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ON TENSOR PRODUCTS OF RINGS AND EXTENSION CONJECTURES

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ABSTRACT. We show that a commutative local version of a conjecture of Tachikawa holds over a Cohen-Macaulay tensor product of rings provided it holds over the rings themselves.

1. Introduction. The purpose of this note is to broaden the context in which some homological conjectures hold.

The following commutative local version of a conjecture of Tachikawa has been of interest recently (see [**4, 9**]).

Conjecture (Tachikawa). *Let A be a Cohen-Macaulay local ring. If A has a canonical module ω and $\mathrm{Ext}^i_A(\omega, A) = 0$ for all $i > 0$, then A is Gorenstein, i.e., $\omega = A$.*

Implying this conjecture of Tachikawa is another conjecture, which is a commutative local version of one of Auslander and Reiten, and which has also been of interest (see [**1, 2, 8, 9**]).

Conjecture (Auslander-Reiten). *Let A be a commutative Noetherian local ring, and let M be a finitely generated A-module. If $\mathrm{Ext}^i_A(M, M \oplus A) = 0$ for all $i > 0$, then M is free.*

Suppose that $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are commutative local rings which are essentially of finite type over the same field k. We also assume that k is the common residue field of both R_1 and R_2. In this note we are concerned with what we call the *local tensor* R of R_1 and R_2, this being the localization of $R_1 \otimes_k R_2$ at the maximal ideal $\mathfrak{m} := \mathfrak{m}_1 \otimes_k R_2 + R_1 \otimes_k \mathfrak{m}_2$. The main point here is that properties of vanishing homology and cohomology for modules over the local tensor

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R are often inherited from those properties for modules over R_1 and R_2. In particular, we show that the Tachikawa's conjecture holds over R assuming that it holds over both R_1 and R_2. This result follows from our proof that the Auslander and Reiten's conjecture holds for a certain class of modules over R provided it holds for modules over R_1 and R_2.

Throughout the remainder of the paper we assume that $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ are essentially of finite type over the same field k, and sharing k as the common residue field. We will usually be assuming that both R_1 and R_2 are Cohen-Macaulay. We also assume throughout that our modules are finitely generated.

What makes our results interesting is that the hypotheses placed on the rings in [**4, 7, 8, 9, 13**] for proving the conjectures, like Artinian with radical cube zero, or codimension ≤ 3, or Golod, are relaxed considerably after taking local tensors of such rings. Since local tensors are again essentially of finite type over k, and having k as the residue field, one may iterate the process to obtain a larger class of rings satisfying Tachikawa's conjecture.

2. Preliminaries. We assume that $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$, $\mathfrak{m} = \mathfrak{m}_1 \otimes_k R_2 + R_1 \otimes_k \mathfrak{m}_2$, and $R = (R_1 \otimes_k R_2)_{\mathfrak{m}}$ are as described in the introduction. The results of this paper rely on the following Künneth relation.

2.1. *Suppose that $\mathbf{C}_1$ is a complex of R_1-modules and $\mathbf{C}_2$ is a complex of R_2-modules. Then $\mathbf{C}_1 \otimes_k \mathbf{C}_2$ is a complex of $R_1 \otimes_k R_2$-modules with*

$$\mathrm{H}_i(\mathbf{C}_1 \otimes_k \mathbf{C}_2) \cong \bigoplus_{p+q=i} \mathrm{H}_p(\mathbf{C}_1) \otimes_k \mathrm{H}_q(\mathbf{C}_2)$$

for all i.

(See, for example, [**6**, Theorem 3.1, in V.3].)

2.2. Let $\mathbf{F}_1$ and $\mathbf{G}_1$ be complexes of free R_1-modules, and let $\mathbf{F}_2$ and $\mathbf{G}_2$ be complexes of free R_2-modules. Then one may check that

$$(\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1) \otimes_k (\mathbf{F}_2 \otimes_{R_2} \mathbf{G}_2)$$

and

$$(\mathbf{F}_1 \otimes_k \mathbf{F}_2) \otimes_{R_1 \otimes_k R_2} (\mathbf{G}_1 \otimes_k \mathbf{G}_2)$$

are isomorphic complexes (as $R_1 \otimes_k R_2$-modules).

