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# INITIAL ALGEBRAS OF PFAFFIAN RINGS

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Dedicated to the 70th birthday of our friend and teacher Jürgen Herzog

ABSTRACT. We compute initial algebras of standard Pfaffian rings via suitable embeddings. This yields simple proofs that these rings are normal Gorenstein domains, with rational singularities in characteristic 0 and $F$-rational in characteristic $p > 0$. We also determine their $a$-invariants. These methods can also be applied to prove the same properties for rings defined by cogenerated Pfaffian ideals.

**1. Introduction.** Since the fundamental results of Buchsbaum and Eisenbud [**7**] on structure theorems of minimal resolutions of ideals of codimension 3, of particular interest are Pfaffian ideals and rings. More precisely, let $X = (X_{ij})$ be an $n \times n$ matrix which is skew-symmetric, let $K$ be a field and $K[X]$ a polynomial ring over $K$ in the variables $X_{ij}$ for $i < j$. Then the Pfaffian ideal $I_{2r+2}(X)$ is generated by all Pfaffians of $X$ of size $2r + 2$. The corresponding Pfaffian ring is $R_{2r+2}(X) = K[X]/I_{2r+2}(X)$. By results in [**22, 23, 25**] the ring $R_{2r+2}(X)$ is a Cohen-Macaulay normal domain of dimension $r(2n-2r-1)$. Moreover, it is proved in [**1**] that $R_{2r+2}(X)$ is factorial, hence Gorenstein. At least two more general classes of Pfaffian ideals have been considered, also motivated by the study of coordinate rings of some Schubert varieties. One is the class of ladder Pfaffians ideals, studied first in [**13**] and more recently in [**15, 16**]. The second is the class of cogenerated Pfaffian ideals that arise naturally when one considers the ASL structure. In [**13**], it has been proved that rings defined by cogenerated Pfaffian ideals are also Cohen-Macaulay normal domains, formulas for the dimension are given and it is characterized which of these rings are Gorenstein. See also [**17**] for related results.

The main goal of this paper is to study the rings $R_{2r+2}(X)$ by initials algebra methods initiated in [**6**]. Although it does not yield new results for these rings, one gets relatively quickly and simple structural

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properties of Pfaffian rings by using standard results from combinatorial commutative algebra; see, e.g., [**5, 26, 28**] for more details on these kinds of methods. More precisely, by explicitly computing a finitely generated initial algebra of $R_{2r+2}(X)$ in a suitable embedding, we can prove that this ring is a normal Gorenstein domain, with rational singularities in characteristic 0 and $F$-rational in characteristic $p > 0$; see Theorem 4.3 and Corollary 5.8. Moreover, we can determine in Corollary 5.8 also its $a$-invariant and thus also related invariants like the Castelnuovo-Mumford regularity. In Section 6 we apply our methods to get structural properties and prove new facts for rings defined by cogenerated Pfaffian ideals. In Theorem 6.1 we prove in particular that these rings have rational singularities in characteristic 0 and are $F$-rational in characteristic $p > 0$.

**2. Preliminaries on Pfaffian ideals.** At first we fix some notation. Let $A = (c_{ij})$ be an arbitrary $m \times n$ matrix with coefficients in some commutative ring. We write $[a_1 \dots a_t \mid b_1 \dots b_t]_A$ for the *minor* of matrix $A$ of *size* $t$ with rows indexed by $a_1 \dots a_t$ and columns indexed by $b_1 \dots b_t$. Here we assume that $1 \leq a_1 < \cdots < a_t \leq m$ and $1 \leq b_1 < \cdots < b_t \leq n$.

Most times we are interested in the case that $A$ is an $n \times n$ *skew-symmetric matrix* for $n \geq 2$, i.e., we have $c_{ij} = -c_{ji}$ and $c_{ii} = 0$ for all $i, j$. For an integer $r$ with $1 \leq 2r \leq n$, let then $\mathrm{Pf}\,(a_1 \dots a_{2r})_A$ be the $2r$-*Pfaffian* of the submatrix of $A$ with rows and columns indexed by $a_1 \dots a_{2r}$; it is not restrictive to assume $1 \leq a_1 < \cdots < a_{2r} \leq n$. The number $2r$ is the *size* of the Pfaffian. We always consider Pfaffians of even size since Pfaffians of odd size are zero. The empty Pfaffian $\mathrm{Pf}\,(\;)_A$ is set to 1.

To study the "generic" case, let $X = (X_{ij})$ be a skew-symmetric $n \times n$ matrix of indeterminates, and let $R = K[X] := K[X_{ij} : 1 \leq i < j \leq n]$ be the polynomial ring over a field $K$. To simplify notation we also write $\mathrm{Pf}\,(a_1 \dots a_{2r})$ for the Pfaffian $\mathrm{Pf}\,(a_1 \dots a_{2r})_X$.

Let $\mathrm{Pf}\,(X)$ be the set of all the Pfaffians of $X$. We denote by $\mathrm{Pf}_{2s}(X)$ the subset of $\mathrm{Pf}\,(X)$ consisting of Pfaffians of size exactly $2s$ where $\mathrm{Pf}_{2s}(X) = \varnothing$ for $2s > n$. Following [**11**, Section 6], we consider on $\mathrm{Pf}\,(X)$ the partial order: if $\alpha = \mathrm{Pf}\,(a_1 \dots a_{2s})$ and $\beta = \mathrm{Pf}\,(b_1 \dots b_{2t})$. Then

$$\alpha \leq \beta \text{ if } s \geq t \text{ and } a_i \leq b_i \text{ for } i = 1, \dots, 2t.$$

A *standard monomial* is a product $\alpha_1 \cdot \ldots \cdot \alpha_h$ of Pfaffians with $\alpha_1 \leq \cdots \leq \alpha_h$.

It is known that the standard monomials are a $K$-basis of $K[X]$, see [**11**, Theorem 6.5] or Theorem 4.1 below. Moreover, $K[X]$ is an ASL on $\mathrm{Pf}\,(X)$; see [**10**, Section 12].

