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STANDARD DECOMPOSITIONS IN GENERIC COORDINATES

DAVID EISENBUD AND IRENA PEEVA

Dedicated to Jürgen Herzog on the occasion of his seventieth birthday.

1. Introduction. Throughout the paper, $S = k[x_1, \dots, x_c]$ is a polynomial ring over an infinite field k, graded with $\deg(x_i) = 1$ for each i. We consider a graded finitely generated S-module M.

Let $\mathcal{A}$ be a subset of the variables $\{x_1, \dots, x_c\}$. Set $k[\mathcal{A}] = k[x_i \mid x_i \in \mathcal{A}]$. We say that a homogeneous element $m \in M$ is $\mathcal{A}$-*standard* if the map

$$\begin{aligned} k[\mathcal{A}] &\longrightarrow M \\ 1 &\longmapsto m \end{aligned}$$

is a monomorphism. Let $m_1, \dots, m_s \in M$ and $\mathcal{A}_1, \dots, \mathcal{A}_s$ be subsets of the variables $\{x_1, \dots, x_c\}$. A direct sum of vector spaces

$$M = \bigoplus_{1 \le i \le s} k[\mathcal{A}_i]\, m_i$$

is called a *standard decomposition* of M if m_i is $\mathcal{A}_i$-standard for each i. We say that the decomposition is *nested* if the $\mathcal{A}_i$ are nested subsets of $\{x_1, \dots, x_c\}$, that is, for each i, j one of $\mathcal{A}_i, \mathcal{A}_j$ is contained in the other. Easy arguments using "prime filtrations" (these are filtrations of M whose quotients have the form S/P for various prime ideals P) show that every module admits a standard decomposition (see [**6**, Section 1].)

A well-known combinatorial conjecture of Richard Stanley [**9**, Conjecture 5.1] asserts that a multigraded finitely generated module M of depth d has a standard decomposition as above where the m_i are multihomogeneous elements and every $\mathcal{A}_i$ has at least d variables. The

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conjecture has been studied from an algebraic point of view by Herzog, Jahan, Vladoiu, Yassemi and Zheng [**4–6**], among others. Jahan [**7**, Corollary 4.1] observes that, by Alexander duality, Stanley's conjecture is equivalent to the statement that every multigraded finitely generated module has a standard decomposition in which the m_i are multihomogeneous elements of degrees $\leq \operatorname{reg}(M)$, where $\operatorname{reg}(M)$ denotes the Castelnuovo-Mumford regularity.

The goal of this note is to show that this form of the conjecture becomes easy if, instead of the variables x_i, we allow ourselves to use generic coordinates. We prove:

Theorem 1. *If M is a graded finitely generated S-module and $z_1, \dots, z_c \in S$ are sufficiently general linear forms, then there is a nested standard decomposition*

$$M = \bigoplus_{1 \leq i \leq s} k[\mathcal{B}_i] m_i$$

such that $\mathcal{B}_i \subset \{z_1, \dots, z_c\}$ and the m_i are homogeneous elements of degrees $\leq \operatorname{reg}(M)$. If M is multigraded with respect to the x_i, then the m_i may be taken to be multihomogeneous with respect to the x_i.

To see the relevance of the regularity, consider the case in which M is a Cohen-Macaulay module of dimension d. By Noether normalization, M is a finite module over $k[z_1, \dots, z_d]$ (and it might happen that M is a finite module over $k[x_1, \dots, x_d]$). In this case M is a free module over $k[z_1, \dots, z_d]$, so there is a standard decomposition of the form $M = \oplus_{1 \leq i \leq s} k[z_1, \dots, z_d] m_i$ and $\operatorname{reg}(M) = \max\{\deg(m_i)\}$. In particular, if M is Artinian, then it is a finite-dimensional vector space and we have a standard decomposition $M = \oplus_{1 \leq i \leq s} k m_i$ with $\operatorname{reg}(M) = \max\{\deg(m_i)\}$.

As the next example illustrates, having a nested standard decomposition is a generalization of a property of Borel fixed ideals; this may suggest the relevance of generic coordinates.

Example 2. A monomial ideal N is 0-*Borel* if, whenever $i < j$ and m is a monomial such that $m x_j \in N$, we have $m x_i \in N$ as well. If m is a monomial, then we set $\max(m) = \max\{i \mid x_i \text{divides } m\}$. It is easy to see that, if N is a 0-Borel ideal generated in one degree p by monomials $m_1, \dots, m_s$, then

$$N = \bigoplus_{1 \le i \le s} k[x_{\max(m_i)}, \dots, x_c]\, m_i$$

is a nested standard decomposition.