2.3. Suppose M_1, N_1 are R_1-modules and M_2, N_2 are R_2-modules. If $\mathbf{F}_1$ is an R_1-free resolution of M_1 and $\mathbf{F}_2$ is an R_2-free resolution of M_2, then $\mathbf{F}_1 \otimes_k \mathbf{F}_2$ is an $R_1 \otimes_k R_2$-free resolution of $M_1 \otimes_k M_2$. Similarly, if $\mathbf{G}_1$ is an R_1-free resolution of N_1 and $\mathbf{G}_2$ is an R_2-free resolution of N_2, then $\mathbf{G}_1 \otimes_k \mathbf{G}_2$ is an $R_1 \otimes_k R_2$-free resolution of $N_1 \otimes_k N_2$. Therefore, from the isomorphism of complexes in 2.2 we have that $\mathrm{H}_i((\mathbf{F}_1 \otimes_{R_1} \mathbf{G}_1) \otimes_k (\mathbf{F}_2 \otimes_{R_2} \mathbf{G}_2)) \cong \mathrm{Tor}_i^{R_1 \otimes_k R_2}(M_1 \otimes_k M_2, N_1 \otimes_k N_2)$, and thus the Künneth relation 2.1 becomes

$$\mathrm{Tor}_i^{R_1 \otimes_k R_2}(M_1 \otimes_k M_2, N_1 \otimes_k N_2) \cong \bigoplus_{p+q=i} \mathrm{Tor}_p^{R_1}(M_1, N_1) \otimes_k \mathrm{Tor}_q^{R_2}(M_2, N_2).$$

2.4. Suppose that M_1 is an R_1-module and M_2 is an R_2-module. Then $M_1 \otimes_k M_2 = 0$ if and only if $(M_1 \otimes_k M_2)_\mathfrak{m} = 0$. From 2.3 this means that

$$\mathrm{Tor}_i^{R_1 \otimes_k R_2}(M_1 \otimes_k M_2, N_1 \otimes_k N_2) = 0$$

if and only if

$$\mathrm{Tor}_i^R\big((M_1 \otimes_k M_2)_\mathfrak{m}, (N_1 \otimes_k N_2)_\mathfrak{m}\big) = 0.$$

Lemma 2.5. *We have the following.*

(1) *If R_1 and R_2 are Cohen-Macaulay with canonical modules ω_1 and ω_2, respectively, then R is a Cohen-Macaulay ring with canonical module*

$$\omega = (\omega_1 \otimes_k \omega_2)_\mathfrak{m}.$$

(2) *If R_1 and R_2 are both Gorenstein, then so is R.*

Proof. That R is Cohen-Macaulay is proved in [**14**]. Write $R_1 = Q_1/I_1$ and $R_2 = Q_2/I_2$ with Q_1 and Q_2 regular local rings, and set

$\mathfrak{n} := \mathfrak{n}_1 \otimes_k Q_2 + Q_1 \otimes_k \mathfrak{n}_2$, where $\mathfrak{n}_i$ in Spec (Q_i) denotes the preimage of $\mathfrak{m}_i$, $i = 1, 2$. Let Q denote the regular local ring $(Q_1 \otimes_k Q_2)_\mathfrak{n}$ (paragraph 2.3 shows that a finite minimal free resolution of k over Q is obtained from those over Q_1 and Q_2. Thus Q is regular).

We have $\omega_i \cong \mathrm{Ext}^{s_i}_{Q_i}(R_i, Q_i)$, where s_i denotes the projective dimension of R_i over Q_i, $i = 1, 2$. Let $\mathbf{F}_i$ be a minimal Q_i-free resolution of R_i. Then $\omega_i \cong \mathrm{H}^{s_i}(\mathrm{Hom}_{Q_i}(\mathbf{F}_i, Q_i))$, and there is a natural isomorphism of complexes

$$\mathrm{Hom}_{Q_1 \otimes_k Q_2}(\mathbf{F}_1 \otimes_k \mathbf{F}_2, Q_1 \otimes_k Q_2) \cong \mathrm{Hom}_{Q_1}(\mathbf{F}_1, Q_1) \otimes_k \mathrm{Hom}_{Q_2}(\mathbf{F}_2, Q_2).$$

