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ON THE DISCRETE COUNTERPARTS OF ALGEBRAS WITH STRAIGHTENING LAWS

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ABSTRACT. We study properties of a poset generating a Cohen-Macaulay algebra with straightening law. We show that if a poset P generates a Cohen-Macaulay algebra with straightening law, then P is pure and, if P is moreover Buchsbaum, then P is Cohen-Macaulay.

1. Introduction. DeConcini, Eisenbud and Procesi defined the notion of Hodge algebra in their article [**4**] and proved many properties of Hodge algebras. They also showed that many algebras appearing in algebraic geometry and commutative ring theory have structures of Hodge algebras. In fact, the theory of Hodge algebras is an abstraction of combinatorial arguments that are used to study those rings.

A Hodge algebra is an algebra with relations which satisfy certain laws regulated by combinatorial data. There exist many Hodge algebras supported on the same combinatorial data; however, there is one which is, in some sense, the simplest Hodge algebra with given combinatorial data, called the *discrete Hodge algebra*. For a given Hodge algebra, we call the discrete Hodge algebra with the same combinatorial data the discrete counterpart of it. DeConcini, Eisenbud and Procesi proved that:

• A Hodge algebra and its discrete counterpart have the same dimension.

• The depth of the discrete counterpart is not greater than the depth of the original Hodge algebra.

It is known that there is a Hodge algebra whose discrete counterpart has strictly smaller depth than the original one [**5**].

But if we restrict our attention to ordinal Hodge algebras (algebras with straightening laws, ASL for short), the influence of the combinatorial data on the ring theoretical properties becomes greater. So there

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may be a restriction to the combinatorial data by the ring theoretical properties of an ASL.

The purpose of this article is to study properties of combinatorial data of a Cohen-Macaulay graded ASL. Since it is equivalent to studying the properties of combinatorial data of an ASL, i.e., the properties of the partially ordered set (poset for short) generating the ASL (see Section 2 for terminology), and to studying the properties of the discrete counterpart, our results are sometimes written in the language of posets and sometimes in the language of commutative rings.

We have to comment upon the result of Terai [**12**]. Let I be a homogeneous ideal in a polynomial ring R and τ a term order. Terai proved in [**12**] that $\operatorname{depth} R/I - \operatorname{depth} R/\operatorname{in}(I) \leq 1$ in the ASL setting. Unfortunately Terai's argument is not correct since it works as well for any initial ideal and there are plenty of examples showing that $\operatorname{depth} R/I - \operatorname{depth} R/\operatorname{in}(I)$ can be larger than 1. Indeed the difference between the depth of R/I and that of $R/\operatorname{in}(I)$ can be arbitrarily big. This happens even if I is a prime ideal defining a smooth projective variety and τ is a degrevlex order as the following example shows.

Example 1.1. Let I be the ideal of the 2-minors of an $n \times n$ generic symmetric matrix $X = (x_{ij})$, and let τ be the revlex order associated with $x_{11} > x_{12} > \cdots > x_{1n} > x_{22} > \cdots > x_{nn}$. The depth of R/I is n and the depth of $R/\operatorname{in}(I)$ is 2.

In Section 3, we note that if P generates a Cohen-Macaulay ASL, then P is pure. In Section 4, we show that if P generates a Cohen-Macaulay ASL, and P itself is Buchsbaum, then P is Cohen-Macaulay.

Consider the following four conditions.

(i) P is a poset.

(ii) P is a pure poset.

(iii) P is a Buchsbaum poset.

(iv) P is a Cohen-Macaulay poset.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial or well known. And the results of Sections 3 and 4 show that with the assumption that there is a Cohen-Macaulay ASL generated by P, (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii) are also valid.

2. Preliminaries. In this article all rings and algebras are commutative with identity. We denote the number of elements of a finite set X by $|X|$ and, for two sets X and Y, we denote by $X \setminus Y$ the set $\{x \in X \mid x \notin Y\}$. The set of integers (respectively non-negative integers) are denoted by $\mathbf{Z}$ (respectively $\mathbf{N}$). Standard terminology on Hodge algebras and Stanley-Reisner rings are used freely. See [**1**, Chapter 5], [**2, 4, 6**] and [**11**, Chapter II] for example. However, we use the term "algebra with straightening laws" (ASL for short) to mean an ordinal Hodge algebra. Therefore, we use the expression like "A is a Hodge algebra over k generated by H governed by Σ" or "A is an ASL over k generated by H." But we sometimes use the expression like "A is an ASL over k supported by H."