As mentioned in the introduction, the main goal of this paper is to study Pfaffian ideals, that is, ideals generated by certain subsets of $\mathrm{Pf}\,(X)$. The classical Pfaffian ideals are the ideals $I_{2r+2}(X)$ generated by all Pfaffians in $\mathrm{Pf}_{\,2r+2}(X)$. More generally, ones defines cogenerated Pfaffian ideals as follows. Given an $\alpha \in \mathrm{Pf}\,(X)$, the ideal *cogenerated by* $\alpha$ is:

$$I_\alpha(X) = (\beta \in \mathrm{Pf}\,(X) : \beta \not\geq \alpha) \subset K[X].$$

Note that, for $\alpha = \mathrm{Pf}\,(1, \ldots, 2r)$, we get back the standard Pfaffian ideal $I_\alpha(X) = I_{2r+2}(X)$. We will study the algebraic properties of the Pfaffian ring $R_\alpha(X) = K[X]/I_\alpha(X)$ in Section 6.

**3. Initial forms of Pfaffians at the generic point.** To study the Pfaffian rings $R_{2r+2}(X)$, we follow the approach of [**6**] where initial algebra methods were used to prove structural properties of standard determinantal rings, in a relatively quick and simple way. One of the key ideas is to consider the generic point of such a ring. Although the main idea is the same as in [**6**], in the Pfaffian case non-trivial technical problems have to be solved, which is partly done in this section.

For this, let $Y = (Y_{ij})$ be an $n \times 2r$-matrix of indeterminates $Y_{ij}$ where $n, r$ are integers such that $1 \leq 2r \leq n$. Let $E(2r)$ be the $2r \times 2r$ matrix with integer coefficients which has the form

$$\begin{pmatrix}
0 & -1 & 0 & 0 & \cdots & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & 1 & 0 & -1 & \cdots & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
0 & \cdots & \cdots & 1 & 0 & -1 & 0 \\
0 & \cdots & \cdots & 0 & 1 & 0 & -1 \\
0 & \cdots & \cdots & 0 & 0 & 1 & 0
\end{pmatrix},$$

i.e., for $E(2r) = (\alpha_{ij})$, we have $\alpha_{i+1\,i} = 1$ for $i = 1, \ldots, 2r-1$, $\alpha_{i-1\,i} = -1$ for $i = 2, \ldots, 2r$ and $\alpha_{ij} = 0$ in all other cases. Observe that $YE(2r)Y^t$ is a skew-symmetric $n \times n$-matrix. The main goal of this section is to present results related to initial forms of Pfaffians of $YE(2r)Y^t$, the matrix corresponding to a generic point of Pfaffian rings as considered in the next section.

Let $K[Y]$ be the polynomial ring over a field $K$ generated by the entries $Y_{ij}$ of $Y$. On $K[Y]$ we consider the reverse-lexicographic monomial order induced by

$$Y_{n\,1} > Y_{n-1\,1} > \cdots > Y_{1\,1} > Y_{n\,2} > \cdots > Y_{1\,2} > \cdots > Y_{n\,2r} > Y_{n-1\,2r} > Y_{1\,2r}.$$

For any $0 \neq f \in K[Y]$, we denote by $\operatorname{in}(f)$ the initial monomial with respect to the chosen monomial order. At first we present some useful equations needed later.

**Lemma 3.1.** (i) $\operatorname{in}([a_1 \ldots a_{2r} \mid 1 \ldots 2r]_Y) = \prod_{i=1}^{2r} Y_{a_i i}$ *for all* $1 \leq a_1 < \ldots < a_{2r} \leq n$.

(ii) $[1 \ldots 2r \mid 1 \ldots 2r]_{E(2r)} = 1$.

*Proof.* (i) Using Laplace expansion with respect to the last column we get

$$\begin{aligned}
&[a_1 \ldots a_{2r} \mid 1 \ldots 2r]_Y \\
&\qquad = \pm Y_{a_{2r} 2r}[a_1 \ldots a_{2r-1} \mid 1 \ldots 2r-1]_Y \\
&\qquad\quad + \sum_{i=1}^{2r-1} (-1)^{a_i+2r} Y_{a_i 2r}[a_1 \ldots a_{i-1} a_{i+1} \ldots a_{2r} \mid 1 \ldots 2r-1]_Y.
\end{aligned}$$

By the chosen monomial order on $K[Y]$ we see that

$$\operatorname{in}([a_1 \ldots a_{2r} \mid 1 \ldots 2r]_Y) = Y_{a_{2r} 2r} \operatorname{in}([a_1 \ldots a_{2r-1} \mid 1 \ldots 2r-1]_Y).$$

Now an induction on the size of the minor considered concludes the proof.

(ii) The equation $[1 \ldots 2r \mid 1 \ldots 2r]_{E(2r)} = 1$ is proved by a direct computation. □

The last result is useful for determining the initial term of a Pfaffian in $K[Y]$.

**Lemma 3.2.** *For all integers $t$ with $1 \le t \le r$ and $1 \le a_1 < \cdots < a_{2t} \le n$, we have*:

$$\operatorname{in}\left(\operatorname{Pf}(a_1 \dots a_{2t})_{YE(2r)Y^t}\right) = \prod_{j=1}^{2t} Y_{a_j j}.$$

*Proof.* We prove the lemma by induction on $r - t \ge 0$.

Let $r = t$. Using Lemma 3.1 (ii), we get

$$\begin{aligned}
&[a_1 \dots a_{2r} | a_1 \dots a_{2r}]_{YE(2r)Y^t} \\
&\qquad = [a_1 \dots a_{2r} | 1 \dots 2r]_Y \cdot [1 \dots 2r | 1 \dots 2r]_{E(2r)} \\
&\qquad\quad \cdot [1 \dots 2r | a_1 \dots a_{2r}]_{Y^t} \\
&\qquad = \left([a_1 \dots a_{2r} | 1 \dots 2r]_Y\right)^2.
\end{aligned}$$

It follows from Lemma 3.1 (i) that

$$\operatorname{in}\left(\operatorname{Pf}(a_1 \dots a_{2r})_{YE(2r)Y^t}\right) = \operatorname{in}\left([a_1 \dots a_{2r} \mid 1 \dots 2r]_Y\right) = \prod_{j=1}^{2r} Y_{a_j j}.$$