From 2.1 we have

$$\mathrm{H}^i\big(\mathrm{Hom}_{Q_1 \otimes_k Q_2}(\mathbf{F}_1 \otimes_k \mathbf{F}_2, Q_1 \otimes_k Q_2)\big) \cong \bigoplus_{p+q=i} \mathrm{Ext}^p_{Q_1}(R_1, Q_1) \otimes_k \mathrm{Ext}^q_{Q_2}(R_2, Q_2).$$

Thus,

$$\mathrm{H}^{s_1+s_2}\big(\mathrm{Hom}_{Q_1 \otimes_k Q_2}(\mathbf{F}_1 \otimes_k \mathbf{F}_2, Q_1 \otimes_k Q_2)\big) \cong \mathrm{Ext}^{s_1}_{Q_1}(R_1, Q_1) \otimes_k \mathrm{Ext}^{s_2}_{Q_2}(R_2, Q_2).$$

Since (by 2.3) $(\mathbf{F}_1 \otimes_k \mathbf{F}_2)_\mathfrak{n}$ is a minimal Q-free resolution of R of length $s_1 + s_2$, we see that $\mathrm{pd}_Q R = s_1 + s_2$, and

$$\omega = \mathrm{Ext}^{s_1+s_2}_Q(R, Q) \cong (\omega_1 \otimes_k \omega_2)_\mathfrak{m}.$$

Property (2) is obvious given part (1) (cf. also [**14**]). □

The main result of this section requires the following lemma.

Lemma 2.6. *Let A be a Cohen-Macaulay local ring with canonical module ω, M a maximal Cohen-Macaulay module and x a non zerodivisor on A, ω, and M. Then*

$$\mathrm{Hom}_A(M, \omega)/x\,\mathrm{Hom}_A(M, \omega) \cong \mathrm{Hom}_{A/(x)}(M/xM, \omega/x\omega).$$

Proof. We let $-^\vee = \operatorname{Hom}_A(-, \omega)$. Consider the short exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$. Applying $-^\vee$, we obtain the exact sequence

$$0 \longrightarrow (M/xM)^\vee \longrightarrow M^\vee \xrightarrow{x} M^\vee \longrightarrow \operatorname{Ext}^1_A(M/xM, \omega) \longrightarrow 0.$$

Thus, $M^\vee/xM^\vee \cong \operatorname{Ext}^1_A(M/xM, \omega)$. Now Rees's formula says that $\operatorname{Ext}^1_A(M/xM, \omega) = \operatorname{Hom}_{A/(x)}(M/xM, \omega/x\omega)$ (see [**11**, page 140, Lemma 2(i)]), and this establishes the lemma. □

The main result of this section allows us to prove statements involving cohomology by proving a corresponding statement for homology.

Theorem 2.7. *Let A be a Cohen-Macaulay local ring with canonical module ω. Then for maximal Cohen-Macaulay A-modules M and N the following are equivalent*:

(1) $\operatorname{Ext}^i_A(M, N) = 0$ *for all* $i > 0$;

(2) $\operatorname{Tor}^A_i(M, \operatorname{Hom}_A(N, \omega)) = 0$ *for all* $i > 0$, *and* $M \otimes_A \operatorname{Hom}_A(N, \omega)$ *is maximal Cohen-Macaulay.*

Proof. We let $-^\vee = \operatorname{Hom}_A(-, \omega)$ and induct on d, the dimension of A. When $d = 0$ the canonical module ω is the injective hull of the residue field k of A, and therefore $\operatorname{Ext}^i_A(M, N)^\vee \cong \operatorname{Tor}^A_i(M, N^\vee)$ (see [**12**, 11.57]). Matlis duality (see [**5**, 3.2.13]) shows that $\operatorname{Ext}^i_A(M, N)^\vee = 0$ if and only if $\operatorname{Ext}^i_A(M, N) = 0$, and this takes care of the $d = 0$ case. Now assume that $d > 0$.