In addition we use the following notation and convention.

• We use the term poset to stand for finite partially ordered set.

• If P is a poset, we denote the set of all the minimal elements of P by $\min P$.

• If P is a poset, a poset ideal of P is a subset Q of P such that $x \in Q$, $y \in P$ and $y < x$ imply $y \in Q$.

• For a poset P, we define the order complex $\Delta(P)$ of P by

$$\Delta(P) := \{\sigma \subseteq P \mid \sigma \text{ is a chain}\},$$

where a chain stands for a totally ordered subset.

• We denote the Stanley-Reisner ring $k[\Delta(P)]$ by $k[P]$, where k is a commutative ring and P is a poset. And if $k[P]$ is Cohen-Macaulay (or Buchsbaum respectively), then we say P is Cohen-Macaulay (Buchsbaum respectively) over k.

• If A is a Hodge algebra over k generated by H governed by Σ, we denote by A_{dis} the discrete Hodge algebra over k generated by H governed by Σ.

• If B is an $\mathbf{N}^m$-graded ring with $B_{(0,\dots,0)}$ a field, then we denote by $\operatorname{depth} B$ the depth of B_M, where M is the unique $\mathbf{N}^m$-graded maximal ideal.

Next we recall the notion of a standard subset [**9**].

Definition 2.1. Let A be a Hodge algebra over k generated by H governed by Σ. A subset Ω of H is called a standard subset of H if for any element $x \in \Omega A$ and for any standard monomial M_i appearing in the standard representation

$$x = \sum_i b_i M_i \quad (0 \neq b_i \in k,\ M_i \text{ standard})$$

of x, $\operatorname{supp} M_i$ meets Ω.

For example, a poset ideal of H is a standard subset by Fact 2.3 below. Note that if Ω is a standard subset of H, then $A/\Omega A$ is a Hodge algebra over k generated by $H \setminus \Omega$ governed by $\{\mu \in \Sigma \mid \operatorname{supp} \mu \cap \Omega = \varnothing\}$.

Now we recall several facts.

Fact 2.2 [**4**, Theorem 6.1 and Corollary 7.2]. *If A is a graded Hodge algebra over a field, then*

$$\dim A_{\text{dis}} = \dim A$$

and

$$\operatorname{depth} A_{\text{dis}} \leq \operatorname{depth} A.$$

Fact 2.3 [**4**, Proposition 1.2]. *If A is a Hodge algebra over k generated by a poset P governed by Σ and Q is a poset ideal of P, then A/QA is a Hodge algebra over k generated by $P \setminus Q$ governed by $\{\mu \in \Sigma \mid \operatorname{supp} \mu \cap Q = \varnothing\}$.*

Like [**11**, II.5], we make the following

Definition 2.4. For a Hodge algebra A over k generated by H governed by Σ, we define

$$\begin{aligned} \operatorname{core} H &:= \bigcup_{N \text{ is a generator of } \Sigma} \operatorname{supp} N, \\ \operatorname{core} \Sigma &:= \{\mu \in \Sigma \mid \operatorname{supp} \mu \subseteq \operatorname{core} H\}, \\ \operatorname{core} A &:= A/(H \setminus \operatorname{core} H)A. \end{aligned}$$

It is obvious that if $\Omega = \{x_1, \dots, x_t\}$ is a subset of H such that $\Omega \cap \operatorname{core} H = \varnothing$, then Ω is a standard subset of H and $x_1, \dots, x_t$ is an A-regular sequence. In particular,

Lemma 2.5. $\operatorname{core} A$ *is a Hodge algebra generated by* $\operatorname{core} H$ *governed by* $\operatorname{core} \Sigma$. *Furthermore, if* $H \setminus \operatorname{core} H = \{x_1, \dots, x_t\}$, *then* $x_1, \dots, x_t$ *is an* A*-regular sequence and* $\operatorname{core} A = A/(x_1, \dots, x_t)$.

Moreover, it is easily verified that

$$(\operatorname{core} A)_{\mathrm{dis}} = \operatorname{core}(A_{\mathrm{dis}}).$$

So we denote both sides by $\operatorname{core} A_{\mathrm{dis}}$.