Next we assume that $r - t > 0$ and thus $r > t$. Let $Y'$ be the matrix obtained by removing the last two columns of $Y$ and set $W' = Y'E(2r-2)(Y')^t$. Observe that $Y'$ is an $n \times 2(r-1)$-matrix. The induction hypothesis yields

$$\operatorname{in}\left(\operatorname{Pf}(a_1 \dots a_{2t})_{W'}\right) = \prod_{j=1}^{2t} Y'_{a_j j} = \prod_{j=1}^{2t} Y_{a_j j}.$$

Let $W = YE(2r)Y^t = (W_{ij})$. We claim that

$$\operatorname{in}\left(\operatorname{Pf}(a_1 \dots a_{2t})_W\right) = \operatorname{in}\left(\operatorname{Pf}(a_1 \dots a_{2t})_{W'}\right)$$

which then concludes the proof. For this, let $Z = YE(2r)$. Note that $Z$ is an $n \times 2r$ matrix and for every $i \in \{1, \dots, n\}$ we see that

$$\begin{aligned}
Z_{i\,1} &= Y_{i\,2}, \\
Z_{i\,2r} &= -Y_{i\,2r-1}, \\
Z_{i\,j} &= Y_{i\,j+1} - Y_{i\,j-1} \quad \text{for } j = 2, \dots, 2r-1.
\end{aligned}$$

To avoid special cases, we set $Y_{i\,0} = Y_{i\,2r+1} = 0$. Since $W = ZY^t$, we obtain

$$\begin{aligned} W_{ij} &= \sum_{k=1}^{2r} Y_{jk}(Y_{i\,k+1} - Y_{i\,k-1}) \\ &= \sum_{k=1}^{2r-3} Y_{jk}(Y_{i\,k+1} - Y_{i\,k-1}) + Y_{j\,2r-2}Y_{i\,2r-1} - Y_{j\,2r-2}Y_{i\,2r-3} + p'_{ij} \end{aligned}$$

where $p'_{ij} \in K[Y]$ is in the ideal $I$ generated by the indeterminates of the last two columns of $Y$. Setting $p_{ij} = Y_{j\,2r-2}Y_{i\,2r-1} + p'_{ij}$, one has

$$W_{ij} = W'_{ij} + p_{ij} \quad \text{with } p_{ij} \in I.$$

Thus, $\mathrm{Pf}\,(a_1 \dots a_{2t})_W = \mathrm{Pf}\,(a_1 \dots a_{2t})_{W'} + p$ with $p \in I$. Since the monomial order is the reverse lexicographic order and the indeterminates in $I$ are smaller than all the other ones, we get the desired result $\mathrm{in}\,(\mathrm{Pf}\,(a_1 \dots a_{2t})_W) = \mathrm{in}\,(\mathrm{Pf}\,(a_1 \dots a_{2t})_{W'})$. □

Now we want to extend the last result to a product of Pfaffians. It is a standard technique to identify products of Pfaffians with *Young tableaux*. More precisely, given Pfaffians $\alpha_i = \mathrm{Pf}\,(a_{i1}, a_{i2}, \dots, a_{i\,2t_i})$ for $i = 1, \dots, h$, with $t_1 \geq t_2 \geq \dots \geq t_h$, one identifies the product $\alpha_1 \cdot \ldots \cdot \alpha_h$ with the tableau whose $i$th row is filled with the indexes of the Pfaffian $\alpha_i$. Such a tableau has two properties. The entries in each row form a strictly increasing sequence of integers, and the size of all of its rows is even. Let $\lambda_j$ be the number of entries in the $j$th row of $T$. The *shape* of $T$ is defined to be the vector $(\lambda_1 \dots \lambda_t)$. The *length* of $T$ is the number of entries of the first row. $T$ is said to be *standard* if the entries in every column form a weakly increasing sequence. It is clear that standard tableaux correspond to standard monomials of Pfaffians. As an example, consider the standard monomial $\alpha_1 \cdot \alpha_2 \cdot \alpha_3 = \mathrm{Pf}\,(1,2,3,4,5,6) \cdot \mathrm{Pf}\,(3,4,5,6) \cdot \mathrm{Pf}\,(3,6)$. The corresponding tableau is:

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |
| 3 | 4 | 5 | 6 | | |
| 3 | 6 | | | | |

which is standard.

Let $\mathcal{S}_{\mathbf{2r}}$ be the set of all standard tableaux with all rows of even size and of length $\leq 2r$. In the following, we will often identify, with some abuse of notation, standard monomials of Pfaffians with standard tableaux.

As in Section 2, let $X$ be a skew-symmetric matrix of indeterminates, and let $K[X]$ be the associated polynomial ring over the field $K$. Consider the homomorphism of rings

$$\Phi_{2r+2} : K[X] \longrightarrow K[Y]$$

which maps $X_{ij}$ to the $(i,j)$th entry of $YE(2r)Y^t$. For a standard tableau $\Sigma \in \mathcal{S}_{2r}$, consider the corresponding product of Pfaffians of $X$ and denote with $\Sigma_{YE(2r)Y^t}$ the image in $K[Y]$. Using this notation, we can state the main result of this section:

**Proposition 3.3.** (i) *Let $\Sigma = (a_{ij}) \in \mathcal{S}_{2r}$ be of shape $(2s_1 \dots 2s_u)$. The initial term of $\Sigma_{YE(2r)Y^t}$ is*

$$\operatorname{in}(\Sigma_{YE(2r)Y^t}) = \prod_{i=1}^{u} \prod_{j=1}^{2s_i} Y_{a_{ij}\, j}.$$

(ii) *For $\Sigma, \Sigma' \in \mathcal{S}_{2r}$, $\Sigma \neq \Sigma'$, we have* $\operatorname{in}(\Sigma_{YE(2r)Y^t}) \neq \operatorname{in}(\Sigma'_{YE(2r)Y^t})$. *In particular, the polynomials $\{\Sigma_{YE(2r)Y^t} \mid \Sigma \in \mathcal{S}_{2r}\}$ are linear independent over $K$.*

*Proof.* (i) Since initial terms commute with products, the equation is a direct consequence of Lemma 3.2.