(1) $\Rightarrow$ (2). Let x be a non zerodivisor on M, N, $N^\vee$ and A. From $0 \to N \xrightarrow{x} N \to N/xN \to 0$, we get the long exact sequence

$$\begin{aligned} (2.7.1) \quad \cdots \longrightarrow \operatorname{Ext}^i_A(M, N) &\xrightarrow{x} \operatorname{Ext}^i_A(M, N) \\ &\longrightarrow \operatorname{Ext}^i_A(M, N/xN) \longrightarrow \operatorname{Ext}^{i+1}_A(M, N) \longrightarrow \cdots . \end{aligned}$$

Therefore, $\operatorname{Ext}^i_A(M, N) = 0$ for all $i > 0$ implies $\operatorname{Ext}^i_A(M, N/xN) = 0$ for all $i > 0$. The isomorphisms (see [**11**, page 140, Lemma 2(ii)])

$$(2.7.2) \qquad \operatorname{Ext}^i_A(M, N/xN) \cong \operatorname{Ext}^i_{A/(x)}(M/xM, N/xN)$$

for all i imply that $\operatorname{Ext}^i_{A/(x)}(M/xM, N/xN) = 0$ for all $i > 0$. Now induction, and the isomorphism $\operatorname{Hom}_{A/(x)}(N/xN, \omega/x\omega) \cong N^\vee/xN^\vee$ of 2.6 yields $\operatorname{Tor}_i^{A/(x)}(M/xM, N^\vee/xN^\vee) = 0$ for all $i > 0$, and that $M/xM \otimes_{A/(x)} N^\vee/xN^\vee$ is a maximal Cohen-Macaulay $A/(x)$-module. The isomorphisms (see [**11**, page 140, Lemma 2(iii)])

$$\text{(2.7.3)} \qquad \operatorname{Tor}_i^A(M, N^\vee/xN^\vee) \cong \operatorname{Tor}_i^{A/(x)}(M/xM, N^\vee/xN^\vee)$$

for all i show that $\operatorname{Tor}_i^A(M, N^\vee/xN^\vee) = 0$ for all $i > 0$. Consider the long exact sequence

(2.7.4)

$$\cdots \longrightarrow \operatorname{Tor}_i^A(M, N^\vee) \xrightarrow{x} \operatorname{Tor}_i^A(M, N^\vee) \longrightarrow \operatorname{Tor}_i^A(M, N^\vee/xN^\vee) \longrightarrow \cdots$$

derived from the short exact sequence $0 \to N^\vee \xrightarrow{x} N^\vee \to N^\vee/xN^\vee \to 0$. It follows that $\operatorname{Tor}_i^A(M, N^\vee) = 0$ for all $i > 0$, and we have the short exact sequence $0 \to M \otimes_A N^\vee \xrightarrow{x} M \otimes_A N^\vee \to M \otimes_A N^\vee/xN^\vee \to 0$. Thus x is a non zerodivisor on $M \otimes_A N^\vee$. Since $(M \otimes_A N^\vee)/x(M \otimes_A N^\vee) \cong M \otimes_A (N^\vee/xN^\vee) \cong M/xM \otimes_{A/(x)} N^\vee/xN^\vee$ is a maximal Cohen-Macaulay module over $A/(x)$, we see that $M \otimes_A N^\vee$ is a maximal Cohen-Macaulay module over A.

(2) $\Rightarrow$ (1). Let x be a non zerodivisor on M, N, $N^\vee$, $M \otimes_A N^\vee$ and A. The long exact sequence 2.7.4 induced by the short exact sequence $0 \to N^\vee \xrightarrow{x} N^\vee \to N^\vee/xN^\vee \to 0$, and the hypothesis shows that $\operatorname{Tor}_i^A(M, N^\vee/xN^\vee) = 0$ for all $i > 0$ (that $\operatorname{Tor}_1^A(M, N^\vee/xN^\vee) = 0$ follows since x is a non zerodivisor on $M \otimes_A N^\vee$). The isomorphisms 2.7.3 for all i show that $\operatorname{Tor}_i^{A/(x)}(M/xM, N^\vee/xN^\vee) = 0$ for all $i > 0$. Since $(M \otimes_A N^\vee)/x(M \otimes_A N^\vee) \cong M \otimes_A (N^\vee/xN^\vee) \cong M/xM \otimes_{A/(x)} N^\vee/xN^\vee$ is a maximal Cohen-Macaulay module over $A/(x)$, induction yields $\operatorname{Ext}^i_{A/(x)}(M/xM, N/xN) = 0$ for all $i > 0$. The isomorphisms 2.7.2 show that $\operatorname{Ext}^i_A(M, N/xN) = 0$ for all $i > 0$. Finally, the long exact sequence 2.7.1 implies that $\operatorname{Ext}^i_A(M, N) = 0$ for all $i > 0$. □