3. Stepping stones. In what follows in this article, we focus our attention on ASL and consider the following

Problem 3.1. *If there is a Cohen-Macaulay* ASL *over* k *generated by a poset* P, *what can be said about* P? *In particular, is* P *Cohen-Macaulay over* k?

To tackle this problem, we state two stepping stones and consider the following four conditions.

(i) P is a poset.

(ii) P is a pure poset.

(iii) P is a Buchsbaum poset.

(iv) P is a Cohen-Macaulay poset.

The implications (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial or well known. And in the following, we state that, under the assumption that there is a Cohen-Macaulay ASL generated by P, (iii) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii) are also valid.

As for (i) $\Rightarrow$ (ii), we first recall the following result of Varbaro.

Fact 3.2 [**13**, Corollary 2.13]. *Let* S *be a polynomial ring over a field with monomial order and* I *an ideal of* S. *If* S/I *is Cohen-Macaulay,* *then* $\operatorname{Spec}(S/\operatorname{in}(I))$ *is connected in codimension* 1*, where* $\operatorname{in}(I)$ *stands for the initial ideal of* I*.*

Assume that A is a graded Hodge algebra over k. Then it is well known that there is a polynomial ring S with monomial order and a graded ideal I of S such that

$$A \simeq S/I \quad \text{and} \quad A_{\mathrm{dis}} \simeq S/\operatorname{in}(I).$$

Therefore, by the result of Varbaro, we see the following result (cf. [**13**, Remark 1.1]).

Corollary 3.3. *Let* A *be a Cohen-Macaulay graded Hodge algebra over a field. Then* $\operatorname{Spec}(A_{\mathrm{dis}})$ *is connected in codimension* 1*. In particular,* A_{dis} *is equidimensional.*

In particular, we have the following

Corollary 3.4. *Let* P *be a poset. If there is a Cohen-Macaulay* ASL *generated by* P*, then* P *is pure.*

4. Buchsbaum posets supporting Cohen-Macaulay **ASL's are** Cohen-Macaulay**.** In this section, we shall prove that if a poset P generates a Cohen-Macaulay ASL and if P itself is Buchsbaum, then P is Cohen-Macaulay.

We begin by noting the graded version of the theorem of Huckaba and Marley [**7**], which is proved by noting Lemma 4.2 below and reducing the proof of [**7**, Proposition 3.2] to the case of bigraded prime ideals.

Theorem 4.1 (The graded version of the theorem of Huckaba-Marley). *Let* A *be a non-negatively graded Noetherian ring with* A_0 *a field and* I *a non-nilpotent graded ideal of* A*. Denote by* R *the Rees algebra with respect to* I *and by* G *the associated graded ring. Suppose that*

$$\operatorname{depth} G < \operatorname{depth} A.$$

Then

$$\operatorname{depth} R = \operatorname{depth} G + 1.$$

Lemma 4.2. *Let R be an $\mathbf{N}^2$-graded ring and I an ideal of R which is homogeneous in the first grading. If we set*

$$I^* := (a \in I \mid a \text{ is homogeneous in the second grading}),$$

then

$$I^* = (x \in I \mid x \text{ is bihomogeneous}).$$

In particular, I^* is a bigraded ideal.

By a standard argument (cf. [**10**] for example), we see the following

Lemma 4.3. *Let k be an infinite field and G an $\mathbf{N}^m$-graded Hodge algebra over k. Then for any $\alpha \in \mathbf{Z}^m$, $[H^i_N(G)]_\alpha$ is a subquotient of $[H^i_{N'}(G_{\mathrm{dis}})]_\alpha$, where N (or N' respectively) is the unique $\mathbf{N}^m$-graded maximal ideal of G (G_{dis} respectively).*

Now we state the following

Theorem 4.4. *Let A be a graded Cohen-Macaulay square-free Hodge algebra over a field k. Suppose that $\operatorname{core} A_{\mathrm{dis}}$ is Buchsbaum. Then A_{dis} is Cohen-Macaulay.*

Proof. Let H be the poset which generates the Hodge algebra A, and let Σ be the ideal of monomials on H which govern A. We may assume that $H \cap \Sigma = \varnothing$ (i.e., h is a standard monomial for any $h \in H$). Set $\Delta := \{\sigma \subseteq H \mid \prod_{x_i \in \sigma} x_i \notin \Sigma\}$. Then $A_{\mathrm{dis}} = k[\Delta]$.