(ii) Let us consider the monomial $m = \operatorname{in}(\Sigma_{YE(2r)Y^t})$ for a standard tableaux $\Sigma \in \mathcal{S}_{2r}$, and fix $j$. We take all the indices $k_1, \dots, k_t \in \{1, \dots, n\}$ such that $Y_{k_i\, j}$ divides $m$, if there exist such indices. Then, by (i), the indices $k_1, \dots, k_t$ are exactly the entries of the $j$th column of $\Sigma$. Since $\Sigma$ is a standard tableaux, there is only one possibility for putting the entries in the column which have to be non-decreasing from top to bottom. Repeating the argument for all columns one sees that there is exactly one standard tableau whose initial term is $m$. The linear independence of the polynomials $\{\Sigma_{YE(2r)Y^t} \mid \Sigma \in \mathcal{S}_{2r}\}$ is an immediate consequence. $\square$

**4. An initial algebra of $R_{2r+2}(X)$.** Using the results of Section 3, we can complete the program suggested in [**6**, Remark 3.10] for the Pfaffian case. Throughout the section, we consider $n, r \in \mathbf{N}$ with $1 \leq 2r \leq n$, and we use the same notation as in the previous sections.

The map $\Phi_{2r+2}: K[X] \to K[Y]$ comes from the above factors through the Pfaffian ring $R_{2r+2}(X) = K[X]/I_{2r+2}(X)$, since rank $(YE(2r)Y^t) = 2r$. Note that $R_{2r+2}(X) = K[X]$ for $r = \lfloor n/2 \rfloor$ since $I_{2r+2}(X) = (0)$ in this case. We partially recover a new proof of a classical result for Pfaffians; see [**11**, Theorem 6.5], whose proof is based on the ideas and results of Hodge [**21**] and Doubilet, Rota and Stein [**18**].

**Theorem 4.1** *The standard monomials of Pfaffians are a $K$-basis of $K[X]$.*

*Proof.* We apply Proposition 3.3 with $r = \lfloor n/2 \rfloor$. It yields that all standard monomials of Pfaffians in $K[X]$ are linearly independent, which is a new proof of Lemma 6.4 in [**11**]. That the standard monomials generate $K[X]$ as a $K$-vector space can be proved in at least two ways. Either one follows the remaining part of the original proof in [**11**, Theorem 6.5]. Another possibility is to use the Knuth-Robinson-Schensted correspondence [**24**] in the version of [**20**] applied to the Pfaffian case, which shows that degreewise we have as many standard tableaux in $\mathcal{S}_{2r}$ as monomials in $K[X]$. $\square$

In the following, we identify Pfaffians and products of Pfaffian (that is tableaux) in $K[X]$ with their classes in $R_{2r+2}(X)$. We obtain well-known results about $R_{2r+2}(X)$.

**Corollary 4.2.** *Let $K[YE(2r)Y^t]$ be the $K$-subalgebra of $K[Y]$ generated by the entries of the matrix $YE(2r)Y^t$.*

(i) *The homomorphism of rings $\Phi_{2r+2}$ induces an isomorphism*

$$\phi_{2r+2} : R_{2r+2}(X) \longrightarrow K[YE(2r)Y^t].$$

(ii) *The ring $R_{2r+2}(X)$ has a $K$-basis given by all standard tableaux in $\mathcal{S}_{2r}$.*

*Proof.* Recall that $\phi_{2r+2}$ is well defined (see above) and obviously it is surjective. Let $J \subset K[X]$ be the ideal generated by all standard tableaux not in $\mathcal{S}_{2r}$. By definition, we see that $J \subseteq I_{2r+2}(X)$. This fact, together with Theorem 4.1, implies that the standard tableaux in $\mathcal{S}_{2r}$ generate $R_{2r+2}(X)$. Applying Proposition 3.3, we see that these standard tableaux and the images in $K[YE(2r)Y^t]$ are $K$-linearly independent. This concludes the proof of Corollary 4.2 (i) and (ii). $\square$

We prove an analogous statement to [**6**, Theorem 3.5] for Pfaffians. Note that this approach via the initial algebras gives new proofs to some known facts on $R_{2r+2}(X)$, see e.g., [**1, 2, 12, 23, 25**].

**Theorem 4.3.** *The initial algebra* $\operatorname{in}(R_{2r+2}(X)) \subseteq K[Y]$ *is generated by*

$$\prod_{j=1}^{2t} Y_{a_j j} \quad \text{where } 1 \leq t \leq r \text{ and } 1 \leq a_1 < \cdots < a_{2t} \leq n.$$

*In particular,* $\operatorname{in}(R_{2r+2}(X))$ *is a normal affine monoid ring. Moreover,* $R_{2r+2}(X)$ *is a normal Cohen-Macaulay domain, with rational singularities in characteristic* $0$ *and* $F$*-rational in characteristic* $p > 0$*.*

*Proof.* The proof is analogous to the one of [**6**, Theorem 3.5] with minor modifications. Note the latter proof is itself based on Corollary 4.2 and the fact that one is able to apply [**8**, Corollary 2.3] and [**8**, Proposition 2.4]. $\square$

**5. The cone associated to the initial algebra of $R_{2r+2}(X)$.** To avoid trivial cases, we assume $2r < n$ and $n \geq 3$ from now on. Let $D_{2r+2} = \operatorname{in}(R_{2r+2}(X))$ be the initial algebra of $R_{2r+2}(X)$ inside $K[Y]$ where we identify $R_{2r+2}(X)$ with its image in $K[YE(2r)Y^t]$ under $\phi_{2r+2}$. In this section, we describe explicitly the cone generated by the exponent vectors of the elements in $D_{2r+2}$. We give an irredundant description of the defining half spaces of this cone and determine its relative interior. Using standard tools of combinatorial commutative algebra this will allow us, for example, to provide a relatively quick proof of the Gorenstein property of the rings $R_{2r+2}(X)$.

