| = 8$ or 16 and $(\Gamma(R))^c$ is planar. With the help of Proposition 4.19, [**5**, Theorem 3.2], and relevant results from [**9, 10**], in Corollary 4.20 we provide, up to isomorphism of rings, a complete list of rings R such that $(\Gamma(R))^c$ is planar.

Let T be a commutative ring with identity which has a unique maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in T. Suppose that $(\Gamma(T))^c$ contains at least one edge. In Section 5, we attempt to answer the question of determining rings T such that $(\Gamma(T))^c$ satisfies (C_1). The main result of this section is Corollary 5.5 in which it is proved that $(\Gamma(T))^c$ satisfies (C_2) if and only if $(\Gamma(T))^c$ satisfies (C_1) and $|P/P^2| \leq 4$.

Let R be a commutative ring with identity which admits at least two nonzero zero-divisors. This article determines, up to isomorphism of rings, all rings R such that $(\Gamma(R))^c$ is planar(see Proposition 2.4, Remark 3.6 and Corollary 4.20). We observe from the results proved in this article that, if a ring R is such that $(\Gamma(R))^c$ is planar, then R must be finite (except in the case where R has exactly one maximal N-prime of (0) such that $(\Gamma(R))^c$ has no edges) and, moreover, $|R| \in \{4, 6, 8, 9, 10, 12, 15, 16, 20, 25\}$.

2. Characterization of rings R such that R has exactly three maximal N-primes of (0) and $(\Gamma(R))^c$ is planar. Let R be a commutative ring with identity which is not an integral domain. The following proposition gives a bound on the maximal N-primes of (0) in R so that $(\Gamma(R))^c$ satisfies either (C_1) or (C_2). We make use of the following observation in the proofs of several results of this article.

Suppose that a graph G contains K_6 as a subgraph. Then G admits K_5 as a subgraph and it also contains $K_{3,3}$ as a subgraph. Hence, G does not satisfy (C_1) and it does not satisfy (C_2).

Proposition 2.1. *Let R be a commutative ring with identity. If R has more than three maximal N-primes of* (0)*, then* $(\Gamma(R))^c$ *does not satisfy* (C_1) *and it does not satisfy* (C_2).

Proof. In view of the hypothesis on the set of all N-primes of (0) in R, it is possible to find a subset $\{P_1, P_2, P_3, P_4\}$ of the set of all maximal N-primes of (0) in R. Let $1 \leq i \leq 4$. Let $x_i \in P_i \setminus (\cup_{j\in\{1,2,3,4\}\setminus\{i\}} P_j)$. Let $x_5 \in (P_1 \cap P_2) \setminus (P_3 \cup P_4)$, and let $x_6 \in (P_2 \cap P_3) \setminus (P_1 \cup P_4)$. It is easy to verify that $x_i \neq x_j$ and $x_i x_j \neq 0$ for all $i, j \in \{1, 2, 3, 4, 5, 6\}$ with $i \neq j$. Hence, the subgraph of $(\Gamma(R))^c$ induced on $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a clique. This implies, as is observed in the beginning of this section, that $(\Gamma(R))^c$ does not satisfy (C_1) and it does not satisfy (C_2). $\square$

Let R be a commutative ring with identity which has exactly three maximal N-primes of (0). Let $\{P_1, P_2, P_3\}$ be the set of all maximal N-primes of (0) in R. Our aim is to determine necessary and sufficient conditions in order that $(\Gamma(R))^c$ be planar. In order to achieve that aim, we next have the following lemma.

Lemma 2.2. *Let R be a commutative ring with identity which has exactly three maximal N-primes of* (0)*. Let $\{P_1, P_2, P_3\}$ be the set of all maximal N-primes of* (0) *in R. If $\cap_{i=1}^3 P_i \neq (0)$, then* $(\Gamma(R))^c$ *does not satisfy* (C_1) *and it does not satisfy* (C_2).

Proof. Let $x \in \cap_{i=1}^3 P_i$, $x \neq 0$. Let $x_i \in P_i \setminus (\cup_{j\in\{1,2,3\}\setminus\{i\}} P_j)$. Let $y_i = x_i + x$. If $i \neq j$, then $x_i \in P_i \setminus P_j$ and $y_i \in P_i \setminus P_j$. Hence, $x_i \neq x_j, y_i \neq y_j$, and $x_i \neq y_j$. Moreover, there is a $k \in \{1, 2, 3\}$ such that $x_i, x_j \notin P_k$ and $y_i, y_j \notin P_k$. Hence, $x_i x_j \neq 0$, $y_i y_j \neq 0$ and $x_i y_j \neq 0$. For any i, as $x \neq 0$, $x_i \neq y_i$. As $x_i, y_i \notin P_j$ for any $j \neq i$, it follows that $x_i y_i \neq 0$. Hence, we obtain that the subgraph of $(\Gamma(R))^c$ induced on $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ is a clique and hence it contains K_5 and $K_{3,3}$ as subgraphs. Thus, if $\cap_{i=1}^3 P_i \neq (0)$, then $(\Gamma(R))^c$ does not satisfy (C_1) and it does not satisfy (C_2). $\square$

Let R, $\{P_1, P_2, P_3\}$ be as in the statement of Lemma 2.2. If $|R/P_i| \geq 3$ for some $i \in \{1, 2, 3\}$, then we show in Proposition 2.4 that $(\Gamma(R))^c$ does not satisfy either (C_1) or (C_2). We need the following result for proving Proposition 2.4.

Lemma 2.3. *Let R_1, R_2, and R_3 be integral domains. Let $T = R_1 \times R_2 \times R_3$. If $|R_i| \geq 3$ for some $i \in \{1,2,3\}$, then $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).*

Proof. We may assume, without loss of generality, that $|R_1| \geq 3$. Let $x \in R_1 \setminus \{0,1\}$. Let $t_1 = (1,0,0)$, $t_2 = (1,1,0)$, $t_3 = (1,0,1)$, $t_4 = (x,0,0)$, $t_5 = (x,1,0)$ and $t_6 = (x,0,1)$. Since R_1 is an integral domain, it follows that $x^2 \neq 0$ and so the subgraph of $(\Gamma(T))^c$ induced on $\{t_1,t_2,t_3,t_4,t_5,t_6\}$ is a clique. Hence, we obtain that the $(\Gamma(T))^c$ contains K_5 and $K_{3,3}$ as subgraphs. Thus, if $|R_i| \geq 3$ for some $i \in \{1,2,3\}$, then $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2). $\square$

Let R be a commutative ring with identity. Suppose that R has exactly three maximal N-primes of (0). The following proposition is the main result of this section, and it determines necessary and sufficient conditions in order that $(\Gamma(R))^c$ be planar. For any $n > 1$, we denote by $\mathbf{Z}_n$, the ring of integers modulo n.

Proposition 2.4. *Let R be a commutative ring with identity which has exactly three maximal N-primes of (0). Let $\{P_1,P_2,P_3\}$ be the set of all maximal N-primes of (0) in R. The following statements are equivalent*:

(i) $(\Gamma(R))^c$ *satisfies* (C_1).

(ii) $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ *as rings.*

(iii) $(\Gamma(R))^c$ *satisfies* (C_2).

(iv) $(\Gamma(R))^c$ *is planar.*

Proof. (i) $\Rightarrow$ (ii). It follows from Lemma 2.2 that $\cap_{i=1}^{3} P_i = (0)$. Since $(\Gamma(R))^c$ satisfies (C_1), $\omega((\Gamma(R))^c) \leq 4$. Now it follows from [**17**, Proposition 4.4] that R is finite. Hence, each prime ideal of R is a maximal ideal of R. Now we obtain from the Chinese remainder theorem that $R = R/(0) = R/\cap_{i=1}^{3} P_i \cong R/P_1 \times R/P_2 \times R/P_3$ as rings. On applying Lemma 2.3, we obtain that $|R/P_i| = 2$ for each $i \in \{1,2,3\}$. Hence, we arrive at $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ as rings.

(ii) $\Rightarrow$ (iii). Let $T = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. Since $R \cong T$ as rings, it is enough to show that $(\Gamma(T))^c$ satisfies (C_2). Note that $Z(T)^* = \{z_1 = (1,0,0), z_2 = (0,1,0), z_3 = (0,0,1), z_4 = (1,1,0), z_5 = (0,1,1), z_6 = (1,0,1)\}$. Since $|Z(T)^*| = 6$ and as $(\Gamma(T))^c$ admits vertices of degree two (indeed $\deg_{(\Gamma(T))^c}(z_i) = 2$ for $i = 1, 2, 3$), it follows that $(\Gamma(T))^c$ satisfies (C_2).

(iii) $\Rightarrow$ (ii). Suppose that (iii) holds. Then $\omega((\Gamma(R))^c) \leq 5$. Hence, it follows from [**17**, Proposition 4.4] that R is finite. Note that (ii) follows as in the proof of (i) $\Rightarrow$ (ii) with the help of Lemmas 2.2 and 2.3.

(ii) $\Rightarrow$ (i). Let T be as in the proof of $(ii) \Rightarrow (iii)$. Since $R \cong T$ as rings, it is enough to show that $(\Gamma(T))^c$ satisfies (C_1). It can be easily verified that $\omega((\Gamma(T))^c) = 3$. Hence we obtain that $(\Gamma(T))^c$ satisfies (C_1).

(iv) $\Rightarrow$ (i) is clear since a subgraph of a planar graph is planar and it is well known that K_5 is nonplanar.

(i) $\Rightarrow$ (iv). It follows from (i) $\Rightarrow$ (ii) that $R \cong \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ as rings. Let $T = \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$. It is easy to verify that $(\Gamma(T))^c$ is the union of the cycles Γ_i for $i = 1, 2$, with $\Gamma_1 : (1,1,0) - (1,0,0) - (1,0,1) - (0,0,1) - (0,1,1) - (0,1,0) - (1,1,0)$ and $\Gamma_2 : (1,1,0) - (1,0,1) - (0,1,1) - (1,1,0)$. The cycle Γ_1 can be represented by means of a hexagon. The vertex set of the cycle Γ_2 is a subset of the vertex set of the hexagon that represents Γ_1. Hence, the cycle Γ_2 can be drawn inside this hexagon in such a way that there is no crossing over of the edges of $(\Gamma(T))^c$. Hence we obtain that $(\Gamma(T))^c$ is planar and so $(\Gamma(R))^c$ is planar.

3. Determination of rings R such that R has exactly two maximal N-primes of (0) and $(\Gamma(R))^c$ is planar. Let R be a commutative ring with identity, and suppose that R has exactly two maximal N-primes of (0). Let $\{P_1, P_2\}$ denote the set of all maximal N-primes of (0) in R. Our aim in this section is to determine ring theoretic necessary and sufficient conditions in order that $(\Gamma(R))^c$ be planar. If $(\Gamma(R))^c$ is planar, then $(\Gamma(R))^c$ must satisfy (C_1) and (C_2). Hence $\omega((\Gamma(R))^c) \leq 5$. Therefore, we obtain from [**17**, Proposition 4.4] that R is finite. Since any prime ideal of a finite ring is maximal, it follows from [**14**, Theorem 84] that any prime ideal of R is a subset of the set of zero-divisors of R. Hence, it follows that $\{P_1, P_2\}$ is the set of all prime ideals of R. Let $(0) = q_1 \cap q_2$ be the irredundant primary decomposition of (0) in R where q_i is P_i-primary for $i = 1, 2$. Since $P_1 + P_2 = R$, it follows that $q_1 + q_2 = R$. Therefore, we obtain from the Chinese remainder theorem [**6**, Proposition 1.10] that $R \cong R/q_1 \times R/q_2$ as rings. In view of the above discussion, in this section we consider rings R of the form $R_1 \times R_2$ where R_i admits exactly one maximal

N-prime of (0) for $i = 1, 2$ and study the problem of when $(\Gamma(R))^c$ is planar. We initially prove several lemmas and present the main result of this section in Remark 3.6. We start with the following lemma.