2. Constructing the decomposition.

Proof of Theorem 1. Let $p = \operatorname{reg}(M)$. Choose a homogeneous vector space basis $\{n_i\}$ for the sum of the homogeneous components of M of degree $\le p$; if M is multigraded (with respect to the variables x_i) we may take the n_i to be multihomogeneous. Let $z_1, \dots, z_c$ be linear forms that are chosen generally with respect to M, in a sense that will be made clear in the construction. We will construct a nested standard decomposition $M = \oplus_{i=0}^{s} k[z_1, \dots, z_{j_i}]g_i$, where the g_i are chosen from among the n_i and the j_i are all bounded by the Krull dimension $d = \dim(M)$.

If $d = 0$, then M is a finite-dimensional vector space, and the n_i form a basis. In this case, $M = \oplus_{i=0}^{s} k\, n_i$ is a decomposition of the desired sort since $\operatorname{reg}(M)$ is equal to the maximal degree of an n_i.

Now suppose that $d > 0$. Since the z_i are chosen generally, the algebra $S/\operatorname{ann}(M)$ is finite over $k[z_1, \dots, z_d]$, and thus M is a finitely generated $k[z_1, \dots, z_d]$-module. Since $(z_1, \dots, z_d) + \operatorname{ann}(M)$ has the same radical as $(z_1, \dots, z_d)$, each local cohomology module $H^i_{(z_1, \dots, z_d)}(M)$ agrees with the local cohomology module $H^i_{(z_1, \dots, z_c)}(M)$, so $\operatorname{reg}(M)$ agrees with the regularity of M as a $k[z_1, \dots, z_d]$-module; in particular, M is generated as a $k[z_1, \dots, z_d]$-module by the elements n_i.

Choose a maximal subset $\{g_1, \dots, g_r\} \subseteq \{n_i\}$ such that the g_i are linearly independent in the vector space $k(z_1, \dots, z_d) \otimes_S M$, and let M' be the submodule of M that they generate. It follows that $M' = \oplus_{i=1}^{r} k[z_1, \dots, z_d]g_i$ is a standard decomposition. It also follows that M/M' is a finitely generated torsion module over $k[z_1, \dots, z_d]$, so $\dim(M/M') < \dim(M)$. By induction, the hypothesis we may choose as a nested standard decomposition of the desired form $M/M' = \oplus_{i=r+1}^{s} k[z_1, \dots, z_{j_i}]\, \overline{g}_i$ using generators $\overline{g}_{r+1}, \dots, \overline{g}_s$ that are images of some of the n_i. It follows at once that $M = \oplus_{i=0}^{s} k[z_1, \dots, z_{j_i}]g_i$ is a nested standard decomposition of M, as required. □

We will give a second proof, which has a different flavor. The *pth truncation* of a finitely generated graded S-module M is the module $M_{\geq p} = \oplus_{i \geq p} M_i$. It is well known that, if $p \geq \operatorname{reg}(M)$, then $\operatorname{reg}(M_{\geq p}) = p$.

Second proof of Theorem 1. Clearly, $M = M_{\geq p} \oplus M_{<p}$ as vector spaces and $M_{<p}$ is a finite-dimensional vector space with basis of elements of degree $< p$. Hence, it suffices to give a nested standard decomposition of the module $L = M_{\geq p}$. Let $n_1, \dots, n_r$ be a vector space basis of L_p.

Let

$$\mathbf{F} : \ \cdots \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 = S^r$$

be a minimal graded free resolution of L over S. Denote by K the first syzygy module $\operatorname{Im}(d_1)$. Since $p = \operatorname{reg}(M)$, it follows that L has a p-linear minimal graded free resolution. Therefore, the module K is generated by elements of degree $p+1$ and has a $(p+1)$-linear minimal free resolution.

Order the variables by $x_1 > \cdots > x_c$, and consider the reverse lex monomial order $\succ$ in S. Let $u_1, \dots, u_r$ be a homogeneous basis of $F_0 = S^r$ such that $d_0(u_i) = n_i$ for each i. Define the reverse lex monomial order $\succ$ in F_0 by setting $u_1 \succ \cdots \succ u_r$ and declaring that, if m and m' are monomials in S, then $mu_i > m'u_j$ if and only if $m \succ m'$, or $m = m'$ and $u_i \succ u_j$. Consider the generic initial ideal $\operatorname{gin}(K)$ with respect to the reverse lex order; so now we suppose we work in generic coordinates $z_1, \dots, z_c$. By [**1**] it follows that