**Proposition 5.1.** *Let $E_{2r+2}$ be the set of vectors $(c_{ij}) \in \mathbf{R}^{n \cdot 2r}$ appearing as exponent vectors of elements in $D_{2r+2} = \operatorname{in}(R_{2r+2}(X)) \subset K[Y]$. Then $E_{2r+2}$ is the set of lattice points of the cone $\mathcal{C}_{2r+2}$ defined by the following inequalities and equalities*:

(i) $c_{ij} \geq 0$ *for all* $i, j$ *with* $1 \leq j \leq 2r$ *and* $j \leq i \leq n$.

(ii) $c_{ij} = 0$ *for all* $i, j$ *with* $1 \leq j \leq 2r$ *and* $j > i \geq 1$.

(iii) $\sum_{i=j}^{k} c_{ij} - \sum_{i=j+1}^{k+1} c_{i\,j+1} \geq 0$ *for all* $j, k$ *with* $1 \leq j \leq 2r-1$ *and* $j \leq k \leq n-1$.

(iv) $\sum_{i=j}^{n} c_{ij} - \sum_{i=j+1}^{n} c_{i\,j+1} = 0$ *for all odd* $j$ *with* $1 \leq j \leq 2r-1$.

*Proof.* Let $(c_{ij})$ be the exponent vector of a monomial in $(\Sigma_{YE(2r)Y^t})$ for a certain standard tableau $\Sigma \in \mathcal{S}_{2r}$. Trivially, $(c_{ij})$ satisfies the equations in (i). Observe that

$$c_{ij} = \#\{\text{entries equal to } i \text{ in the } j\text{th column of } \Sigma\}.$$

Note that, by the standard tableau property, we have $c_{ij} = 0$ for $j > i$. In particular, $(c_{ij})$ satisfies the equations in (ii). Moreover, we see that

$$\sum_{i=j}^{k} c_{ij} = \#\{a \in \mathbf{N} : a \leq k,\ a \text{ is in the } j\text{th column of } \Sigma\}.$$

Since $\Sigma$ is a standard tableau, the number of entries in the $(j+1)$th column which are less than or equal to $k+1$ is at most the number of entries in the $j$th column which are less than or equal to $k$. The latter number might be larger. Hence, $(c_{ij})$ satisfies the equations in (iii).

We have the additional property that the rows of $\Sigma$ have even length. Thus, we see that, for an odd $j$, the number of entries in the $j$th column is equal to the number of entries in the $(j+1)$th column, and so $(c_{ij})$ satisfies also the equations in (iv). Hence, we get that $E_{2r+2}$ is a subset of all lattice points in $C_{2r+2}$.

Next, let $(c_{ij})$ be an arbitrary lattice point in $C_{2r+2}$. We have to construct a standard tableau $\Sigma \in \mathcal{S}_{2r}$ such that $(c_{ij})$ is the exponent vector of the monomial in $(\Sigma_{YE(2r)Y^t})$. For this, consider an integer $j$ with $1 \leq j \leq 2r$. Note that, by (i) and (ii), the number $c_{kj}$ is non-negative. We insert $c_{kj}$ many entries $k$ in the $j$th column such that the entries of that column are increasing from the top to the bottom and obtain a candidate $\Sigma = (a_{ij})$ for the standard tableau. By definition, we have that

$$\sum_{i=j}^{k} c_{ij} = \#\{a \in \mathbf{N} : a \leq k,\ a \text{ is in the } j\text{th column of } \Sigma\}.$$

Note that, by (ii), the number $j$ is the first one which might occur in the $j$th column. Since $\sum_{i=j}^{n-1} c_{i\,j} \geq \sum_{i=j+1}^{n} c_{i\,j+1}$, we see that there are at least as many elements in the $(j+1)$th column as in the $j$th column, which assures that $\Sigma$ is a tableau.

It remains to prove that the entries in each row form a strictly increasing sequence. Suppose this is not the case, and choose $i$ the maximal index such that there exists $j$ with $a_{ij} \geq a_{i\,j+1}$.

Then, $a_{lj} < a_{l\,j+1}$ for $l > i$ if there is an entry at position $(l, j+1)$. Let $k+1 = a_{i\,j+1}$. Assume that $a_{i+1\,j+1} = k+1$. Then, we get the contradiction

$$k+1 = a_{i+1\,j+1} > a_{i+1\,j} \geq a_{ij} \geq a_{i\,j+1} = k+1.$$

Hence, $a_{i+1\,j+1} > k+1$ or the $(j+1)$th column has only $i$ entries. In particular, we see that

$$\sum_{i=j+1}^{k+1} c_{i\,j+1} = \#\{a \in \mathbf{N} : a \leq k+1,\ a \text{ is in the } (j+1)\text{th column of } \Sigma\} = i.$$

Since $a_{ij} \geq a_{i\,j+1} = k+1$, we also have

$$\sum_{i=j}^{k} c_{ij} = \#\{a \in \mathbf{N} : a \leq k,\ a \text{ is in the } j\text{th column of } \Sigma\} \leq i-1.$$

This is a contradiction to the fact that $(c_{ij})$ satisfies the equation in (iii). This shows that every row of $\Sigma$ is a strictly increasing sequence of integers and, by construction, we know already that every column forms a weakly increasing sequence of integers. Condition (iv) implies that we deal with a tableau whose rows have even length, that is, the $j$th and the $(j+1)$th columns have the same number of entries for every odd $j$. By definition, the length of $\Sigma$ is bounded by $2r$. Hence, $\Sigma \in \mathcal{S}_{2r}$, and we see that $(c_{ij})$ is indeed the exponent vector of the monomial in $(\Sigma_{YE(2r)Y^t})$. This concludes the proof. $\square$

The inequalities describing the cone in Proposition 5.1 are not irredundant, as we will see below. Before continuing the discussion, we prove the following result:

**Lemma 5.2.** *Let* $(c_{ij}) \in \mathbf{R}^{n \cdot 2r}$ *be a vector satisfying conditions* (i)–(iv) *in Proposition* 5.1. *Then*:

(a) $c_{11} \geq c_{22} \geq \cdots \geq c_{2r\,2r}$.

(b) $c_{n\,j} = 0$ *for all odd* $j$ *with* $1 \leq j \leq 2r-1$.