Lemma 2.8. *Suppose $M = (M_1 \otimes_k M_2)_\mathfrak{m}$ and that R_1 and R_2 are Cohen-Macaulay. Then M is a maximal Cohen-Macaulay R-module if and only if M_1 is a maximal Cohen-Macaulay R_1-module and M_2 is a maximal Cohen-Macaulay R_2-module.*

Proof. Let $R_i = Q_i/I_i$ with $(Q_i, \mathfrak{n}_i)$ regular local rings $i = 1, 2$, and set $Q = (Q_1 \otimes_k Q_2)_\mathfrak{n}$ with $\mathfrak{n} = \mathfrak{n}_1 \otimes_k Q_2 + Q_1 \otimes_k \mathfrak{n}_2$. By 2.1 and

2.4 we see that $\mathrm{pd}_Q R = \mathrm{pd}_{Q_1} R_1 + \mathrm{pd}_{Q_2} R_2$ and $\mathrm{pd}_Q M = \mathrm{pd}_{Q_1} M_1 + \mathrm{pd}_{Q_2} M_2$. Thus from the Auslander-Buchsbaum-Serre formula we have $\mathrm{depth}_R R = \mathrm{depth}_Q R = \mathrm{depth}_{Q_1} R_1 + \mathrm{depth}_{Q_2} R_2 = \mathrm{depth}_{R_1} R_1 + \mathrm{depth}_{R_2} R_2$, and $\mathrm{depth}_R M = \mathrm{depth}_Q M = \mathrm{depth}_{Q_1} M_1 + \mathrm{depth}_{Q_2} M_2 = \mathrm{depth}_{R_1} M_1 + \mathrm{depth}_{R_2} M_2$. Therefore $\mathrm{depth}_R M = \mathrm{depth}_R R$ if and only if both $\mathrm{depth}_{R_1} M_1 = \mathrm{depth}_{R_1} R_1$ and $\mathrm{depth}_{R_2} M_2 = \mathrm{depth}_{R_2} R_2$. $\square$

Definition. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ denote classes of R_1-modules and R_2-modules, respectively. Then we define $\mathrm{LT}\,(\mathcal{C}_1, \mathcal{C}_2)$ to be the class of all R-modules which are finite direct sums of modules of the form $(M_1 \otimes_k M_2)_{\mathfrak{m}}$, with $M_i \in \mathcal{C}_i$, $i = 1, 2$.

Let $\mathbf{mod}\,(R_i)$ denote the category of finitely generated modules over R_i. The Künneth relation 2.1 says that if M and N are R-modules belonging to $\mathrm{LT}\,(\mathbf{mod}\,(R_1), \mathbf{mod}\,(R_2))$, then computing $\mathrm{Tor}^R(M, N)$ amounts to computing Tors over R_1 and R_2. In Section 5 we give a simple example illustrating that an R-module may not be in $\mathrm{LT}\,(\mathbf{mod}\,(R_1), \mathbf{mod}\,(R_2))$, although one of its syzygies is. This motivates the hypothesis in the theorem of the next section. Note that every module is the 0th syzygy of itself.

5. On the conjecture of Auslander and Reiten. We assume that $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$, $\mathfrak{m} = \mathfrak{m}_1 \otimes_k R_2 + R_1 \otimes_k \mathfrak{m}_2$, and $R = (R_1 \otimes_k R_2)_{\mathfrak{m}}$ are as described in the introduction.

Theorem 3.1. *Assume that the rings R_1 and R_2 are Cohen-Macaulay, and that the Auslander-Reiten conjecture holds for classes $\mathcal{C}_i$ of maximal Cohen-Macaulay modules over R_i, $i = 1, 2$. Then the Auslander-Reiten conjecture holds for R-modules with a syzygy belonging to* $\mathrm{LT}\,(\mathcal{C}_1, \mathcal{C}_2)$.

Remark. In [**9**] it is shown that modules over an Artinian local ring which are annihilated by the square of the maximal ideal satisfy the Auslander-Reiten conjecture. Thus if R_1 and R_2 are Artinian, M_i is an R_i-module with $\mathfrak{m}_i^2 M_i = 0$, $i = 1, 2$, then the R-module $M := (M_1 \otimes_k M_2)_{\mathfrak{m}}$ satisfies the Auslander-Reiten conjecture even though it may be that $\mathfrak{m}^2 M \neq 0$.