Lemma 3.1. *Let T_1 and T_2 be commutative rings with identity such that each of them admits a unique maximal N-prime of* (0) *(that is, equivalently, $Z(T_i)$ is an ideal of T_i for $i = 1, 2$). Let $T = T_1 \times T_2$. Then the following hold*:

(i) *If both T_1 and T_2 are not domains, then $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).*

(ii) *If T_i is a domain for some $i \in \{1, 2\}$ with $|T_i| \geq 6$, then $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).*

Proof. (i) Let us denote $Z(T_i)$ by N_i for $i = 1, 2$. Since T_i is not an integral domain for $i = 1, 2$, it follows that $|N_i| \geq 2$ for $i = 1, 2$. Let $a \in N_1 \setminus \{0\}$ and $b \in N_2 \setminus \{0\}$. Note that $1 + a \notin N_1 = Z(T_1)$ and $1 + b \notin N_2 = Z(T_2)$. Hence, it follows that the subgraph of $(\Gamma(T))^c$ induced on $\{(a, 1), (a, b), (a, 1 + b), (1, 0), (1, b), (1 + a, 0)\}$ is a clique. Thus, $(\Gamma(T))^c$ contains K_6 as a subgraph. Hence, $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).

This proves that, if both T_1 and T_2 are not domains, then $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).

(ii) We may assume, without loss of generality, that T_1 is a domain with $|T_1| \geq 6$. We claim that $|T_1|$ must exceed 6. This is clear if T_1 is infinite. If T_1 is finite, then as the number of elements in any finite field is a prime power, it follows that $|T_1| > 6$. Let $\{x_i | i \in \{1, 2, 3, 4, 5, 6\}\} \subseteq T_1 \setminus \{0\}$. Observe that the subgraph of $(\Gamma(T))^c$ induced on $\{(x_i, 0) \mid i \in \{1, 2, 3, 4, 5, 6\}\}$ is a clique. Hence, $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2). This proves (ii). □

Remark 3.2. Let $R = R_1 \times R_2$ where R_i is a finite local ring with unique maximal ideal M_i for $i = 1, 2$. Note that $Z(R_i) = M_i$ for $i = 1, 2$. Suppose that $(\Gamma(R))^c$ satisfies either (C_1) or (C_2). Then it follows immediately from Lemma 3.1 (i) that either R_1 or R_2 is a field. □

Let F_1 and F_2 be fields, and let $R = F_1 \times F_2$. The following lemma provides necessary and sufficient conditions on F_1 and F_2 so that $(\Gamma(R))^c$ is planar.

Lemma 3.3. *Let F_1 and F_2 be fields. Let $R = F_1 \times F_2$. Then the following statements are equivalent*:

(i) $(\Gamma(R))^c$ *satisfies* (C_1).

(ii) $|F_i| \leq 5$ *for* $i = 1, 2$.

(iii) $(\Gamma(R))^c$ *satisfies* (C_2).

(iv) $(\Gamma(R))^c$ *is planar.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii) follow immediately from Lemma 3.1 (ii).

(ii) $\Rightarrow$ (iv). Observe that $(\Gamma(R))^c$ consists of exactly two components H_1, H_2 with $V(H_1) = \{(\alpha, 0) \mid \alpha \in F_1 \setminus \{0\}\}$ and $V(H_2) = \{(0, \beta) \mid \beta \in F_2 \setminus \{0\}\}$. Since $|F_i| \leq 5$, it follows that $|V(H_i)| \leq 4$ for $i = 1, 2$. Hence, H_i is planar for $i = 1, 2$ and so we obtain that $(\Gamma(R))^c$ is planar.

(iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) follow because any subgraph of a planar graph is planar and, moreover, K_5 and $K_{3.3}$ are nonplanar. $\square$

Let F_1 be a field and R_2 a commutative ring with identity which is not an integral domain. Suppose that R_2 has exactly one maximal N-prime of (0). Let $R = F_1 \times R_2$. We determine in Proposition 3.5 when $(\Gamma(R))^c$ is planar. We make use of the following lemma in the proof of Proposition 3.5.

Lemma 3.4. *Let F be a field and S a commutative ring with identity which is not an integral domain. Suppose that S has exactly one maximal N-prime of* (0). *Let P be the unique maximal N-prime of (*0) *in S. Let $T = F \times S$. Then the following hold*:

(i) *If* $|F| \geq 4$, *then* $(\Gamma(T))^c$ *does not satisfy either* (C_1) *or* (C_2).

(ii) *If* $|S \setminus P| \geq 4$, *then* $(\Gamma(T))^c$ *does not satisfy either* (C_1) *or* (C_2).

Proof. (i) As $|F| \geq 4$, it is possible to find a subset $\{\alpha, \beta, \gamma\}$ of $F \setminus \{0\}$. Let $x \in P \setminus \{0\}$. Let $A = \{(\alpha, 0), (\beta, 0), (\gamma, 0), (\alpha, x), (\beta, x), (\gamma, x)\}$. Note that $A \subseteq Z(T)^*$ and the subgraph of $(\Gamma(T))^c$ induced on A is a clique. Hence, $(\Gamma(T))^c$ contains K_6 as a subgraph. Therefore, $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2).

(ii) By assumption, $|S \setminus P| \geq 4$. Hence it follows from [**12**, Theorem 1] that $P = Z(S)$ must contain at least three elements. One can also appeal to [**5**] to conclude that $|P| \geq 3$. Indeed, if $|P| \leq 2$, then since S is not an integral domain, $|P| = 2$. Hence, it follows from [**5**] that S is isomorphic either to $\mathbf{Z}_4$ or to $\mathbf{Z}_2[x]/x^2\mathbf{Z}_2[x]$. This implies that $|S \setminus P| = 2$. This is a contradiction. Hence, $|P| \geq 3$. Let $\{s_1, s_2, s_3, s_4\} \subseteq S \setminus P$, and let $\{x, y\} \subseteq P \setminus \{0\}$. Let $B = \{(0, s_1), (0, s_2), (0, s_3), (0, s_4), (1, x), (1, y)\}$. Observe that $B \subseteq Z(T)^*$. Since $s_i \notin Z(S)$ for all $i \in \{1, 2, 3, 4\}$, it follows that the subgraph of $(\Gamma(T))^c$ induced on B is a clique. Thus, if $|S \setminus P| \geq 4$, then $(\Gamma(T))^c$ contains K_6 as a subgraph, and hence we obtain that $(\Gamma(T))^c$ does not satisfy either (C_1) or (C_2). $\square$

The following proposition characterizes rings $R = F_1 \times R_2$ such that $(\Gamma(R))^c$ is planar where F_1 and R_2 are as in the paragraph just preceding the statement of Lemma 3.4.

Proposition 3.5. *Let F_1 be a field and R_2 a commutative ring with identity which is not an integral domain. Suppose that R_2 has exactly one maximal N-prime of* (0). *Let $R = F_1 \times R_2$. Then the following statements are equivalent*:

(i) $(\Gamma(R))^c$ *satisfies* (C_1).

(ii) $|F_1| \leq 3$ *and* $|R_2| = 4$.

(iii) $(\Gamma(R))^c$ *satisfies* (C_2).

(iv) $(\Gamma(R))^c$ *is planar.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii). Let N_2 denote the unique maximal N-prime of (0) in R_2. If either (i) or (iii) holds, then it follows from Lemma 3.4 that $|F_1| \leq 3$ and $|R_2 \setminus N_2| \leq 3$. Since $1 + x \in R_2 \setminus N_2$, for all $x \in N_2$, it follows that $|N_2| \leq 3$. Thus, $|R_2| = |N_2| + |R_2 \setminus N_2| \leq 3 + 3 = 6$. Since R_2 is not a field, $4 \leq |R_2|$. Hence, $4 \leq |R_2| \leq 6$. Now R_2 is a finite ring with N_2 as its unique maximal N-prime of (0). Since any prime ideal of a finite ring is maximal and any prime ideal of a finite ring is contained in its set of all zero-divisors, it follows that R_2 is a finite local ring with N_2 as its unique maximal ideal. Therefore, $|R_2|$ must be a prime power. Hence, we obtain that $|R_2| = 4$. This proves (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iv). Observe that $|Z(R_2)^*| = 1$, $|R_2 \setminus Z(R_2)| = 2$. Let $Z(R_2) = \{0, a\}$. Note that $R_2 = \{0, 1, a, 1 + a\}$. Now either $|F_1| = 2$ or $|F_1| = 3$.

Let $|F_1| = 2$. Then $F_1 = \{0, 1\}$. Now $Z(R)^* = \{(0, a), (0, 1), (0, 1 + a), (1, 0), (1, a)\}$. It is easy to verify that $(\Gamma(R))^c$ is the union of the cycle $\Gamma : (0,1) - (0,a) - (0,1+a) - (1,a) - (0,1)$ and two edges e_1, e_2 where $e_1 : (0,1) - (0,1+a)$ and $e_2 : (1,a) - (1,0)$. The cycle Γ can be represented by means of a rectangle. Note that e_1 is a diagonal of the rectangle that represents Γ. Now it is clear that $(\Gamma(R))^c$ can be drawn in a plane in such a way that there is no crossing over of the edges, and hence we obtain that $(\Gamma(R))^c$ is planar.

Let $|F_1| = 3$. Let $F_1 = \{0, 1, \alpha\}$. Note that $Z(R)^* = \{(0,1), (0,a), (0,1+a), (1,0), (1,a), (\alpha,0), (\alpha,a)\}$. Observe that the subgraph H_1 of $(\Gamma(R))^c$ induced on $\{(1,a), (\alpha,a), (1,0), (\alpha,0)\}$ is a complete graph on four vertices and the subgraph H_2 of $(\Gamma(R))^c$ induced on $\{(1,a), (\alpha,a), (0,1), (0,1+a)\}$ is also a complete graph on four vertices. Moreover, $(\Gamma(R))^c$ is the union of H_1, H_2 and the cycle $\Gamma : (0,1+a) - (0,a) - (0,1) - (0,1+a)$. Observe that the edge $(1,a) - (\alpha,a)$ is the only edge which is common to both H_1 and H_2. Note that the cycle Γ has no edge in common with H_1 and $(0,1) - (0,1+a)$ is the only edge which is common to both H_2 and Γ. Since K_4 is planar, it follows from the above description of $(\Gamma(R))^c$ that it can be drawn in a plane in such a way that there is no crossing over of the edges. This proves that $(\Gamma(R))^c$ is planar.

(iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) follow as in the argument given for (iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) of Lemma 3.3. □

The following remark determines up to isomorphism of rings, the commutative rings R with identity such that R has exactly two maximal N-primes of (0) and $(\Gamma(R))^c$ is planar.

Remark 3.6. Let R be a commutative ring with identity. Suppose that R has exactly two maximal N-primes of (0). If $(\Gamma(R))^c$ satisfies either (C_1) or (C_2), then it is noted in the beginning of this section that R is necessarily finite and, moreover, if $(0) = q_1 \cap q_2$ is the irredundant primary decomposition of (0) in R, then it follows from the Chinese remainder theorem that $R \cong R/q_1 \times R/q_2$ as rings. Observe that it follows from Lemma 3.1, Remark 3.2, Lemmas 3.3 and 3.4 and Proposition 3.5 that $(\Gamma(R))^c$ satisfies (C_1) if and only if $(\Gamma(R))^c$ satisfies (C_2) if and only if $(\Gamma(R))^c$ is planar if and only if R is isomorphic to exactly one of the rings given below:

(i) $F_1 \times F_2$ where F_1 and F_2 are fields with $|F_i| \leq 5$ for $i = 1, 2$.

(ii) $\mathbf{Z}_2 \times \mathbf{Z}_4$.

(iii) $\mathbf{Z}_2 \times \mathbf{Z}_2[x]/x^2\mathbf{Z}_2[x]$ where $\mathbf{Z}_2[x]$ is the polynomial ring in one variable over $\mathbf{Z}_2$.

(iv) $\mathbf{Z}_3 \times \mathbf{Z}_4$.

(v) $\mathbf{Z}_3 \times \mathbf{Z}_2[x]/x^2\mathbf{Z}_2[x]$. $\square$

4. Characterization of rings R such that R has exactly one maximal N-prime of (0) and $(\Gamma(R))^c$ is planar. Let R be a commutative ring with identity and suppose that R has exactly one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in R. We assume that $(\Gamma(R))^c$ contains at least one edge. That is, there exist $a, b \in P$ such that $a \neq b$ and $ab \neq 0$. It is useful to mention here that $(\Gamma(R))^c$ contains at least one edge equivalent to the condition that $P^2 \neq (0)$ [**5**, Theorem 2.8]. The aim of this section is to determine all such rings R such that $(\Gamma(R))^c$ is planar. We prove in this section that, for such rings R, $(\Gamma(R))^c$ is planar if and only if $(\Gamma(R))^c$ satisfies both (C_1) and (C_2). It is worth mentioning here that [**11**, Figure 5.9 (a)] illustrates that a graph satisfying both (C_1) and (C_2) need not be planar.