(c) *For all odd* $j$ *with* $1 \leq j \leq 2r-1$ *and* $k = n-1$, *the inequalities in* (iii) *are satisfied by* $(c_{ij})$ *as equalities.*

*Proof.* (a) Proposition 5.1 (iii) for $k = j$ is just $c_{j\,j} \geq c_{j+1\,j+1}$ for all $j$ with $1 \leq j \leq 2r-1$.

(b) and (c). Let $j$ be an odd integer in $\{1, \ldots, 2r-1\}$. By Proposition 5.1, applying (iii) for $k = n-1$ together with (iv), we see that

$$\sum_{i=j}^{n-1} c_{ij} \geq \sum_{i=j+1}^{n} c_{i\,j+1} = \sum_{i=j}^{n} c_{ij}.$$

Since $c_{ij} \geq 0$ by (i) and (ii), this implies $c_{nj} = 0$, which is (b). Moreover, we see that the first inequality has to be an equation, and this proves (c). $\square$

Using Lemma 5.2, we rewrite the inequalities and equalities in Proposition 5.1 to describe $\mathcal{C}_{2r+2}$ in a way which is more suitable in the following.

**Lemma 5.3.** *The cone* $\mathcal{C}_{2r+2}$ *in* $\mathbf{R}^{n \cdot 2r}$ *is defined by the following system of inequalities and equalities*:

(i) $c_{ij} \geq 0$ *for all* $i, j$ *with* $1 \leq j \leq 2r$ *and*

(a) $j < i \leq n-1$ *if* $j$ *is odd,*

(b) $j < i \leq n$ *if* $j$ *is even.*

(ii) $c_{2r\,2r} \geq 0$.

(iii) $c_{nj} = 0$ *for all odd* $j$ *with* $1 \leq j \leq 2r - 1$.

(iv) $c_{ij} = 0$ *for all* $i, j$ *with* $1 \leq j \leq 2r$ *and* $j > i \geq 1$.

(v) $\sum_{i=j}^{k} c_{ij} - \sum_{i=j+1}^{k+1} c_{i\,j+1} \geq 0$ *for all* $j, k$ *with* $1 \leq j \leq 2r - 1$, *and*

(a) $k = j, \ldots, n - 2$ *if* $j$ *is odd,*

(b) $k = j, \ldots, n - 1$ *if* $j$ *is even.*

(vi) $\sum_{i=j}^{n-1} c_{i\,j} - \sum_{i=j+1}^{n} c_{i\,j+1} = 0$ *for all odd* $j$ *with* $1 \leq j \leq 2r - 1$.

*Proof.* Let $(c_{ij}) \in \mathbf{R}^{n \cdot 2r}$ satisfy (i)–(iv) of Proposition 5.1. Using Lemma 5.2, we see that this element also satisfies the new inequalities and equalities.

Conversely, (i), (ii), (iii) and (v) for $k = j$ imply the inequalities of Proposition 5.1 (i). The equations in (iv) are exactly the ones in Proposition 5.1 (ii). We see also that (v) and (vi) imply the inequalities of Proposition 5.1 (iii). Finally, (iii) and (vi) give us the equations in Proposition 5.1 (iv). Thus, we see that (i)–(vi) also define $\mathcal{C}_{2r+2}$. □

The next goal is to describe the lattice points in the relative interior of $\mathcal{C}_{2r+2}$. For this, and to prove later the main result of this section, we construct a special tableau in $\mathcal{S}_{2r}$.

**Definition 5.4.** Let $T_{2r} = (t_{sp})$ be the tableau with $2r$ columns, obtained as follows. For every $p = 1, \ldots 2r$, let

$$t_{sp} = \begin{cases} p & \text{for } s = 1, \ldots, 2r + 1 - p, \\ 2p + s - 2r - 1 & \text{for } s = 2r + 2 - p, \ldots, 2r + n - 2p. \end{cases}$$

If $p$ is even, set $t_{sp} = n$ for $s = 2r + n - 2p + 1,\ 2r + n - 2p + 2$.

**Example 5.5.** Here are $T_2$ with $n = 6$, $T_4$ with $n = 6$, and $T_6$ with $n = 9$.

$$\begin{array}{|c|c|}\hline 1&2\\\hline 1&3\\\hline 2&4\\\hline 3&5\\\hline 4&6\\\hline 5&6\\\hline\end{array}
\qquad
\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 1&2&3&5\\\hline 1&2&4&6\\\hline 1&3&5&6\\\hline 2&4\\\cline{1-2} 3&5\\\cline{1-2} 4&6\\\cline{1-2} 5&6\\\cline{1-2}\end{array}
\qquad
\begin{array}{|c|c|c|c|c|c|}\hline 1&2&3&4&5&6\\\hline 1&2&3&4&5&7\\\hline 1&2&3&4&6&8\\\hline 1&2&3&5&7&9\\\hline 1&2&4&6&8&9\\\hline 1&3&5&7\\\cline{1-4} 2&4&6&8\\\cline{1-4} 3&5&7&9\\\cline{1-4} 4&6&8&9\\\cline{1-4} 5&7\\\cline{1-2} 6&8\\\cline{1-2} 7&9\\\cline{1-2} 8&9\\\cline{1-2}\end{array}$$

**Lemma 5.6.** *The tableau $T_{2r}$ is in $\mathcal{S}_{2r}$, and it has $2nr$ elements. The corresponding lattice point $(c_{ij})$ in $\mathcal{C}_{2r+2}$ has the following properties*:

(i) $c_{jj} = 2r+1-j$ *for all* $j = 1, \dots, 2r$.

(ii) $c_{ij} = 1$ *for all* $i, j$ *with* $1 \le j \le 2r$ *and* $1 \le j < i \le n-1$.

(iii) $c_{nj} = 2$ *for all even* $j$ *with* $2 \le j \le 2r$.

(iv) $\sum_{i=j}^{k} c_{ij} - \sum_{i=j+1}^{k+1} c_{i\,j+1} = 1$ *for all* $j, k$ *with* $1 \le j \le 2r-1$ *and* $j \le k \le n-2$.

(v) $\sum_{i=j}^{n-1} c_{ij} - \sum_{i=j+1}^{n} c_{i\,j+1} = 2$ *for all even* $j$ *with* $2 \le j \le 2r-2$.

*In particular, $(c_{ij})$ satisfies strictly the inequalities in* (i), (ii), (v) *of Lemma* 5.3.