In order to prove the theorem, we may assume, by tensoring an infinite field containing k, that k is an infinite field. And by considering $\operatorname{core} A$ instead of A, we may assume that $A_{\mathrm{dis}} = k[\Delta]$ is Buchsbaum.

We prove the theorem by induction on $|\operatorname{ind} A|$, where $\operatorname{ind} A$ stands for the indiscrete part of A (cf. [**4**, page 16]). If $\operatorname{ind} A = \varnothing$, then $A_{\mathrm{dis}} = A$ and the assertion is clear. So we assume that $\operatorname{ind} A \neq \varnothing$.

Take a minimal element x of $\operatorname{ind} A$ and set $I = xA$. Denote by R the Rees algebra with respect to I and by G the associated graded ring. Then R is a bigraded ring and G is a bigraded Hodge algebra over k such that $\operatorname{ind} G \subseteq \operatorname{ind} A \setminus \{x\}$ with structure map $H \ni h \mapsto h^* \in G$, where h^* denotes the leading form of h with respect to I ([**4**, Theorem 3.1]).

If G is Cohen-Macaulay, then by the inductive hypothesis, we see that $A_{\mathrm{dis}} = G_{\mathrm{dis}}$ is Cohen-Macaulay. So we assume that G is not Cohen-Macaulay. Set $\operatorname{depth} G = e$ and $\dim A = d$. And let M (respectively $\mathfrak{m}$) be the unique bigraded (respectively graded) maximal ideal of R (respectively A).

Since R and G are bigraded rings, there are two entries in the degrees of these rings. From now on, we denote the original degree inherited from A as the first entry and the newly defined degree by the Rees algebra structure as the second entry. Then $\deg x^* = (\deg_A x, 1)$ and $\deg y^* = (\deg_A y, 0)$ for any $y \in H \backslash \{x\}$, where $\deg_A$ denotes the degree as an element of A. Then, since A is concentrated in degree $\mathbf{Z} \times \{0\}$ and $H^i_M(A) = H^i_{\mathfrak{m}}(A)$, we see by the long exact sequence of the local cohomology modules obtained by the short exact sequence

$$0 \longrightarrow R_+ \longrightarrow R \longrightarrow A \longrightarrow 0 \tag{4.1}$$

that

$$[H^i_M(R_+)]_{(u,n)} \simeq [H^i_M(R)]_{(u,n)} \tag{4.2}$$

for any i, u, $n \in \mathbf{Z}$ with $n \neq 0$, where $R_+ = \oplus_{(u,n)\in \mathbf{Z}\times\mathbf{Z}, n>0} R_{(u,n)}$.

On the other hand, since $IR = R_+(0,1)$, by the long exact sequence obtained by

$$0 \longrightarrow IR \longrightarrow R \longrightarrow G \longrightarrow 0,$$

we see that there is an exact sequence

(4.3)

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & [H^{i-1}_M(G)]_{(u,n)} \\ \longrightarrow & [H^i_M(R_+)]_{(u,n+1)} & \longrightarrow & [H^i_M(R)]_{(u,n)} & \longrightarrow & [H^i_M(G)]_{(u,n)} \\ \longrightarrow & [H^{i+1}_M(R_+)]_{(u,n+1)} & \longrightarrow & \cdots & & \end{array}$$

for any u, $n \in \mathbf{Z}$.

Now we recall the following result of Hochster.

Theorem 4.5 (see [**11**, Chapter II, 4.1 Theorem]). *Let Δ be a simplicial complex with vertex set $\{x_1, \dots, x_n\}$. Then the $\mathbf{Z}^n$-graded*

Hilbert series of $H^i_{\mathfrak{m}}(k[\Delta])$ *is*

$$\sum_{\sigma\in\Delta}\left(\dim_k \widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\Delta(\sigma);k)\right)\prod_{x_i\in\sigma}\frac{\lambda_i^{-1}}{1-\lambda_i^{-1}}$$

where $\mathfrak{m}$ *is the unique graded maximal ideal. In particular, if* $k[x_1,\ldots,x_n]$ *is equipped with* $\mathbf{N}^2$*-grading such that* $\deg x_i=(a_i,b_i)$ *with* $(a_i,b_i)\in\mathbf{N}^2\setminus\{(0,0)\}$ *for any* i*, then the* $\mathbf{Z}^2$*-graded Hilbert series of* $H^i_{\mathfrak{m}}(k[\Delta])$ *is*

$$\sum_{\sigma\in\Delta}\left(\dim_k \widetilde{H}^{i-|\sigma|-1}(\mathrm{link}_\Delta(\sigma);k)\right)\prod_{x_i\in\sigma}\frac{\lambda^{-a_i}\mu^{-b_i}}{1-\lambda^{-a_i}\mu^{-b_i}}.$$