It is noted in Section 2 that, if T is a commutative ring with identity with exactly three maximal N-primes of (0), then $(\Gamma(T))^c$ satisfies (C_1) if and only if it satisfies (C_2) if and only if $(\Gamma(T))^c$ is planar. In Section 3, it is proved that, for any commutative ring S with identity which has exactly two maximal N-primes of (0), $(\Gamma(S))^c$ satisfies (C_1) if and only if it satisfies (C_2) if and only if $(\Gamma(S))^c$ is planar. The following example illustrates that there exists a commutative ring R with identity such that R has exactly one maximal N-prime of (0) with the property that $(\Gamma(R))^c$ satisfies (C_1) but it does not satisfy (C_2).

Example 4.1. Let $T = \mathbf{Z}_2[x_1, x_2, x_3]$ be the polynomial ring in three variables over $\mathbf{Z}_2$. Let I be the ideal of T generated by $\{x_1^2, x_2^2, x_3^2, x_2x_3, x_3x_1\}$. Let $R = T/I$. Then $(\Gamma(R))^c$ satisfies (C_1) but it does not satisfy (C_2).

Proof. Observe that R is a finite local ring with unique maximal ideal $M = N/I$ where $N = Tx_1 + Tx_2 + Tx_3$. Moreover, $M^2 \neq (0)$, but $M^3 = (0)$. We show that $(\Gamma(R))^c$ satisfies (C_1) by proving that $\omega((\Gamma(R))^c) = 3$.

For any element $f(x_1, x_2, x_3) \in T$, we denote $f(x_1, x_2, x_3) + I$ by $\overline{f(x_1, x_2, x_3)}$. Note that $\overline{x_1 x_2} \neq \overline{0}$. Let $r_1 = \overline{x_1}, r_2 = \overline{x_1 + x_2}, r_3 = \overline{x_2 + x_3}$. Observe that the subgraph of $(\Gamma(R))^c$ induced on $\{r_1, r_2, r_3\}$ is a clique. Hence, $\omega((\Gamma(R))^c) \geq 3$. Let $A \subseteq Z(R)^* = M \setminus \{\overline{0}\}$ be such that A contains exactly 4 elements. We claim that the subgraph of $(\Gamma(R))^c$ induced on A is not a clique. Suppose that the subgraph of $(\Gamma(R))^c$ induced on A is a clique. Let $A = \{a_1, a_2, a_3, a_4\}$. Note that $a_i = \overline{a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i12}x_1x_2}$ where $a_{i1}, a_{i2}, a_{i3}, a_{i12} \in \mathbf{Z}_2$ for $i = 1, 2, 3, 4$. Since $a_i a_j \neq \overline{0}$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$, it follows that a_{i2} must be 1 for at least three values of $i \in \{1, 2, 3, 4\}$. We may assume, without loss of generality, that $a_{12} = a_{22} = a_{32} = 1$. Since $a_1 a_2 \neq \overline{0}$, it follows that exactly one between a_{11} and a_{21} must be 0. We may assume that $a_{11} = 1$ and $a_{21} = 0$. Now either $a_{31} = 1$ or $a_{31} = 0$. If $a_{31} = 1$, then $a_1 a_3 = \overline{0}$ and this is impossible. If $a_{31} = 0$, then $a_2 a_3 = \overline{0}$. This is also impossible. This proves that the subgraph of $(\Gamma(R))^c$ induced on A cannot be a clique. Hence, $\omega((\Gamma(R))^c) \leq 3$ and so $\omega((\Gamma(R))^c) = 3$. Therefore, we obtain that $(\Gamma(R))^c$ satisfies (C_1).

We next verify that $(\Gamma(R))^c$ does not satisfy (C_2). Let $r_1 = \overline{x_1}$, $r_2 = \overline{x_2}$, $r_3 = \overline{x_2 + x_3}$, $r_4 = \overline{x_1 + x_2}$, $r_5 = \overline{x_1 + x_2 + x_1x_2}$ and $r_6 = \overline{x_1 + x_2 + x_3}$. Let $V_1 = \{r_1, r_2, r_3\}$, $V_2 = \{r_4, r_5, r_6\}$. Observe that the subgraph of $(\Gamma(R))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. Hence, $(\Gamma(R))^c$ does not satisfy (C_2). $\square$

We prove in Proposition 4.15 that if a commutative ring R with identity is such that R has exactly one maximal N-prime of (0), $(\Gamma(R))^c$ contains at least one edge and, if it satisfies (C_2), then it satisfies (C_1).

Let T be a commutative ring with identity with exactly one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in T. It was shown in [**17**, Proposition 4.21] that if $\omega((\Gamma(T))^c) < \infty$, then P is nilpotent. Indeed, it was shown in [**17**, Proposition 4.21] that $P^n = (0)$ where $n = (\omega((\Gamma(T))^c))^2 + 1$. The following lemma is an improvement of [**17**, Proposition 4.21] and, moreover, we need this lemma to prove other results in this note.

Lemma 4.2. *Let T be a commutative ring with identity which has exactly one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in T. Let $n \geq 4$. If $\omega((\Gamma(T))^c) \leq n$, then $P^{n-1} = (0)$.*

Proof. Note that $Z(T) = P$. We first show that, for any $a, a_1, \ldots, a_{n-3} \in P$, $a^2 a_1 \cdots a_{n-3} = 0$. Suppose that $a^2 a_1 \cdots a_{n-3} \neq 0$. Let

$$\begin{aligned} t_1 &= a, \\ t_2 &= a + aa_1 = a(1 + a_1), \\ t_3 &= a + aa_1 + aa_1a_2 = a(1 + a_1 + a_1a_2), \ldots \\ t_{n-2} &= a + aa_1 + aa_1a_2 + \cdots + aa_1a_2 \cdots a_{n-3} \\ &= a(1 + a_1 + a_1a_2 + \cdots + a_1a_2 \cdots a_{n-3}), \\ t_{n-1} &= a + aa_1 + aa_1a_2 + \cdots + aa_1a_2 \cdots a_{n-3} + a^2a_1a_2 \cdots a_{n-3} \\ &= a(1 + a_1 + a_1a_2 + \cdots + a_1a_2 \cdots a_{n-3} + aa_1a_2 \cdots a_{n-3}), \end{aligned}$$

$t_n = a + a^2a_1a_2 \cdots a_{n-3} = a(1 + aa_1a_2 \cdots a_{n-3})$, and $t_{n+1} = aa_1$. Using the assumption that $a^2 a_1 \cdots a_{n-3} \neq 0$, and the fact that, for any $x \in P$, $1 + x \notin P = Z(T)$, it can be easily verified that $t_i \neq t_j$ and $t_i t_j \neq 0$ for all $i, j \in \{1, 2, \ldots, n+1\}$ with $i \neq j$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $\{t_k | k = 1, 2, \ldots, n+1\}$ is a clique. This is impossible since, by hypothesis, $\omega((\Gamma(T))^c) \leq n$. Therefore, we obtain that $a^2 a_1 \cdots a_{n-3} = 0$ for all $a, a_1, \ldots, a_{n-3} \in P$.

Let $x_1, x_2, \ldots, x_{n-1} \in P$ be such that $x_i \neq x_j$ for all $i, j \in \{1, 2, \ldots, n-1\}$ with $i \neq j$. We claim that $x_1 x_2 \cdots x_{n-1} = 0$. Suppose that $x_1 x_2 \cdots x_{n-1} \neq 0$. We assert that $x_k^2 = 0$ for each $k = 1, 2, \ldots, n-1$. Suppose that $x_k^2 \neq 0$ for some $k \in \{1, \ldots, n-1\}$. We may assume without loss of generality that $x_1^2 \neq 0$. For $i \in \{1, 2, \ldots, n+1\}$, let $y_i \in P$ be defined by $y_i = x_i$ for $i = 1, 2, \ldots, n-1$, $y_n = x_1 + x_1x_2 \cdots x_{n-2} = x_1(1 + x_2 \cdots x_{n-2})$, and $y_{n+1} = x_1 + x_1x_2 \cdots x_{n-1} = x_1(1 + x_2 \cdots x_{n-1})$. We know from the first paragraph of this proof that $a^2 a_1 \cdots a_{n-3} = 0$ for any $a, a_1, \ldots, a_{n-3} \in P$. Using this and the assumptions that $x_1^2 \neq 0, x_1x_2 \cdots x_{n-1} \neq 0$, and the fact that, for any $a \in P$, $1 + a \notin P = Z(T)$, it can be easily verified that $y_i \neq y_j$ for all distinct $i, j \in \{1, 2, \ldots, n+1\}$ and the subgraph of $(\Gamma(T))^c$ induced on $\{y_k \mid k = 1, 2, \ldots, n+1\}$ is a clique. This contradicts the hypothesis that $\omega((\Gamma(T))^c) \leq n$. Hence, we obtain that $x_k^2 = 0$ for each $k = 1, 2, \ldots, n-1$.

Let $z_i \in P$ for $i = 1, 2, \ldots, n+1$ be defined by $z_i = x_i$ for $i = 1, 2, \ldots, n-1$, $z_n = x_1 + x_2x_3 \cdots x_{n-2}$ and $z_{n+1} = x_1 + x_2 \cdots x_{n-1}$. Using the fact that $x_k^2 = 0$ for each $k = 1, 2, \ldots, n-1$ and the assumption that $x_1x_2 \cdots x_{n-1} \neq 0$, it can be shown that $z_i \neq z_j$, and $z_i z_j \neq 0$ for all $i, j \in \{1, 2, \ldots, n+1\}$ with $i \neq j$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $\{z_i \mid i = 1, 2, \ldots, n+1\}$ is a clique. This is impossible since $\omega((\Gamma(T))^c) \leq n$. Thus, for any $(n-1)$ distinct elements $x_i \in P$ for $i = 1, 2, \ldots, n-1$, $x_1 x_2 \cdots x_{n-1} = 0$.

This proves that, if $\omega((\Gamma(T))^c) \leq n$, then $P^{n-1} = (0)$. □

As an immediate consequence of Lemma 4.2, we have the following corollary. We state this as a separate corollary for the sake of convenient reference.

Corollary 4.3. *Let T be a commutative ring with identity which has exactly one maximal N-prime of* (0). *Let P be the unique maximal N-prime of* (0) *in T. Then the following hold*:

(i) *If* $(\Gamma(T))^c$ *satisfies* (C_1), *then* $P^3 = (0)$.

(ii) *If* $(\Gamma(T))^c$ *satisfies* (C_2), *then* $P^4 = (0)$.

Proof. (i) Assume that $(\Gamma(T))^c$ satisfies (C_1). Hence, $(\Gamma(T))^c$ does not contain K_5 as a subgraph and so $\omega((\Gamma(T))^c) \leq 4$. Now, on applying Lemma 4.2 with $n = 4$, we obtain that $P^3 = (0)$.

(ii) Assume that $(\Gamma(T))^c$ satisfies (C_2). Therefore, $(\Gamma(T))^c$ does not contain $K_{3,3}$ as a subgraph. As K_6 admits $K_{3,3}$ as a subgraph, it follows that $(\Gamma(T))^c$ does not contain K_6 as a subgraph. Hence, $\omega((\Gamma(T))^c) \leq 5$ and so, on applying Lemma 4.2 with $n = 5$, we obtain that $P^4 = (0)$. □

Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ does not contain any infinite clique, then it was shown in [**17**, Lemma 4.14 (iii)] that $P = ((0) :_T c)$, for some $c \in P \setminus \{0\}$, and in [**17**, Lemma 4.14 (ii)], it was shown that T/P is finite. We now proceed to show below that $|T/P| = 2$ if $(\Gamma(T))^c$ satisfies either (C_1) or (C_2). Towards that goal, we first have the following result.