*Proof.* If $p$ is odd, the $p$th column contains $2r+n-2p$ elements, and the $(p+1)$th column contains $2r+n-2(p+1)+2 = 2r+n-2p$ elements. Thus, $T_{2r}$ is a tableau with rows of even length. We also see that the length of the first row is $2r$. It is easy to check that, by definition, $T_{2r}$ is a standard tableau. Hence, $T_{2r} \in \mathcal{S}_{2r}$. Observe that the shape of $T_{2r}$ is

$$(\underbrace{2r, 2r, \dots, 2r}_{n-2r+2 \text{ times}}, \underbrace{2r-2, \cdots, 2r-2}_{4\text{times}}, \underbrace{2r-4, \dots, 2r-4}_{4 \text{ times}}, \dots,$$
$$4, 4, 4, 4, 2, 2, 2, 2).$$

So we get that the number of the entries of $T$ is equal to

$$2r(n-2r+2)+4\sum_{i=1}^{r-1}2i = 2r(n-2r+2)+4(r-1)r = 2nr.$$

Let $(c_{ij})$ be the lattice point corresponding to $T_{2r}$. Recall that, by Proposition 5.1 and Lemma 5.3, this element satisfies all equalities and inequalities of Lemma 5.3. Moreover, the corresponding proofs show that $c_{ij} = \#\{t_{sj} : t_{sj} = i\}$.

By Definition 5.4, one has $t_{1j} = t_{2\,j} = \cdots = t_{2r+1-j\,j} = j$, $t_{2r+2-j\,j} = j+1$, $t_{2r+3-j\,j} = j+2, \ldots, t_{2r+n-2j\,j} = n-1$. If $j$ is even, then $t_{2r+n-2j+1\,j} = t_{2r+n-2j+2\,j} = n$. Thus, we see that (i), (ii) and (iii) hold. Moreover, $\sum_{i=j}^{k} c_{ij} = \#\{t_{sj} : t_{sj} \le k\}$. Hence, (iv) and (v) follow by comparing the entries of the $j$th column with the entries of the $(j+1)$th column. We also see that $(c_{ij})$ satisfies strictly the inequalities in (i), (ii), (v) of Lemma 5.3. This concludes the proof. □

Using the results so far, we can describe the lattice points in the relative interior of $\mathcal{C}_{2r+2}$.

**Proposition 5.7.** *The relative interior of* $\mathcal{C}_{2r+2}$ *is defined by strict inequalities in* (i), (ii), (v) *of Lemma* 5.3. *Moreover, if* $(c_{ij}) \in \mathbf{R}^{n\cdot 2r} \in \mathcal{C}_{2r+2}$ *is a lattice point in the relative interior of* $\mathcal{C}_{2r+2}$, *then*:

(a) $c_{jj} \ge 2r+1-j$ *for all* $j = 1, \ldots, 2r$.

(b) $c_{nj} \ge 2$ *for all even* $j$ *with* $1 \le j \le 2r$.

(c) $\sum_{i=j}^{n-1} c_{i\,j} - \sum_{i=j+1}^{n} c_{i\,j+1} \ge 2$ *for all even* $j$ *with* $2 \le j \le 2r-2$.

*Proof.* Let $U$ be the subspace of $\mathbf{R}^{n\cdot 2r}$ defined by the equations in (iii), (iv) and (vi) of Lemma 5.3. Then, clearly, $\mathcal{C}_{2r+2} \subseteq U$. Moreover, in Lemma 5.6, we observed that $T_{2r}$ corresponds to a lattice point which strictly satisfies the inequalities in (i), (ii), (v) of Lemma 5.3. This implies that $\mathcal{C}_{2r+2}$ must be full dimensional inside $U$ and $T_{2r}$ must be an interior point of $\mathcal{C}_{2r+2}$ inside $U$, that is a relative interior point of $\mathcal{C}_{2r+2}$ inside $\mathbf{R}^{n\cdot 2r}$.

Moreover, we can conclude that the relative interior of $\mathcal{C}_{2r+2}$ is defined by strict inequalities in (i), (ii), (v) of Lemma 5.3. Now let $(c_{ij}) \in \mathbf{R}^{n\cdot 2r}$ be a lattice point in the relative interior of $\mathcal{C}_{2r+2}$.

(a) By the strict inequalities in Lemma 5.3 (ii) and (v) for $k = j$ we get

$$c_{11} > c_{22} > \cdots > c_{2r\,2r} \geq 1.$$

It follows that $c_{jj} \geq 2r + 1 - j$ for $j = 1, \ldots, 2r$.

(b) Let $j$ be even with $1 \leq j \leq 2r$. By Lemma 5.3 (vi) one has $\sum_{i=j}^{n} c_{ij} = \sum_{i=j-1}^{n-1} c_{i\,j-1}$. Thus,

$$c_{nj} = \sum_{i=j-1}^{n-1} c_{i\,j-1} - \sum_{i=j}^{n-1} c_{ij} = c_{n-1\,j-1} + \left[\sum_{i=j-1}^{n-2} c_{i\,j-1} - \sum_{i=j}^{n-1} c_{ij}\right] \geq 2.$$

The last inequality follows since $c_{n-1\,j-1} \geq 1$ by the strict inequalities in Lemma 5.3 (i) and $\sum_{i=j-1}^{n-2} c_{i\,j-1} > \sum_{i=j}^{n-1} c_{ij}$ by the ones of Lemma 5.3 (v) for $k = n - 2$.

(c) Lemma 5.3 (v) with $k = n - 2$ implies that $\sum_{i=j}^{n-2} c_{i\,j} - \sum_{i=j+1}^{n-1} c_{i\,j+1} \geq 1$. Moreover, $c_{n\,j+1} = 0$ by Lemma 5.3 (iii) and $c_{n-1\,j} \geq 1$ by Lemma 5.3 (i). Hence,

$$\sum_{i=j}^{n-1} c_{i\,j} - \sum_{i=j+1}^{n} c_{i\,j+1} = c_{n-1\,j} + \left[\sum_{i=j}^{n-2} c_{ij} - \sum_{i=j+1}^{n-1} c_{i\,j+1}\right] \geq 2. \qquad \square$$

We are ready to prove the main result of this section:

**Corollary 5.8.** *The ring $R_{2r+2}(X)$ is Gorenstein with $a$-invariant $-nr$.*

*Remark* 5.9. Note that the Gorenstein property of $R_{2r+2}(X)$ is a classical result. Avramov [**1**] showed that $R_{2r+2}(X)$ is factorial. Since it is Cohen-Macaulay, it is also Gorenstein by a result of Murthy (see, e.g., [**19**, 12.31]). See also [**12**]. Here we get it as an application of tools from combinatorial commutative algebra.