We return to the proof of Theorem 4.4. By Theorem 4.5 and Lemma 4.3, we see that

$$[H^i_M(G)]_{(u,n)}=0 \text{ if } n>0. \tag{4.4}$$

So we see by (4.3) that the map

$$[H^i_M(R_+)]_{(u,n+1)}\longrightarrow[H^i_M(R)]_{(u,n)}$$

is an epimorphism for any i, u, $n\in\mathbf{Z}$ with $n>0$. On the other hand,

$$[H^i_M(R_+)]_{(u,n)}=0 \text{ for } n\gg 0,$$

since $H^i_M(R_+)$ is an Artinian module. Therefore, we see by (4.2) that

$$[H^i_M(R_+)]_{(u,n)}\simeq[H^i_M(R)]_{(u,n)}=0$$

for any i, u, $n\in\mathbf{Z}$ with $n>0$.

So by (4.3), we see that

$$[H^e_M(G)]_{(u,0)}\simeq[H^e_M(R)]_{(u,0)}$$

for any $u\in\mathbf{Z}$. Since $\operatorname{depth}R=\operatorname{depth}G+1=e+1$ by Theorem 4.1, it follows that $[H^e_M(R)]_{(u,0)}=0$ and therefore $[H^e_M(G)]_{(u,0)}=0$ for any $u\in\mathbf{Z}$. On the other hand, since $e=\operatorname{depth}G$, we see by (4.4) that there are u and $n\in\mathbf{Z}$ such that

$$[H^e_M(G)]_{(u,n)}\neq 0 \quad\text{and}\quad n<0.$$

It follows from Theorem 4.5 and Lemma 4.3 that

$$\widetilde{H}^{e-|\sigma|-1}(\mathrm{link}_\Delta(\sigma);k) \neq 0 \quad \text{for some } \sigma \in \Delta \text{ with } \sigma \ni x.$$

But this contradicts the assumption that $A_{\mathrm{dis}} = k[\Delta]$ is Buchsbaum. (For characterizations of Buchsbaum complexes, see e.g., [**8**].) □

Remark 4.6. Let k be a field, $B = k[y_1, y_2, y_3]$ the polynomial ring over k with 3 variables and A the second Veronese subring of B. Then A is a Cohen-Macaulay domain of dimension 3. Set $H = \{h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33}\}$ and define the order on H by $h_{11} > h_{12} > h_{13} > h_{22} > h_{23} > h_{33}$. Then A is a homogeneous Hodge algebra over k generated by H with structure map $h_{ij} \mapsto y_i y_j$ governed by the ideal of monomials on H generated by $\{h_{12}^2, h_{12}h_{13}, h_{13}^2, h_{13}h_{22}, h_{13}h_{23}, h_{23}^2\}$. And core A_{dis} is isomorphic to

$$k[X_{12}, X_{13}, X_{22}, X_{23}]/I$$

where X_{12}, X_{13}, X_{22} and X_{23} are indeterminates and

$$I = (X_{12}^2, X_{12}X_{13}, X_{13}^2, X_{13}X_{22}, X_{13}X_{23}, X_{23}^2).$$

Since

$$I = (X_{12}, X_{13}, X_{22}, X_{23})^2 \cap (X_{12}^2, X_{13}, X_{23}^2)$$

and

$$I\colon (X_{12}, X_{13}, X_{22}, X_{23}) = (X_{12}^2, X_{13}, X_{23}^2)$$

is the primary component corresponding to the unique minimal prime ideal $(X_{12},\ X_{13},\ X_{23})$ of I, it is easily verified that

$$I\colon a = (X_{12}^2, X_{13}, X_{23}^2)$$

for any system of parameter a of core A_{dis}. So core A_{dis} is Buchsbaum. Therefore, the square-free assumption in Theorem 4.4 is essential.

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