Lemma 4.4. *Let T be a commutative ring with identity. Suppose that T has only one maximal N-prime of* (0), *and let it be P. If* $\omega((\Gamma(T))^c) < \infty$, *and if there exists an $a \in P$ such that $a^2 \neq 0$, and* $|T/P| > 3$, *then* $\omega((\Gamma(T))^c) \geq 6$.

Proof. By hypothesis, $|T/P| > 3$. Hence, there exist $t, s \in T \setminus P$ such that $t - 1 \notin P$, $s - 1 \notin P$ and $t - s \notin P$. As $\omega((\Gamma(T))^c) < \infty$, it follows from Lemma 4.2 that P is nilpotent. Moreover, by hypothesis, $a^2 \neq 0$ for some $a \in P$. Hence, $P^2 \neq (0)$. Let $m \geq 3$ be least with the property that $P^m = (0)$. Then, for any $c \in P^{m-1} \setminus \{0\}$, $P = ((0) :_T c)$.

Let $v_1 = a$, $v_2 = at$, $v_3 = as$, $v_4 = a + c$, $v_5 = a + ct$ and $v_6 = c + at$ where $c \in P \setminus \{0\}$ is such that $P = ((0) :_T c)$. Using the assumption that $a^2 \neq 0$, it follows from the choice of t, s that, for all distinct $i, j \in \{1, 2, 3, 4, 5, 6\}$, $v_i \neq v_j$ and $v_i v_j \neq 0$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $\{v_k | k \in \{1, 2, 3, 4, 5, 6\}\}$ is a clique. This implies that $\omega((\Gamma(T))^c) \geq 6$. □

We next have the following useful lemma which is a consequence of Lemma 4.4.

Lemma 4.5. *Let T, P be as in the statement of Lemma* 4.2*. Suppose that $P^2 \neq (0)$. If $\omega(\Gamma(T))^c \leq 5$ and if $|T/P| > 3$, then $2 \in P$.*

Proof. By hypothesis, $P^2 \neq (0)$. Hence, by [**3**, Theorem 2.8], there exist $a, b \in P$ such that $a \neq b$ and $ab \neq 0$. Moreover, by assumption, $\omega((\Gamma(T))^c) \leq 5$ and $|T/P| > 3$. Hence, it follows from Lemma 4.4, that $x^2 = 0$ for each $x \in P$. Therefore, $a^2 = b^2 = (a + b)^2 = 0$. Hence, we obtain that $2ab = 0$. Since $ab \neq 0$, it follows that $2 \in Z(T) = P$. □

The following lemma is also used in proving that if $(\Gamma(T))^c$ satisfies (C_1), then $|T/P| = 2$.

Lemma 4.6. *Let T, P be as in the statement of Lemma* 4.2*. Suppose that $P^2 \neq (0)$. If $\omega((\Gamma(T))^c) \leq 4$, then $|T/P| \leq 3$.*

Proof. Since $P^2 \neq (0)$, there exist $a, b \in P$ such that $a \neq b$ and $ab \neq 0$. Suppose that $|T/P| > 3$. Let $t, s \in T \setminus P$ be such that $\{t - 1, s - 1, t - s\} \subseteq T \setminus P$. Since $\omega((\Gamma(T))^c) \leq 4$, it follows from Lemma 4.4 that $a^2 = b^2 = 0$. Consider the elements of P given by $w_1 = a$, $w_2 = a + bt$, $w_3 = a + bs$, $w_4 = b$ and $w_5 = a + b$. Since $a^2 = b^2 = 0$ and $ab \neq 0$, it follows from the choice of t, s that $w_i \neq w_j$ for all distinct $i, j \in \{1, 2, 3, 4, 5\}$. Moreover, it follows from Lemma 4.5 that $2 \in P$ and, as $\{t - 1, s - 1, t - s\} \subseteq T \setminus P$, we obtain that $\{t + 1, s + 1, t + s\} \subseteq T \setminus P$. Using these facts, it can be easily verified that $w_i w_j \neq 0$ for all distinct $i, j \in \{1, 2, 3, 4, 5\}$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $\{w_k | k \in \{1, 2, 3, 4, 5\}\}$ is a clique. This contradicts the assumption that $\omega((\Gamma(T))^c) \leq 4$. This proves that $|T/P| \leq 3$. $\square$

We next have the following lemma which we use in the proof of Propositions 4.8 and 4.10.

Lemma 4.7. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$ and $\omega((\Gamma(T))^c) < \infty$. If $|T/P| = 3$, then $\omega((\Gamma(T))^c) \geq 6$.*

Proof. By hypothesis, $\omega((\Gamma(T))^c) < \infty$. Hence, it follows as in the proof of Lemma 4.4 that there exists a $c \in P \setminus \{0\}$ such that $P = ((0) :_T c)$. Moreover, by assumption, $|T/P| = 3$. Therefore, there exists a $t \in T \setminus P$ such that $t - 1 \notin P$. Since $P^2 \neq (0)$, there exist $a, b \in P$ with $a \neq b$ and $ab \neq 0$. As $|T/P| = 3$, $2 \notin P$. Hence from $ab \neq 0$, it follows that $2ab \neq 0$. Now it is clear that one among a^2, b^2 and $(a+b)^2$ must be nonzero. Either renaming a and b or on replacing a by $a+b$, if necessary, we may assume without loss of generality that $a, b \in P$ are such that $a \neq b$, $ab \neq 0$ and $a^2 \neq 0$. Consider the elements of P given by $z_1 = a$, $z_2 = ta$, $z_3 = a + c$, $z_4 = ta + c$, $z_5 = a + tc$ and $z_6 = ta + tc$. Since $a^2 \neq 0$, it follows from the choice of t that the elements z_1, z_2, z_3, z_4, z_5, and z_6 are distinct and, moreover, $z_i z_j \neq 0$ for all distinct $i, j \in \{1, 2, 3, 4, 5, 6\}$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $\{z_1, z_2, z_3, z_4, z_5, z_6\}$ is a clique. This implies that $\omega((\Gamma(T))^c) \geq 6$. $\square$

Proposition 4.8. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_1), then $|T/P| = 2$.*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_1). That is, equivalently, $\omega((\Gamma(T))^c) \leq 4$. It now follows from Lemmas 4.6 and 4.7 that $|T/P| = 2$. $\square$

The following lemma is also used in the proof of Proposition 4.10.

Lemma 4.9. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_2), then $|T/P| \leq 3$.*

Proof. By hypothesis, $(\Gamma(T))^c$ satisfies (C_2). Hence, $\omega((\Gamma(T))^c) \leq 5$. Suppose that $|T/P| > 3$. Then, it follows from Lemma 4.4 that $a^2 = 0$ for each $a \in P$. Let $t, s \in T \setminus P$ be such that $\{t-1, s-1, t-s\} \subseteq T \setminus P$. From Lemma 4.5, we obtain that $2 \in P$.

Since $P^2 \neq (0)$, there exist $a, b \in P$ such that $a \neq b$ and $ab \neq 0$. Let $W_1 = \{a, a+tb, a+sb\}$, and let $W_2 = \{b, a+b, a+b+c\}$ where $c \in P \setminus \{0\}$ is such that $P = ((0) :_T c)$. Since $2 \in P$, it follows that $\{t+1, s+1, t+s\} \subseteq T \setminus P$. Since $a^2 = b^2 = 0$ and $ab \neq 0$, it can be easily verified with the help of the choice of the elements t, s that $W_1 \cap W_2 = \varnothing$ and, moreover, each element of W_1 is adjacent to each element of W_2 in $(\Gamma(T))^c$. Hence the subgraph of $(\Gamma(T))^c$ induced on $W_1 \cup W_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $(\Gamma(T))^c$ satisfies (C_2). Hence, we obtain that $|T/P| \leq 3$.

This completes the proof of Lemma 4.9. $\square$

Proposition 4.10. *Let T, P be as in the statement of Lemma* 4.2. *Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_2), then $|T/P| = 2$.*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_2). Then it follows immediately from Lemmas 4.9 and 4.7 that $|T/P| = 2$. $\square$

The following lemma is another step that we need to prove that if $(\Gamma(T))^c$ satisfies (C_2), then it satisfies (C_1).

Lemma 4.11. *Let T, P be as in the statement of Lemma* 4.2. *Suppose that $(\Gamma(T))^c$ satisfies (C_2). If there exists an ideal I of T with $I \subseteq ((0) :_T P)$ and $|I| = 2$, then $(\Gamma(T))^c$ satisfies (C_1).*

Proof. Suppose that $(\Gamma(T))^c$ does not satisfy (C_1). Then there exists a subset $A = \{a_1, a_2, a_3, a_4, a_5\} \subseteq P \setminus \{0\}$ such that the subgraph of $(\Gamma(T))^c$ induced on A is a clique. Hence, $a_i a_j \neq 0$ for all distinct $i, j \in \{1, 2, 3, 4, 5\}$.

Let $i \in \{1, 2, 3, 4, 5\}$. We assert that there exists a $j \in \{1, 2, 3, 4, 5\} \setminus \{i\}$ such that $a_i - a_j \in I$. Suppose that it does not hold. Then, for some $i \in \{1, 2, 3, 4, 5\}$, $a_i - a_j \notin I$ for all $j \in \{1, 2, 3, 4, 5\} \setminus \{i\}$. Without loss of generality, we may assume that $a_1 - a_j \notin I$ for all $j \in \{2, 3, 4, 5\}$. Let $I = \{0, c\}$. Let $V_1 = \{a_1, a_1 + c, a_2\}$ and $V_2 = \{a_3, a_4, a_5\}$. Then it is clear that $V_1 \cap V_2 = \varnothing$ and, moreover, using the fact that $a_i a_j \neq 0$ for all distinct $i, j \in \{1, 2, 3, 4, 5\}$ and the hypothesis that $I \subseteq ((0) :_T P)$, it follows that the subgraph of $(\Gamma(T))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $(\Gamma(T))^c$ satisfies (C_2). Thus, for each $i \in \{1, 2, 3, 4, 5\}$, there exists a $j \in \{1, 2, 3, 4, 5\} \setminus \{i\}$ such that $a_i - a_j \in I$.

Since $|I \setminus \{0\}| = 1$, it follows that such a j is necessarily unique. We may assume, without loss of generality, that $a_1 - a_2 \in I$. Then $a_3 - a_1$

and $a_3 - a_2$ cannot be in I. Hence, either $a_3 - a_4 \in I$ or $a_3 - a_5 \in I$. We may assume, without loss of generality, that $a_3 - a_4 \in I$. Observe that $a_5 - a_j \notin I$ for all $j \in \{1, 2, 3, 4\}$. This is a contradiction. This proves that, if there exists an ideal I of T with $I \subseteq ((0) :_T P)$, $|I| = 2$, and if $(\Gamma(T))^c$ satisfies (C_2), then $(\Gamma(T))^c$ satisfies (C_1). □

Lemma 4.12. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If P is a principal ideal of T, and if $(\Gamma(T))^c$ satisfies (C_2), then $P^3 = (0)$ and $|P^2| = 2$, and $(\Gamma(T))^c$ satisfies (C_1).*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_2). We know from Corollary 4.3 (ii) that $P^4 = (0)$ and, moreover, from Proposition 4.10, we know that $|T/P| = 2$. We are assuming that P is a principal ideal of T. Let $x \in P$ be such that $P = Tx$. We first show that $P^3 = (0)$.

Suppose that $P^3 \neq (0)$. Then $x^3 \neq 0$. Let $V_1 = \{x, x + x^2, x + x^3\}$ and $V_2 = \{x^2, x^2 + x^3, x + x^2 + x^3\}$. It is clear that $V_1 \cup V_2 \subseteq P \setminus \{0\}$, $V_1 \cap V_2 = \varnothing$, and the subgraph of $(\Gamma(T))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $(\Gamma(T))^c$ satisfies (C_2). Hence, $P^3 = (0)$.

So, the T-module structure on P^2 induces a T/P-module structure on P^2, and hence P^2 is a vector space over the field T/P. Since P^2 is a nonzero principal ideal of T, it follows that $\dim_{T/P} P^2 = 1$. As $|T/P| = 2$, we obtain that $|P^2| = 2$. Now $P^3 = (0)$, and so $P^2 \subseteq ((0) :_T P)$. Since $|P^2| = 2$, it follows from Lemma 4.11 that $(\Gamma(T))^c$ satisfies (C_1). □

We also need the following lemma in the proof of the assertion that if $(\Gamma(T))^c$ satisfies (C_2), then $(\Gamma(T))^c$ satisfies (C_1).