The $a$-invariant was computed in [**4**, Corollary 1.7] by different methods. See also [**14**]. Observe that the $a$-invariant determines the highest shift in the resolution of $R_{2r+2}(X)$. Thus, this determines, e.g., also the Castelnuovo-Mumford regularity of $R_{2r+2}(X)$.

*Proof of Corollary* 5.8. By Theorem 4.3, we know that $D_{2r+2}$ is a normal affine monoid ring. Since it is Cohen-Macaulay, it has a (graded) canonical module and it is known from results of Danilov [**9**] and Stanley [**27**] that this module is the ideal generated by the monomials corresponding to the lattice points in the relative interior of $C_{2r+2}$, see also [**5**, subsection 6.3]. We know by Lemma 5.6 and Proposition 5.7 that $T_{2r}$ corresponds to a lattice point $c$ in the relative interior of $C_{2r+2}$. Moreover, for any lattice point $d$ in the relative interior of $C_{2r+2}$, we see by Proposition 5.7 that $d-c$ is again a lattice point in $C_{2r+2}$. Hence, $T_{2r}$ is a minimal lattice point in the relative interior of $C_{2r+2}$ and, equivalently, the canonical module is isomorphic to $D_{2r+2}$ shifted by $2nr$.

Thus, $D_{2r+2}$ is Gorenstein with $a$-invariant $-2nr$. This implies that $R_{2r+2}(X)$ is Gorenstein; see, e.g., [**3, 8**]. Note that in the deformation from $R_{2r+2}(X)$ to $D_{2r+2}$ the $X$ variables, which have degree 1, are mapped to degree 2 polynomials. Hence, the $a$-invariant of $R_{2r+2}(X)$ is $-nr$. This concludes the proof. $\square$

**6. Pfaffian rings $R_\alpha(X)$.** In Section 2, we have introduced the cogenerated Pfaffian ring $R_\alpha(X)$, for some Pfaffian $\alpha \in \mathrm{Pf}(X)$. We have not constructed a generic point for the rings of this form. But we are able to study them using the generic point of $K[X]$ and initial algebra methods. At first, we recall the following result:

*Remark* 6.1. Let $\alpha \in \mathrm{Pf}(X)$. It is a well-known fact in ASL theory that, since $I_\alpha(X)$ is an order ideal, it has a basis over $K$ given by all standard monomials of Pfaffians $\alpha_1 \cdot \ldots \cdot \alpha_u$ such that $\alpha_1 \not\geq \alpha$.

Note that the fact that these standard monomials are linearly independent also follows easily by Theorem 4.1.

For the next result, we introduce the following notation. Let $D = \mathrm{in}\,(K[X])$ be the initial algebra of $K[X]$ inside $K[Y]$ where we identify $K[X]$ with its image $K[YE(2r)Y^t]$ under $\phi_{2r+2}$ for $r = \lfloor n/2 \rfloor$. For $\alpha \in \mathrm{Pf}(X)$, let $J_\alpha(X) = \mathrm{in}\,(I_\alpha(X))$ be the initial ideal of $I_\alpha(X)$ inside $D$. Now we are ready to prove the main result of this section.

**Theorem 6.1.** *Let $\alpha \in Pf(X)$. Then $J_\alpha(X)$ is a monomial prime ideal in $D$. In particular, $R_\alpha(X)$ is a normal Cohen-Macaulay* *domain, with rational singularities in characteristic* 0 *and* $F$*-rational in characteristic* $p > 0$.

That $R_\alpha(X)$ is a normal Cohen-Macaulay domain was observed, for example, in [**12**, Proposition 1.1, Corollary 2.2 and Proposition 2.4].

*Proof.* It follows from Lemma 6.1 (i) that, for $I_\alpha(X)$, a $K$-basis is given by all standard monomials of Pfaffians $\alpha_1 \cdot \ldots \cdot \alpha_u$ such that $\alpha_1 \not\geq \alpha$. Hence, $J_\alpha(X)$ is generated by all initial forms of such $\alpha_1 \cdot \ldots \cdot \alpha_u$ which we computed explicitly in Theorem 4.3.

More precisely, each of these monomials has the form $\prod_{j=1}^{2t} Y_{a_j j}$ where $1 \leq t \leq \lfloor n/2 \rfloor$ and $1 \leq a_1 < \cdots < a_{2t} \leq n$. If $\alpha = \mathrm{Pf}\,(b_1 \ldots b_{2s})$, then $\mathrm{Pf}\,(a_1 \ldots a_{2t}) \not\geq \alpha$ if and only if $t > s$ or $t \leq s$ and $a_i < b_i$ for some $i \in \{1, \ldots, s\}$. The latter monomials generate $J_\alpha(X)$. Since $D$ is a normal affine monoid ring, we see that $J_\alpha(X)$ is a monomial prime ideal and $D/J_\alpha(X)$ is again a normal affine monoid ring; for details see, e.g., [**5**, Chapter 6]. This concludes the proof by applying again [**8**, Corollary 2.3 and Proposition 2.4]. □

*Remark* 6.2. As in Section 5, one could compute the defining equations of the normal affine monoid ring $T/J_\alpha(X)$ to determine its canonical module and to investigate the Gorenstein property of $R_\alpha(X)$. We saw already in Corollary 5.8 that $R_{2r+2}(X)$ is Gorenstein; besides them, very few of the rings $R_\alpha(X)$ are Gorenstein, see [**12**, Theorem 3.5] for a full classification. We leave the proof of the general case with the techniques used in the previous section to the interested reader or refer to the original proof in [**12**].

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