$$\operatorname{reg}(\operatorname{gin}(K)) = \operatorname{reg}(K) = p + 1.$$

In particular, $\operatorname{gin}(K)$ is generated in degree $p+1$. Since $\operatorname{gin}(K)$ is an initial module, we have that it has the form $\operatorname{gin}(K) = B_1 u_1 \oplus \cdots \oplus B_r u_r$, where each B_i is a Borel monomial ideal. As $\deg(u_i) = \deg(n_i) = p$, it follows that each B_i is generated by variables. A linear monomial ideal I in S is Borel if, whenever $q < j$ and $z_j \in I$, we have $z_q \in I$ as well. Therefore, for each i, there exists a q_i such that $B_i = (z_1, \dots, z_{q_i - 1})$. Hence, $S^r/\operatorname{gin}(K) = \oplus_{1 \leq i \leq u}\ k[z_{q_i}, \dots, z_c]\, u_i$ as vector spaces. The quotient S^r/K has the same basis (as a vector space) as $S^r/\mathrm{gin}\,(K)$. Therefore,

$$S^r/K = \oplus_{1\leq i\leq r}\, k[z_{q_i}, \ldots , z_c]\, u_i.$$

Finally, note that $L \cong S^r/K$, and the isomorphism maps u_i to n_i for each i. □

Corollary 3. *Let M be a finitely generated graded S-module with a linear free resolution. If $z_1, \ldots , z_c \in S$ are sufficiently general linear forms, then there is a nested standard decomposition*

$$M = \bigoplus_{i=1}^{s} k[\mathcal{B}_i] m_i,$$

where $\mathcal{B}_i \subset \{z_1, \ldots , z_c\}$ and $\{m_i\}$ is a homogeneous basis of $M_{\mathrm{reg}\,(M)}$. In particular, if V is any finitely generated graded S-module and $p \geq \mathrm{reg}\,(V)$, then $V_{\geq p}$ has a nested standard decomposition as above involving a basis of V_p.

3. Hilbert polynomials. Corollary 3 leads to a representation of the Hilbert polynomial of a graded finitely generated S-module different from the well-known Macaulay representation. Recall that $h_j(t) = \binom{j-1+t}{j-1}$ is the Hilbert function of $k[z_1, \ldots , z_j]$, where we interpret $\binom{-1+t}{-1}$ as being 1 for $t = 0$ and 0 otherwise.

Corollary 4. *Let V be a graded finitely generated S-module. For any $p \geq \mathrm{reg}\,(V)$, there exist unique nonnegative integers $\gamma_0, \ldots , \gamma_c$ such that, for $t \geq p$,*

$$\dim V_t = \sum_{0\leq j\leq c} \gamma_j h_j(t-p) = \sum_{0\leq j\leq c} \gamma_j \binom{j-1+t-p}{j-1}.$$

In particular, the Hilbert polynomial of V is $\sum_{1\leq j\leq c} \gamma_j h_j(t-p)$.

Proof. Existence follows from the existence of a standard decomposition, while uniqueness holds because, for $t > p$, the $h_j(t-p)$ are polynomials in t of different degrees. □

We say that the representation in Corollary 4 is the *p-representation* of the Hilbert polynomial of V, and we call the numbers $\gamma_1, \dots, \gamma_c$ the *p-representation coefficients.*

Macaulay's theorem [**8**] characterizes all possible Hilbert functions of graded ideals in the polynomial ring S. The key idea is that for every graded ideal there exists a lex ideal with the same Hilbert function. Thus, in order to study the Hilbert polynomials of graded ideals, it suffices to study the Hilbert polynomials of lex ideals. A monomial ideal T is called a *lex-segment ideal* if it is generated by the monomials in an initial lex segment in some fixed degree (that is, generated by lex-consecutive monomials in a fixed degree that are starting with a power of x_1). If Q is a lex ideal and

$$p = \operatorname{reg}(Q) = \text{ maximal degree of a minimal monomial generator of } Q,$$

then the truncation ideal $Q_{\geq p}$ is a lex-segment ideal. Hence, in order to study the Hilbert polynomials of graded ideals, it suffices to study the Hilbert polynomials of lex-segment ideals. Let T be a lex-segment ideal generated in degree p. We will compare the p-representation and the Macaulay representation of the Hilbert polynomial.

The Macaulay representation of the Hilbert polynomial of T is constructed as follows. Let $q = \dim_k(S_p/T_p)$. There exist unique numbers $s_p > \cdots > s_1 \geq 0$ such that

$$q = \binom{s_p}{p} + \binom{s_{p-1}}{p-1} + \cdots + \binom{s_1}{1}.$$

This is called the *pth Macaulay representation of the number* q. Set $a_i = s_i - i$ for every i. Then (cf. [**1**]), the Hilbert polynomial of S/T is

$$\binom{t+a_p}{a_p} + \binom{t+a_{p-1}-1}{a_{p-1}} + \cdots + \binom{t+a_1-p+1}{a_1},$$

and $a_p \geq \cdots \geq a_1 \geq 0$; thus, the Hilbert polynomial of T is:

$$h_T(t) = \binom{c-1+t}{c-1} - \left[\binom{t+a_p}{a_p} + \binom{t+a_{p-1}-1}{a_{p-1}} + \cdots + \binom{t+a_1-p+1}{a_1}\right].$$