Lemma 4.13. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If T/P is a field, P is nilpotent (hence P is the only prime ideal of T), and if $\dim_{T/P}(P/P^2) \geq m$ with $m \geq 2$, then there exist $a_1, a_2, \ldots, a_m \in P$ such that $a_1 a_2 \neq 0$ and $\{a_1 + P^2, a_2 + P^2, \ldots, a_m + P^2\}$ is linearly independent over T/P.*

Proof. Let $\{a_\alpha\}_{\alpha \in \Lambda} \subseteq P \setminus \{0\}$ be such that $\{a_\alpha + P^2\}_{\alpha \in \Lambda}$ forms a basis of P/P^2 as a vector space over T/P. Let $I = \Sigma_{\alpha \in \Lambda} T a_\alpha$. Note that $P = I + P^2$. Hence, $P = I + (I + P^2)^2 = I + P^4$. Proceeding in this way, we obtain that $P = I + P^{2t}$ for all $t \geq 1$, and hence $P = I + P^k$

for all $k \geq 2$. We are assuming that P is nilpotent. Hence, $P^n = (0)$ for some $n \in \mathbf{N}$. As $P^2 \neq (0)$, it follows that $n \geq 3$. As $P = I + P^n$, we obtain that $P = I$.

Suppose that $a_{\alpha_1}a_{\alpha_2} \neq 0$ for some distinct $\alpha_1, \alpha_2 \in \Lambda$. Let $A \subseteq \Lambda \setminus \{\alpha_1, \alpha_2\}$ be such that $|A| = m - 2$. Let $A = \{\alpha_3, \ldots, \alpha_m\}$. Let $a_i = a_{\alpha_i}$ for $i = 1, 2, \ldots, m$. Then it is clear that $a_1a_2 \neq 0$ and $\{a_1 + P^2, a_2 + P^2, \ldots, a_m + P^2\}$ is linearly independent over T/P.

Suppose that $a_\alpha a_\beta = 0$ for all distinct $\alpha, \beta \in \Lambda$. Then, from $P = I = \Sigma_{\alpha\in\Lambda} T a_\alpha$, we obtain that $P^2 = I^2 = \Sigma_{\alpha\in\Lambda} T(a_\alpha)^2$. Since $P^2 \neq (0)$, it follows that $(a_{\alpha_1})^2 \neq 0$ for some $\alpha_1 \in \Lambda$. Let $\{\alpha_2, \ldots, \alpha_m\} \subseteq \Lambda \setminus \{\alpha_1\}$. Let $a_1 = a_{\alpha_1}, a_2 = a_{\alpha_1} + a_{\alpha_2}, a_i = a_{\alpha_i}$ for $i = 3, \ldots, m$. Then it is clear that the elements $a_1, a_2, \ldots, a_m$ satisfy the conclusion of the lemma. $\square$

We next have the following lemma which is also needed for proving some results of this paper.

Lemma 4.14. *Let T, P be as in the statement of Lemma* 4.2. *Let I be an ideal of T such that $I \subseteq ((0) :_T P)$. If $(\Gamma(T))^c$ satisfies (C_2), and if there exist $a, b \in P \setminus \{0\}$ such that $ab \neq 0$ and $a - b \notin I$, then $|I| \leq 2$.*

Proof. Suppose that $|I| \geq 3$. Let $c_1, c_2 \in I \setminus \{0\}$ be such that $c_1 \neq c_2$. Let $W_1 = \{a, a + c_1, a + c_2\}$ and $W_2 = \{b, b + c_1, b + c_2\}$. Since $a - b \notin I$, it is clear that $a \neq b$. Using the hypotheses that $ab \neq 0, a - b \notin I$ and $I \subseteq ((0) :_T P)$, it is easy to verify that the subgraph of $(\Gamma(T))^c$ induced on $W_1 \cup W_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $(\Gamma(T))^c$ satisfies (C_2). Hence, we obtain that $|I| \leq 2$. $\square$

With T, P as in the statement of Lemma 4.2, we prove below in the following proposition that, if $(\Gamma(T))^c$ satisfies (C_2), then it satisfies (C_1).

Proposition 4.15. *Let T, P be as in the statement of Lemma* 4.2. *Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_2), then $P^3 = (0)$, $|P^2| = 2$, and moreover, $(\Gamma(T))^c$ satisfies (C_1).*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_2). We know from Corollary 4.3 (ii) that $P^4 = (0)$. Since $P \neq (0)$, it follows that $P \neq P^2$. It is shown in Proposition 4.10 that $|T/P| = 2$. Note that P/P^2 is a nonzero vector space over the field T/P. Let $\{a_\alpha\}_{\alpha\in\Lambda} \subseteq P$ be such that $\{a_\alpha + P^2\}_{\alpha\in\Lambda}$ is a basis of P/P^2 as a vector space over T/P. Then $P = \Sigma_{\alpha\in\Lambda} T a_\alpha$, as is noted in the proof of Lemma 4.13. We consider the following two cases.

Case (i). $\dim_{T/P}(P/P^2) = 1$. In this case, P is principal. Hence, we obtain from Lemma 4.12 that $P^3 = (0)$, $|P^2| = 2$ and, furthermore, $(\Gamma(T))^c$ satisfies (C_1).

Case (ii). $\dim_{T/P}(P/P^2) \geq 2$. We know from Lemma 4.13 that there exist $a, b \in P$ such that $ab \neq 0$ and $\{a + P^2, b + P^2\}$ is linearly independent over T/P. Hence, $a - b \notin P^2$. Since $P^4 = (0)$, it follows that $P^3 \subseteq ((0) :_T P)$.

Suppose that $P^3 \neq (0)$. Then $|P^3| \geq 2$. As $a - b \notin P^2$, it follows that $a - b \notin P^3$. Now we obtain from Lemma 4.14 that $|P^3| = 2$. On applying Lemma 4.11 with $I = P^3$, it follows that $(\Gamma(T))^c$ satisfies (C_1). This implies by Corollary 4.3 (i) that $P^3 = (0)$. This is a contradiction. Hence, $P^3 = (0)$.

So $P^2 \subseteq ((0) :_T P)$. Now $a, b \in P, ab \neq 0$ and $a - b \notin P^2$. Hence, on applying Lemma 4.14 with $I = P^2$, we obtain that $|P^2| = 2$. It now follows immediately from Lemma 4.11 that $(\Gamma(T))^c$ satisfies (C_1).

This shows that, if $P^2 \neq (0)$ and if $(\Gamma(T))^c$ satisfies (C_2), then $P^3 = (0)$, $|P^2| = 2$, and, moreover, $(\Gamma(T))^c$ satisfies (C_1). □

Recall that a commutative ring R with identity is called a *chained ring* if the principal ideals of R are comparable under the inclusion relation (equivalently, the ideals of R are comparable under the inclusion relation). Let R be a chained ring with M as its unique maximal ideal. If $M^2 \neq (0)$, then it is known that $(\Gamma(R))^c$ does not contain any infinite clique if and only if $\omega((\Gamma(R))^c)$ is finite if and only if R is finite [**17**, Proposition 4.16]. Thus, if a chained ring R is such that $(\Gamma(R))^c$ contains at least one edge and if it satisfies either (C_1) or (C_2) (and hence satisfies (C_1) by Proposition 4.15), then R must be finite.

Let R be a chained ring with M as its unique maximal ideal. Suppose that $M^2 \neq (0)$. The following proposition characterizes when $(\Gamma(R))^c$ is planar.

Proposition 4.16. *Let R be a chained ring which is not an integral domain. Let M denote the unique maximal ideal of R. Suppose that $M^2 \neq (0)$. The following statements are equivalent*:

(i) $(\Gamma(R))^c$ *satisfies* (C_1).

(ii) $|R| = 8$.

(iii) $(\Gamma(R))^c$ *satisfies* (C_2).

(iv) $(\Gamma(R))^c$ *is planar.*

Proof. (i) $\Rightarrow$ (ii). We know from Corollary 4.3 (i) that $M^3 = (0)$. Moreover, we know from Proposition 4.8 that $|R/M| = 2$. Since $\omega((\Gamma(R))^c) \leq 4$, it follows from [**17**, Proposition 4.16] that R is finite. As R is a finite chained ring, M must be a principal ideal of R. Since $M^3 = (0)$, M^2 is a vector space over the field R/M. As M is principal and M^2 is nonzero, it follows that $\dim_{R/M} M^2 = 1$ and, moreover, as $M \neq M^2$, we obtain that $\dim_{R/M}(M/M^2) = 1$. Since $|R/M| = 2$, it follows that $|M^2| = 2$ and $|M/M^2| = 2$. As $|M| = |M/M^2||M^2|$, we obtain that $|M| = 4$. Thus, $|R| = |R/M||M| = 8$.

(ii) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv). Since $|R| = 8$, Lagrange's theorem implies that $|M| \leq 4$. As $M^2 \neq (0)$, it follows from [**5**, Theorem 2.8] that there exist $a, b \in M$ with $a \neq b$ such that $ab \neq 0$. It is clear that $a \neq ab$ and $b \neq ab$. Hence, M contains at least 4 elements and so $|M| = 4$. Thus $(\Gamma(R))^c$ is a graph on 3 vertices. Now it is clear that $(\Gamma(R))^c$ satisfies (C_2) and $(\Gamma(R))^c$ is planar.

(iv) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) follow immediately since K_5 and $K_{3,3}$ are nonplanar and a subgraph of a planar graph is planar.

(iii) $\Rightarrow$ (i). This follows immediately from Proposition 4.15.

This completes the proof of Proposition 4.16. $\square$

With the help of Proposition 4.16 and [**4**, Theorem 3.2], we determine in the following remark all chained rings R such that $(\Gamma(R))^c$ contains at least one edge and is planar.

Remark 4.17. Observe that if R is one of the rings from the collection $\{\mathbf{Z}_8, \mathbf{Z}_2[x]/x^3\mathbf{Z}_2[x], \mathbf{Z}_4[x]/(2x\mathbf{Z}_4[x] + (x^2 - 2)\mathbf{Z}_4[x])\}$ (where $\mathbf{Z}_2[x]$, respectively $\mathbf{Z}_4[x]$, denotes the polynomial ring in one variable over $\mathbf{Z}_2$, respectively over $\mathbf{Z}_4$), then it is easy to verify that R is a chained ring with the property that the number of elements in its unique maximal ideal equals 4, square of its unique maximal ideal is nonzero and $|R| = 8$. Hence, $(\Gamma(R))^c$ is a graph with at least one edge and as it is a graph on three vertices, it is clearly planar.

We next verify with the help of [**4**, Theorem 3.2] that if R is any chained ring with the property that $(\Gamma(R))^c$ admits at least one edge and is planar, then R is isomorphic to one of the rings mentioned in the preceding paragraph. Let R be chained such that $(\Gamma(R))^c$ admits at least one edge and is planar. Hence, $(\Gamma(R))^c$ satisfies (C_1) and (C_2). As was remarked before the statement of Proposition 4.16, we obtain that R is finite. Let M denote the unique maximal ideal of R. It follows from the proof of (i) $\Rightarrow$ (ii) of Proposition 4.16 that $M^3 = (0)$, $|M| = 4$, $|R| = 8$. Let $a, b \in M$ be such that $a \neq b$ and $ab \neq 0$. Observe that $ab \notin \{a, b\}$. Thus, $M = \{0, a, b, ab\}$. Moreover, as $ab \neq 0$ and $a^2b = ab^2 = 0$, it follows that $\omega((\Gamma(R)) = 2$. Now it follows from [**4**, Theorem 3.2] that R is isomorphic to exactly one of the following rings: $\mathbf{Z}_8, \mathbf{Z}_2[x]/x^3\mathbf{Z}_2[x]$, $\mathbf{Z}_4[x]/(2x\mathbf{Z}_4[x] + (x^2-2)\mathbf{Z}_4[x])$. Thus, if R is a chained ring satisfying the conditions

(i) $(\Gamma(R))^c$ contains at least one edge and

(ii) $(\Gamma(R))^c$ is planar,

then R is isomorphic to one of the rings mentioned above. $\square$

Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. It is proved in Proposition 4.8 that, if $(\Gamma(T))^c$ satisfies (C_1), then $|T/P| = 2$ and in Proposition 4.10 that, if $(\Gamma(T))^c$ satisfies (C_2), then $|T/P| = 2$. If $(\Gamma(T))^c$ satisfies (C_2), then in the following lemma we show that $\dim_{T/P}(P/P^2) \leq 2$.

Lemma 4.18. *Let T be a commutative ring with identity, and suppose that T has only one maximal N-prime of* (0). *Let P be the unique maximal N-prime of* (0) *in T. Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_2), then $\dim_{T/P}(P/P^2) \leq 2$.*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_2). We know from Proposition 4.10 that $|T/P| = 2$. Moreover, we know from Proposition 4.15 that $P^3 = (0)$, $|P^2| = 2$ and $(\Gamma(T))^c$ satisfies (C_1).

Suppose that $\dim_{T/P}(P/P^2) \geq 3$. Then it follows from Lemma 4.13 that there exist $a, b, c \in P$ such that $ab \neq 0$ and $\{a+P^2, b+P^2, c+P^2\}$ is linearly independent over T/P. Observe that we have one of the following possibilities:

(i) $bc = ca = 0$,

(ii) $bc \neq 0, ca = 0$,

(iii) $bc = 0, ca \neq 0$, and

(iv) $bc \neq 0, ca \neq 0$.

We now discuss the above-mentioned possibilities separately.

Case (i). $ab \neq 0$, $bc = ca = 0$. In this case, we assert that $a^2 \neq 0$, $b^2 \neq 0$ and $c^2 \neq 0$. Suppose that $a^2 = 0$. Let $V_1 = \{a, a+ab, a+c\}$, $V_2 = \{a+b, b, b+ab\}$. It follows from the choice of the elements a, b, and c that $V_1 \cap V_2 = \varnothing$. Now, using the fact that $P^3 = (0)$ and the assumptions that $ab \neq 0$, $bc = ca = a^2 = 0$, it can easily be verified that each element of V_1 is adjacent to every element of V_2 in $(\Gamma(T))^c$. Hence, we obtain that the subgraph of $(\Gamma(T))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $(\Gamma(T))^c$ satisfies (C_2). Thus, $a^2 \neq 0$. Similarly, if $b^2 = 0$, it can be shown that the subgraph of $(\Gamma(T))^c$ induced on $W_1 \cup W_2$ (where $W_1 = \{b, b+ab, b+c\}$ and $W_2 = \{a, a+b, a+ab\}$) contains $K_{3,3}$ as a subgraph. This is impossible and so $b^2 \neq 0$. Similarly, if $c^2 = 0$, then the subgraph of $(\Gamma(T))^c$ induced on $A_1 \cup A_2$ (where $A_1 = \{b, b+ab, b+c\}$ and $A_2 = \{a, a+c, a+ab\}$) contains $K_{3,3}$ as a subgraph. This is impossible, and so $c^2 \neq 0$. Let $U_1 = \{a, a+c, b+c\}$ and $U_2 = \{b, b+b^2, a+c^2\}$.

It can easily be verified that the subgraph of $(\Gamma(T))^c$ induced on $U_1 \cup U_2$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $(\Gamma(T))^c$ satisfies (C_2). This shows that Case (i) cannot hold.

Case (ii). $ab \neq 0$, $bc \neq 0$, $ca = 0$. We claim that $a^2 \neq 0$ and $c^2 \neq 0$. Suppose that $a^2 = 0$. Let $U_1 = \{a, c, a+ab\}$ and $U_2 = \{b, a+b, b+bc\}$. Using the choice of the elements a, b and c, it is straightforward to show that $U_1 \cap U_2 = \varnothing$. Moreover, using the fact that $P^3 = (0)$ and the assumptions that $a^2 = ac = 0$, $ab \neq 0$ and $bc \neq 0$, it follows that each element of U_1 is adjacent to every element of U_2 in $(\Gamma(T))^c$. Hence, the subgraph of $(\Gamma(T))^c$ induced on $U_1 \cup U_2$ contains $K_{3.3}$ as a subgraph. This is impossible. Hence, $a^2 \neq 0$. Similarly, it follows that $c^2 \neq 0$. Let $V_1 = \{a, c, a+ab\}$ and $V_2 = \{a+c, b, b+ab\}$. It is easy to show that the subgraph of $(\Gamma(T))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is impossible. Hence, Case (ii) cannot hold.

Case (iii). $ab \neq 0$, $ca \neq 0$, $bc = 0$. Proceeding as in Case (ii), it can be shown that Case (iii) is impossible.

Case (iv). $ab \neq 0$, $bc \neq 0$, $ca \neq 0$. We claim that at least one among a^2, b^2 and c^2 must be equal to 0. Suppose that $a^2 \neq 0$,

$b^2 \neq 0$ and $c^2 \neq 0$. Then the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, c, a+ab, b+ab, c+ab\}$ is a clique. This is impossible since $(\Gamma(T))^c$ satisfies (C_2). Hence, at least one among a^2, b^2 and c^2 must be equal to 0.

We may assume, without loss of generality, that $a^2 = 0$. We next claim that either $b^2 = 0$ or $c^2 = 0$. Suppose that $b^2 \neq 0$ and $c^2 \neq 0$. Let $V_1 = \{a+b, a+c, c\}$ and $V_2 = \{a, a+ab, a+b+c\}$. Now we have $a^2 = 0$, $b^2 \neq 0$ and $c^2 \neq 0$. Moreover, we know from Proposition 4.15 that $|P^2| = 2$. Thus, $ab = bc = ca$. We know from Proposition 4.10 that $|T/P| = 2$. As $P^3 = (0)$, it follows that $ab+bc = bc+ca = ca+ab = 0$. Using these facts, it can be easily verified that the subgraph of $(\Gamma(T))^c$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $(\Gamma(T))^c$ satisfies (C_2). Hence, either $b^2 = 0$ or $c^2 = 0$.

We may assume without loss of generality that $b^2 = 0$. Let $W_1 = \{a, b, a+ab\}$ and $W_2 = \{a+b, c, c+ab\}$. It can be easily verified that the subgraph of $(\Gamma(T))^c$ induced on $W_1 \cup W_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the hypothesis that $(\Gamma(T))^c$ satisfies (C_2). This shows that Case (iv) cannot hold.

It is now clear from the above discussion that $\dim_{T/P}(P/P^2) \leq 2$. $\square$

Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. The following proposition characterizes when $(\Gamma(T))^c$ satisfies (C_2). We verify in the following proposition, among other equivalent conditions, that $(\Gamma(T))^c$ satisfies (C_2) if and only if $(\Gamma(T))^c$ is planar. Moreover, the following proposition is one of the main results in this paper.

Proposition 4.19. *Let T be a commutative ring with identity which admits exactly one maximal N-prime of* (0). *Let P be the unique maximal N-prime of* (0) *in T. Suppose that $(\Gamma(T))^c$ admits at least one edge. Then the following statements are equivalent*:

(i) $(\Gamma(T))^c$ *satisfies* (C_2).

(ii) $(\Gamma(T))^c$ *satisfies both* (C_1) *and* (C_2).

(iii) $P^3 = (0)$, $|T/P| = 2$, $|P^2| = 2$ *and* $\dim_{T/P}(P/P^2) \leq 2$.

(iv) $|T|$ *is either* 8 *or* 16 *and* $(\Gamma(T))^c$ *is planar.*

(v) $(\Gamma(T))^c$ *is planar.*

Proof. (i) $\Rightarrow$ (ii). We know from Proposition 4.15 that $(\Gamma(T))^c$ satisfies (C_1). Hence, (i) $\Rightarrow$ (ii) holds.

(ii) $\Rightarrow$ (iii). By hypothesis, $(\Gamma(T))^c$ admits at least one edge. Hence, there are elements $a, b \in P, a \neq b$, and $ab \neq 0$. Therefore, $P^2 \neq (0)$. It is now clear that (ii) $\Rightarrow$ (iii) follows immediately from Propositions 4.8 and 4.15 and Lemma 4.18.

(iii) $\Rightarrow$ (iv). Since $\dim_{T/P}(P/P^2) \leq 2$ and $|T/P| = 2$, it follows that $|P/P^2| \leq 4$. Since $P \neq P^2$, and $|T/P| = 2$, we obtain that either $|P/P^2| = 2$ or $|P/P^2| = 4$. We have $|P^2| = 2$. Thus, either $|P| = 4$ or $|P| = 8$. As $|T/P| = 2$, we obtain that either $|T| = 8$ or $|T| = 16$. If $|P| = 4$, then $(\Gamma(T))^c$ is a graph on three vertices and is clearly planar. Suppose that $|P| = 8$. Then $\dim_{T/P}(P/P^2) = 2$. Let $a, b \in P$ be such that $\{a + P^2, b + P^2\}$ forms a basis of P/P^2 as a vector space over T/P with $ab \neq 0$. Thus, $P = \{0, a, b, a+b, ab, a+ab, b+ab, a+b+ab\}$ and $P^2 = \{0, ab\}$. Since $P^3 = (0)$, ab is an isolated vertex of $(\Gamma(T))^c$. We need to consider the following cases.

Case (i). $a^2 = b^2 = 0$. In this case it is easy to verify that, except for the isolated vertex ab, each of the other vertices are of degree 4 in $(\Gamma(T))^c$. Moreover, it is easy to show that $(\Gamma(T))^c$ is the union of the cycles $\Gamma_1 : a - b - (a+b+ab) - (b+ab) - (a+b) - a$, $\Gamma_2 : a - (a+b+ab) - (b+ab) - a$, $\Gamma_3 : (b+ab) - (a+b+ab) - (a+ab) - (b+ab)$, the edges $e_1 : (a+b) - b, e_2 : b - (a+ab)$, $e_3 : (a+ab) - (a+b)$, and the isolated vertex ab. The cycle Γ_1 can be represented by means of a pentagon. Note that the cycle Γ_2 is a triangle enclosed by one side and two diagonals of the pentagon representing Γ_1. Observe that Γ_3 has only one edge in common with Γ_1. The vertex $a + ab$ of Γ_3 can be plotted outside this pentagon. It is easy to see that the cycle Γ_3 and the edges e_1, e_2, e_3 can be drawn outside this pentagon without any crossing over of the edges. The above discussion shows that $(\Gamma(T))^c$ can be drawn in a plane without any crossing over of the edges. This proves that $(\Gamma(T))^c$ is planar.

Case (ii). $a^2 \neq 0, b^2 = 0$. Now $a^2 + ab \in P^2 = \{0, ab\}$ and, as $a^2 \neq 0$, it follows that $a^2 + ab = 0$. In this case, in $(\Gamma(T))^c$, it can be easily verified that $\deg(a) = 3$, $\deg(b) = 4$, $\deg(a+b) = 3$, $\deg(a+ab) = 3$, $\deg(b+ab) = 4$, $\deg(a+b+ab) = 3$, and $\deg(ab) = 0$. It is easy to verify that $(\Gamma(T))^c$ is the union of the cycles $\Gamma_1 : a - b - (a+b+ab) - (b+ab) - a$, $\Gamma_2 : b - (a+b) - (b+ab) - (a+ab) - b$, the edges $e_1 : a - (a+ab)$,

$e_2 : (a+b+ab) - (a+b)$, and the isolated vertex ab. Observe that the cycle Γ_1 can be represented by means of a rectangle. It is not hard to verify that the vertices $a+b$, $a+ab$ can be plotted inside the rectangle representing Γ_1 in such a way that the cycle Γ_2 and the edges e_1 and e_2 can be drawn inside this rectangle so that there is no crossing over of the edges. This proves that $(\Gamma(T))^c$ is planar.

Case (iii). $a^2 = 0, b^2 \neq 0$. It follows as in Case (ii) that $(\Gamma(T))^c$ is planar.

Case (iv). $a^2 \neq 0, b^2 \neq 0$. Note that, as $a^2 \neq 0, b^2 \neq 0$, it follows as in Case (ii) that $a^2 + ab = b^2 + ab = 0$. Moreover, in $(\Gamma(T))^c$, it can be easily verified that $\deg(a) = \deg(b) = \deg(a+ab) = \deg(b+ab) = 3$, and $\deg(ab) = \deg(a+b) = \deg(a+b+ab) = 0$. Furthermore, it is easy to verify that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, a+ab, b+ab\}$ is a clique. Hence, $(\Gamma(T))^c$ is K_4 together with three isolated vertices. Hence, $(\Gamma(T))^c$ is planar.