We next consider the p-representation of the Hilbert polynomial of T. The p-representation is easier to obtain than the Macaulay representation because, if γ_j is the number of monomials m in T_p with $\max(m) = j$, then by Example 2, it follows that the Hilbert polynomial of T is

$$h_T(t) = \sum_{1 \le j \le c} \gamma_j h_{c-j+1}(t-p),$$

where $h_{c-j+1}(t)$ is the Hilbert polynomial of $k[x_j, \dots, x_c]$. Note that, since T_p is Borel-fixed, it follows that $\gamma_1 = 1$. Thus, in order to obtain the p-representation, we just need to count how many monomials there are in T_p with a fixed maximal variable. In contrast, in order to obtain the Macaulay representation, we need to construct the pth-Macaulay representation of the number $\dim_k(T_p)$. We will illustrate the difference in the following example.

Example 5. Let $S = k[x_1, \dots, x_6]$, and let T be the lex-segment ideal $(x_1^3, x_1^2x_2, x_1^2x_3)$. The 3-representation of the Hilbert polynomial of T is:

$$\begin{aligned} h_T(t) &= \sum_{1 \le j \le 6} \gamma_j \binom{6-j+t-3}{t-3} \\ &= 1\binom{6-1+t-3}{t-3} + 1\binom{6-2+t-3}{t-3} + 1\binom{6-3+t-3}{t-3} \\ &= \binom{t+2}{5} + \binom{t+1}{4} + \binom{t}{3}. \end{aligned}$$

Next we compute the Macaulay representation of the Hilbert polynomial of S/T. We have

$$\dim_k(S/T)_3 = \dim_k(S)_3 - \dim_k(T_3) = 56 - 3 = 53.$$

We have to compute the third Macaulay representation of 53. Since $53 \le \binom{7}{4}$ and $53 \ge \binom{7}{3}$, we obtain $53 = \binom{7}{3} + 18$, and we have to compute the second Macaulay representation of 18. Since $18 \le \binom{6}{3}$ and $18 \ge \binom{6}{2}$, we get $18 = \binom{6}{2} + 3$ and we have to compute the first Macaulay representation of 3. This is $3 = \binom{3}{1}$. Therefore, the third Macaulay representation of 53 is:

$$53 = \binom{7}{3} + \binom{6}{2} + \binom{3}{1}.$$

Thus, $s_3 = 7$, $s_2 = 6$, $s_1 = 3$. Now $a_i = s_i - i$, so $a_3 = 4$, $a_2 = 4$, $a_1 = 2$. Thus, Macaulay's representation of the Hilbert polynomial of S/T is

$$\binom{t+a_3}{a_3} + \binom{t+a_2-1}{a_2} + \binom{t+a_1-2}{a_1} = \binom{t+4}{4} + \binom{t+3}{4} + \binom{t}{2}.$$

Hence,

$$h_T(t) = \binom{t+5}{5} - \binom{t+4}{4} - \binom{t+3}{4} - \binom{t}{2}.$$

A similar approach was used in [**3**].

REFERENCES

1. D. Bayer and M. Stillman, *A criterion for detecting m-regularity*, Invent. Math. **87** (1987), 1–11.

2. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambr. Stud. Adv. Math. **39**, Cambridge University Press, Cambridge, 1993.

3. Marc Chardin and Guillermo Moreno-Socias, *Regularity of lex-segment ideals: some closed formulas and applications*, Proc. Amer. Math. Soc. **131** (2003), 1093–1102.

4. J. Herzog, A. Jahan and S. Yassemi, *Stanley decompositions and partitionable simplicial complexes*, J. Alg. Combin. **27** (2008), 113–125.

5. J. Herzog, D. Popescu and M. Vladoiu, *Stanley depth and size of a monomial ideal*, Proc. Amer. Math. Soc. **140** (2012), 493–504.

6. J. Herzog, M. Vladoiu and X. Zheng, *How to compute the Stanley depth of a monomial ideal*, J. Algebra **322** (2009), 3151–3169.

7. A. Jahan, *Prime filtrations and Stanley decompositions of squarefree modules and Alexander duality*, Manuscr. Math. **130** (2009), 533–550.

8. F. Macaulay, *Some properties of enumeration in the theory of modular systems*, Proc. Lond. Math. Soc. **26** (1927), 531–555.

9. R. Stanley, *Linear diophantine equations and local cohomology*, Invent. Math. **68** (1982), 175–193.

Mathematics Department, University of California at Berkeley, Berkeley, CA 94720
Email address: de@msri.org

Mathematics Department, Cornell University, Ithaca, NY 14853
Email address: irena@math.cornell.edu