(iv) $\Rightarrow$ (v). This is clear.

(v) $\Rightarrow$ (i). Since a planar graph cannot contain $K_{3,3}$ as a subgraph, (v) $\Rightarrow$ (i) follows immediately. $\square$

The following corollary is an immediate consequence of Proposition 4.19, [**4**, Theorem 3.2] and the list of all finite commutative local rings with identity of order 16 given in [**9**, page 475]. The list of rings given in [**9**, page 475] are listed with the help of the theorems proved in [**10**]. Let $n \geq 2$. In the list of rings given below $\mathbf{Z}_n[x]$ (respectively $\mathbf{Z}_n[x,y]$) denotes the polynomial ring in one variable (respectively the polynomial ring in two variables) over $\mathbf{Z}_n$.

Corollary 4.20. *Let R be a commutative ring with identity. Suppose that R has exactly one maximal N-prime of (0), and let it be P. Suppose that $(\Gamma(R))^c$ contains at least one edge. Then $(\Gamma(R))^c$ is planar if and only if R is isomorphic to exactly one of the following rings. In particular, if R is infinite, then $(\Gamma(R))^c$ is not planar.*

(a) $\mathbf{Z}_8$.

(b) $\mathbf{Z}_2[x]/x^3\mathbf{Z}_2[x]$.

(c) $\mathbf{Z}_4[x]/(2x\mathbf{Z}_4[x] + (x^2-2)\mathbf{Z}_4[x])$.

(d) $\mathbf{Z}_2[x,y]/(x^2\mathbf{Z}_2[x,y] + y^2\mathbf{Z}_2[x,y])$.

(e) $\mathbf{Z}_4[x,y]/(x^2\mathbf{Z}_4[x,y]+y^2\mathbf{Z}_4[x,y]+(xy-2)\mathbf{Z}_4[x,y])$.

(f) $\mathbf{Z}_4[x]/x^2\mathbf{Z}_4[x]$.

(g) $\mathbf{Z}_2[x,y]/((x^2-y^2)\mathbf{Z}_2[x,y]+xy\mathbf{Z}_2[x,y])$.

(h) $\mathbf{Z}_4[x,y]/((x^2-2)\mathbf{Z}_4[x,y]+xy\mathbf{Z}_4[x,y]+(y^2-2)\mathbf{Z}_4[x,y]+2x\mathbf{Z}_4[x,y])$.

(i) $\mathbf{Z}_4[x]/(x^2-2x)\mathbf{Z}_4[x]$.

(j) $\mathbf{Z}_8[x]/((x^2-4)\mathbf{Z}_8[x]+2x\mathbf{Z}_8[x])$.

(k) $\mathbf{Z}_2[x,y]/(x^3\mathbf{Z}_2[x,y]+xy\mathbf{Z}_2[x,y]+y^2\mathbf{Z}_2[x,y])$.

(l) $\mathbf{Z}_4[x,y]/((x^2-2)\mathbf{Z}_4[x,y]+xy\mathbf{Z}_4[x,y]+y^2\mathbf{Z}_4[x,y]+2x\mathbf{Z}_4[x,y])$.

(m) $\mathbf{Z}_4[x]/(x^3\mathbf{Z}_4[x]+2x\mathbf{Z}_4[x])$.

(n) $\mathbf{Z}_8[x]/(x^2\mathbf{Z}_8[x]+2x\mathbf{Z}_8[x])$.

Proof. Observe that the each of the rings given in the list from (a)–(n) in the statement of the corollary is a finite local ring where the square of its unique maximal ideal is nonzero whereas the cube of its maximal ideal is zero. Hence, each one of the above rings admits exactly one maximal N-prime for its zero ideal. Moreover, the complement of the zero-divisor graph of each of the above rings admits at least one edge. Furthermore, it can easily be verified that each of the rings mentioned in the list from (a)–(n) satisfies condition (iii) of Proposition 4.19. Hence, it follows from (iii) $\Rightarrow$ (iv) of Proposition 4.19 that the complement of the zero-divisor graph of each one of them is planar.

Suppose that R is a commutative ring with identity which has exactly one maximal N-prime of (0). Let P be the unique maximal N-prime of (0) in R. Suppose that $(\Gamma(R))^c$ admits at least one edge and is planar. We now verify with the help of [**4**, Theorem 3.2] and the results from [**9, 10**] that R is isomorphic to exactly one of the rings from the list of rings given in the statement of the corollary. Since $(\Gamma(R))^c$ admits at least one edge, there exist $a,b\in P$ such that $a\neq b$ and $ab\neq 0$. Thus, $P^2\neq 0$. As $(\Gamma(R))^c$ is planar, it must satisfy (C_2). Hence, we obtain from (i) $\Rightarrow$ (iii) of Proposition 4.19 that $P^3=(0)$, $|P^2|=2$, $|R/P|=2$, and $\dim_{R/P}(P/P^2)\leq 2$. Since $P\neq P^2$, $\dim_{R/P}(P/P^2)\geq 1$.

If $\dim_{R/P}(P/P^2)=1$, then R is a chained ring and, in this case, it is already verified in Remark 4.17 that R is isomorphic to exactly one among the rings mentioned in the statement of the corollary against (a), (b) or (c).

Suppose that $\dim_{R/P}(P/P^2) = 2$. Now it is clear from Lemma 4.13 that there exist $a, b \in P$ such that $\{a + P^2, b + P^2\}$ forms a basis of P/P^2 as a vector space over R/P with $ab \neq 0$. Then, as noted in the proof of (iii) $\Rightarrow$ (iv) of Proposition 4.19, $P = \{0, a, b, a+b, ab, a+ab, b+ab, a+b+ab\}$. Note that R is a finite local ring with $|R| = 16$.

We now consider the following cases.

Case (A). $a^2 = b^2 = 0$. In this case it follows from the list of finite local rings of order 16 given in [**9**, page 475] that R is isomorphic to one of the following rings:

$\mathbf{Z}_2[x,y]/(x^2\mathbf{Z}_2[x,y] + y^2\mathbf{Z}_2[x,y])$,

$\mathbf{Z}_4[x,y]/(x^2\mathbf{Z}_4[x,y] + y^2\mathbf{Z}_4[x,y] + (xy-2)\mathbf{Z}_4[x,y])$,

$\mathbf{Z}_4[x]/x^2\mathbf{Z}_4[x]$.

Thus, in this case, R is isomorphic to one among the rings mentioned in (d), (e) or (f).

Case (B). $a^2 \neq 0$, $b^2 = 0$. It follows from [**9**, page 475] that in this case R is isomorphic to one of the following rings:

$\mathbf{Z}_2[x,y]/((x^2-y^2)\mathbf{Z}_2[x,y] + xy\mathbf{Z}_2[x,y])$,

$\mathbf{Z}_4[x,y]/((x^2-2)\mathbf{Z}_4[x,y] + xy\mathbf{Z}_4[x,y] + (y^2-2)\mathbf{Z}_4[x,y] + 2x\mathbf{Z}_4[x,y])$,

$\mathbf{Z}_4[x]/(x^2-2x)\mathbf{Z}_4[x]$,

$\mathbf{Z}_8[x]/((x^2-4)\mathbf{Z}_8[x] + 2x\mathbf{Z}_8[x])$.

Thus in this case R is isomorphic to exactly one of the rings mentioned in (g), (h), (i) or (j).

Case (C). $a^2 \neq 0$, $b^2 \neq 0$. Now it follows from [**9**, page 475] that in this case R is isomorphic to exactly one of the following rings:

$\mathbf{Z}_2[x,y]/(x^3\mathbf{Z}_2[x,y] + xy\mathbf{Z}_2[x,y] + y^2\mathbf{Z}_2[x,y])$,

$\mathbf{Z}_4[x,y]/((x^2-2)\mathbf{Z}_4[x,y] + xy\mathbf{Z}_4[x,y] + y^2\mathbf{Z}_4[x,y] + 2x\mathbf{Z}_4[x,y])$,

$\mathbf{Z}_4[x]/(x^3\mathbf{Z}_4[x] + 2x\mathbf{Z}_4[x])$,

$\mathbf{Z}_8[x]/(x^2\mathbf{Z}_8[x] + 2x\mathbf{Z}_8[x])$.

Hence, in this case R is isomorphic to exactly one among the rings mentioned against (k), (l), (m) or (n).

Thus, if R is a commutative ring with identity which has exactly one maximal N-prime of (0), $(\Gamma(R))^c$ admits at least one edge, then

$(\Gamma(R))^c$ is planar if and only if R is isomorphic to exactly one of the rings mentioned against (a)–(n) in the statement of the corollary. Now it is clear that, if R is an infinite ring which has exactly one maximal N-prime of (0), and, if $(\Gamma(R))^c$ admits at least one edge, then $(\Gamma(R))^c$ is not planar. This completes the proof of Corollary 4.21. $\square$

5. Rings T with exactly one maximal N-prime of (0) such that $(\Gamma(T))^c$ satisfies (C_1). Let T, P be as in the statement of Lemma 4.2. That is, T is a commutative ring with identity which has P as its unique maximal N-prime of (0). Suppose that $(\Gamma(T))^c$ admits at least one edge. It is shown in Proposition 4.19 that, if $(\Gamma(T))^c$ satisfies (C_2), then T is necessarily a finite ring and, indeed, $|T|$ is either 8 or 16. The following example illustrates that this is not the case for a ring that satisfies (C_1).

Example 5.1. Let $S = \mathbf{Z}_2[x_1, x_2, x_3, \ldots]$ be the polynomial ring in an infinite number of variables over $\mathbf{Z}_2$. Let I be the ideal of S generated by $\{(x_k)^2 | k = 1, 2, 3, \ldots\} \cup \{x_1 x_k \mid k = 3, 4, 5, \ldots\} \cup \{x_i x_j | i, j \in \mathbf{N}, 2 \leq i < j\}$. Let $T = S/I$. Let M be the maximal ideal of S generated by $\{x_i | i = 1, 2, 3, \ldots\}$. Let $P = M/I$. Note that T is an infinite quasi-local ring with P as its unique maximal ideal. Moreover, $x_1 + I, x_2 + I$ are distinct zero-divisors of T with $x_1 x_2 \notin I$. Hence, it follows that $(\Gamma(T))^c$ admits at least one edge. Observe that $P^3 = (0)$. It was verified in [**17**, Example 4.22] that $\omega((\Gamma(T))^c) = 3$. Hence, $(\Gamma(T))^c$ satisfies (C_1). $\square$

Let T, P be as in the statement of Lemma 4.2. Suppose that $(\Gamma(T))^c$ admits at least one edge. That is, equivalently, $P^2 \neq (0)$ by [**5**, Theorem 2.8]. If $(\Gamma(T))^c$ satisfies (C_1), then it is proved in Corollary 4.3 (i) that $P^3 = (0)$. However, the following question remains.

Question 5.2. For which rings T will $(\Gamma(T))^c$ satisfy (C_1)?

In Proposition 5.4 we provide a necessary condition for a ring in order that the complement of its zero-divisor graph satisfies (C_1). We begin with the following lemma which is used in the proof of Proposition 5.4.

Lemma 5.3. *Let T, P be as in the statement of Lemma 4.2. Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_1), then the following hold*:

(i) $P^3 = (0)$, $|T/P| = 2$.

(ii) *Let $A \subseteq P/P^2$ be such that A is linearly independent over T/P. If $a, b \in P$ are such that $\{a + P^2, b + P^2\} \subseteq A$ and if $ab \neq 0$, then $a^2, b^2 \in \{0, ab\}$.*

(iii) *Let A be as in* (ii)*. Let $a, b, c \in P$ be such that$\{a+P^2, b+P^2, c+P^2\} \subseteq A$. Suppose that $ab \neq 0$. If $c^2 \neq 0$, then $c^2 = ab$. Moreover, $ac, bc \in \{0, ab\}$.*

(iv) *Let A be as in* (ii)*. Let $a, b, c \in P$ be such that $\{a+P^2, b+P^2, c+P^2\} \subseteq A$ and $ab \neq 0$. Then $ac, bc \in \{0, ab\}$.*

(v) *Let A be as in* (ii)*. Let $a, b, c, d \in P$. If $\{a + P^2, b + P^2, c + P^2, d + P^2\} \subseteq A$ with $ab \neq 0$, then $cd \in \{0, ab\}$.*

Proof. (i) By hypothesis, $(\Gamma(T))^c$ satisfies (C_1). Hence, it follows from Corollary 4.3 (i) that $P^3 = (0)$. Moreover, we know from Proposition 4.8 that $|T/P| = 2$.

(ii) Suppose that $a^2 \notin \{0, ab\}$. Then $\{0, a^2, ab\} \subseteq P^2$, and hence $|P^2| \geq 3$. Since $P^3 = (0)$, P^2 is a vector space over the field T/P and as $|T/P| = 2$, it follows that $\dim_{T/P} P^2 \geq 2$ and so $|P^2| \geq 4$. Let $t \in P^2 \setminus \{0, a^2, ab\}$. Since $\{a + P^2, b + P^2\}$ is linearly independent over T/P, and as $\{a^2, ab, t\} \subseteq P^2 \setminus \{0\}$, it follows that the elements $a, b, a + ab, a + a^2, a + t$ are all distinct. Moreover, it is easy to verify that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, a + ab, a + a^2, a + t\}$ is a clique. This is in contradiction to the assumption that $(\Gamma(T))^c$ satisfies (C_1). Hence, we obtain that $a^2 \in \{0, ab\}$. Similarly, it follows that $b^2 \in \{0, ab\}$.

(iii) By assumption, $\{a + P^2, b + P^2, c + P^2\}$ is linearly independent over T/P with $ab \neq 0$. Suppose that $c^2 \neq 0$. We claim that $c^2 = ab$. Suppose that $c^2 \neq ab$. Then $\{0, ab, c^2\} \subseteq P^2$. Hence, it follows as in the proof of (ii) that there exists a $t \in P^2 \setminus \{0, ab, c^2\}$. If $ac \neq 0$, then the subgraph of $(\Gamma(T))^c$ induced on $\{a, c, c + ab, c + c^2, c + t\}$ is a clique. This contradicts the assumption that $(\Gamma(T))^c$ satisfies (C_1). Hence, $ac = 0$. Similarly, if $bc \neq 0$, then the subgraph of $(\Gamma(T))^c$ induced on $\{b, c, c + ab, c + c^2, c + t\}$ is a clique. This is impossible since $\omega((\Gamma(T))^c) \leq 4$. Thus, $bc = 0$. Observe that the subgraph of $(\Gamma(T))^c$ induced on $\{c, c + ab, c + c^2, c + t, a + b + c\}$ is a clique. This is in contradiction to the assumption that $(\Gamma(T))^c$ satisfies (C_1). Hence, it follows that $c^2 = ab$.

We next verify that $ac, bc \in \{0, ab\}$. Suppose that $ac \neq 0$. Now $\{a+P^2, c+P^2\}$ is linearly independent over T/P and, by assumption, $ac \neq 0$. Hence, we obtain from (ii) that $c^2 \in \{0, ac\}$. As $c^2 \neq 0$, it follows that $c^2 = ac$. Thus, $ab = c^2 = ac$. This shows that $ac \in \{0, ab\}$. Since $\{b + P^2, c + P^2\}$ is linearly independent over T/P, it follows similarly that $bc \in \{0, ab\}$.

(iv) Suppose that $ac \notin \{0, ab\}$. Hence, it follows from (iii) that $c^2 = 0$. Moreover, if $b^2 \neq 0$, then it follows from (iii) that $ab, bc \in \{0, ac\}$. Since $ab \neq 0$, it follows that $ab = ac$. This is in contradiction to the assumption that $ac \neq ab$. Hence, it follows that $b^2 = 0$. If $a^2 \neq 0$, then it follows from (ii) that $a^2 = ab = ac$. This again contradicts the assumption that $ac \neq ab$. So it follows that $a^2 = 0$. Note that either $bc = 0$ or $bc \neq 0$.

Case (I). Suppose that $bc = 0$. Since $P^3 = (0)$ and $2 \in P$ (indeed, $|T/P| = 2$), it follows that $x + x = 0$ for each $x \in P^2$. As $ac \neq ab$, by assumption, it follows that $ab + ac \neq 0$. Observe that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, a+b, a+b+c, a+c+ab\}$ is a clique. This contradicts the hypothesis that $(\Gamma(T))^c$ satisfies (C_1). Hence, this case cannot happen.

Case (II). Suppose that $bc \neq 0$. We assert that $bc \in \{ab, ac\}$. Suppose that $bc \notin \{ab, ac\}$. Then $ab + bc \neq 0, bc + ac \neq 0$, and already by assumption, $ab + ac \neq 0$. This implies that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, c, a+b, a+b+c\}$ is a clique. This contradicts the hypothesis that $(\Gamma(T))^c$ satisfies (C_1). Hence, either $bc = ab$ or $bc = ac$. If $bc = ab$, then $bc \neq ac$ and hence $bc + ac \neq 0$. Note that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, c, a+b, b+c\}$ is a clique. This is impossible. If $bc = ac$, then $bc + ab \neq 0$. Moreover, it is easy to verify that the subgraph of $(\Gamma(T))^c$ induced on $\{a, b, c, a+c, b+c\}$ is a clique. This is impossible. Hence, Case (II) cannot happen.

Thus, if $ac \notin \{0, ab\}$, we arrive at a contradiction. Therefore, we obtain that $ac \in \{0, ab\}$. Similarly, it can be shown that $bc \in \{0, ab\}$.

(v) Suppose that $cd \notin \{0, ab\}$. We claim that $ac = bc = ad = bd = 0$. Suppose that $ac \neq 0$. Since $\{a + P^2, b + P^2, c + P^2\}$ is linearly independent over T/P with $ab \neq 0$, it follows from (iv) that $ac = ab$. Note that $\{a + P^2, c + P^2, d + P^2\}$ is linearly independent over T/P with $ac \neq 0$. Hence, it follows from (iv) that $ad, cd \in \{0, ac\}$. Since $cd \neq 0$, by assumption, we obtain that $cd = ac$. Hence, we arrive at $cd = ac = ab$. This contradicts the assumption that $cd \neq ab$. Hence, $ac = 0$. Similarly, as $ab \neq cd$, it follows using (ii) and (iii) that $b^2 = c^2 = d^2 = 0$. In addition, it can be shown with the help of (iv) and the assumption that $cd \neq 0$ that $bc = ad = bd = 0$. Moreover, if $a^2 \neq 0$, it follows from (ii) and (iii) that $a^2 = ab = cd$. This contradicts the assumption that $cd \neq ab$. Hence, $a^2 = 0$. Since, by assumption, $ab \neq cd$, we obtain that $ab + cd \neq 0$. Using the facts that $ac = bc = ad = bd = a^2 = b^2 = c^2 = d^2 = 0$ and $ab + cd \neq 0$, it follows that the subgraph of $(\Gamma(T))^c$ induced on $\{a + c, a + d, a + b + c, a + b + d, a + b + c + d\}$ is a clique. This is in contradiction to the hypothesis that $(\Gamma(T))^c$ satisfies (C_1). This proves that $cd \in \{0, ab\}$.

This completes the proof of Lemma 5.3. $\square$

The following proposition provides a necessary condition in order that $(\Gamma(T))^c$ satisfy (C_1).

Proposition 5.4. *Let T, P be as in the statement of Lemma* 4.2. *Suppose that $P^2 \neq (0)$. If $(\Gamma(T))^c$ satisfies (C_1), then $|P^2| = 2$.*

Proof. Assume that $(\Gamma(T))^c$ satisfies (C_1). We know from Corollary 4.3 (i) that $P^3 = (0)$. Moreover, we know from Proposition 4.8 that $|T/P| = 2$. Observe that P^2 is a nonzero vector space over the field T/P. If P is a principal ideal of T, then it is clear that $|P^2| = 2$. Hence, we may assume that P is not a principal ideal of T. Let $\{a_\alpha + P^2\}_{\alpha\in\Lambda}$ be a basis of P/P^2 as a vector space over T/P. Since $P^3 = (0)$, it follows as in the proof of Lemma 4.13 that $P = \sum_{\alpha\in\Lambda} Ta_\alpha$. As P is not principal, it follows that $\dim_{T/P}(P/P^2) = |\Lambda| \geq 2$. Moreover, since $P^2 \neq (0)$, it follows as in the proof of Lemma 4.13 that there exists a basis $\{a_\alpha + P^2\}_{\alpha\in\Lambda}$ of the (T/P)-vector space P/P^2 such that $a_{\alpha_1}a_{\alpha_2} \neq 0$ for some distinct $\alpha_1, \alpha_2 \in \Lambda$. Since $P = \sum_{\alpha\in\Lambda} Ta_\alpha$, it follows that $P^2 = \sum_{\alpha,\beta\in\Lambda} Ta_\alpha a_\beta$. By hypothesis, $(\Gamma(T))^c$ satisfies (C_1). Hence, we obtain from Lemma 5.3 (ii) and (iii) that $a_\alpha^2 \in \{0, a_{\alpha_1}a_{\alpha_2}\}$ for each $\alpha \in \Lambda$. Moreover, it follows from Lemma 5.3 (iii) and (iv) that $a_{\alpha_i}a_\alpha \in \{0, a_{\alpha_1}a_{\alpha_2}\}$ for any $\alpha \in \Lambda$ and for $i = 1, 2$. Furthermore, it follows from Lemma 5.3 (v) that, for any distinct $\alpha, \beta \in \Lambda \setminus \{\alpha_1, \alpha_2\}$, $a_\alpha a_\beta \in \{0, a_{\alpha_1}a_{\alpha_2}\}$.

It is now clear that $P^2 = T(a_{\alpha_1}a_{\alpha_2})$. Thus, P^2 is a one-dimensional vector space over the field T/P and, as $|T/P| = 2$, it follows that $|P^2| = 2$. This completes the proof of Proposition 5.4. $\square$

Suppose that $(\Gamma(T))^c$ satisfies (C_1). The following corollary determines necessary and sufficient conditions in order that $(\Gamma(T))^c$ satisfy (C_2). (It is shown in Example 4.1 that $(\Gamma(T))^c$ satisfies (C_1) need not imply that it satisfies (C_2).)

Corollary 5.5. *Let T, P be as in the statement of Lemma 4.2. Suppose that $(\Gamma(T))^c$ contains at least one edge. Then the following statements are equivalent*:

(i) $(\Gamma(T))^c$ *satisfies* (C_2).

(ii) $(\Gamma(T))^c$ *satisfies* (C_1) *and* $|P/P^2| \leq 4$.

(iii) $(\Gamma(T))^c$ *is planar.*

Proof. (i) $\Leftrightarrow$ (iii) is the same as (i) $\Leftrightarrow$ (v) of Proposition 4.19.

(i) $\Rightarrow$ (ii). We know from Proposition 4.15 that $(\Gamma(T))^c$ satisfies (C_1). Moreover, it follows from (i) $\Rightarrow$ (iii) of Proposition 4.19 that $P^3 = (0)$, $|T/P| = 2$, $|P^2| = 2$ and $\dim_{T/P}(P/P^2) \leq 2$. Hence, we obtain that $|P/P^2| \leq |T/P|^2 \leq 4$.

(ii) $\Rightarrow$ (i). We know from Corollary 4.3 (i) that $P^3 = (0)$. Since $(\Gamma(T))^c$ admits at least one edge, it follows that $P^2 \neq (0)$. We know from Proposition 4.8 that $|T/P| = 2$. Moreover, we obtain from Proposition 5.4 that $|P^2| = 2$. As $|T/P| = 2$ and, by assumption, $|P/P^2| \leq 4$, it is clear that $\dim_{T/P}(P/P^2) \leq 2$. Thus, if (ii) holds, then $P^3 = (0)$, $|T/P| = 2$, $|P^2| = 2$ and $\dim_{T/P}(P/P^2) \leq 2$. Now it follows from (iii) $\Rightarrow$ (i) of Proposition 4.19 that $(\Gamma(T))^c$ satisfies (C_2).

This completes the proof of Corollary 5.5. $\square$

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