The proof of 3.1 will require a lemma:

Lemma 3.2. *Suppose that R_1 and R_2 are Cohen-Macaulay with canonical modules ω_1 and ω_2. Let ω denote the canonical module of R, and suppose M_1 is a maximal Cohen-Macaulay R_1-module and M_2 is maximal Cohen-Macaulay R_2-module. Then*

$$\operatorname{Hom}_R\big((M_1 \otimes_k M_2)_{\mathfrak{m}}, \omega\big) \cong \big(\operatorname{Hom}_{R_1}(M_1, \omega_1) \otimes_k \operatorname{Hom}_{R_2}(M_2, \omega_2)\big)_{\mathfrak{m}}.$$

Proof of Theorem 3.1. Note that R is Cohen-Macaulay by 2.5. We let $-^{\vee}$ denote the dual $\operatorname{Hom}_R(-, \omega)$, where ω is the canonical module of R. Let M be an R-module with $\operatorname{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, and assume that M has a syzygy $N \in \operatorname{LT}(\mathcal{C}_1,\ C_2)$. Following the proof of Lemma 1.4 of [**3**] we have $\operatorname{Ext}^i_R(N, N \oplus R) = 0$ for all $i > 0$. It suffices to assume that $N \cong (N_1 \otimes_Q N_2)_{\mathfrak{m}}$, with $N_i \in \mathcal{C}_i$, $i = 1, 2$. By 2.7 and 2.8 we have $\operatorname{Tor}^R_i(N, (N \oplus R)^{\vee}) = 0$ for all $i > 0$, and $N \otimes_R (N \oplus R)^{\vee}$ is maximal Cohen-Macaulay. By Lemma 3.2,

$$\begin{aligned}(N \oplus R)^{\vee} \cong &\big(\operatorname{Hom}_{R_1}(N_1, \omega_1) \otimes_k \operatorname{Hom}_{R_2}(N_2, \omega_2)\big)_{\mathfrak{m}} \\ &\oplus \big(\operatorname{Hom}_{R_1}(R_1, \omega_1) \otimes_k \operatorname{Hom}_{R_2}(R_2, \omega_2)\big)_{\mathfrak{m}}.\end{aligned}$$

Now 2.3, 2.4 and 2.8 yield $\operatorname{Tor}^{R_1}_i(N_1, \operatorname{Hom}_{R_1}(N_1 \oplus R_1, \omega_1)) = 0$ for all $i > 0$ with $N_1 \otimes_{R_1} \operatorname{Hom}_{R_1}(N_1 \oplus R_1, \omega_1)$ maximal Cohen-Macaulay over R_1, and $\operatorname{Tor}^{R_2}_i(N_2, \operatorname{Hom}_{R_2}(N_2 \oplus R_2, \omega_2)) = 0$ for all $i > 0$ with $N_2 \otimes_{R_2} \operatorname{Hom}_{R_2}(N_2 \oplus R_2, \omega_2)$ maximal Cohen-Macaulay over R_2. By 2.7 again, we have $\operatorname{Ext}^i_{R_1}(N_1, N_1 \oplus R_1) = 0$ for all $i > 0$ and $\operatorname{Ext}^i_{R_2}(N_2, N_2 \oplus R_2) = 0$ for all $i > 0$. Since N_1 and N_2 satisfy the Auslander-Reiten conjecture over R_1 and R_2, respectively, we have that N_i is a free R_i-module, $i = 1, 2$. Thus N is a free R-module, and M has finite projective dimension over R. Finally, it is shown in [**10**] that if M has finite projective dimension, then $\sup\{i | \operatorname{Ext}^i_R(M, X) \neq 0\}$, for any finitely generated R-module X, is equal to $\operatorname{depth}_R R - \operatorname{depth}_R M$. Since $\operatorname{Ext}^i_R(M, R) = 0$ for all $i > 0$ we conclude that M is free, by the Auslander-Buchsbaum formula. □

Proof of Theorem 3.2. First note that the result is true if M_1 and M_2 are free. In the general case, let $R_i^{n_i} \to R_i^{m_i} \to M_i \to 0$ be an R_i-free presentation of M_i, $i = 1, 2$. Let

$$\begin{aligned} F &:= R_1^{m_1} \otimes_k R_2^{m_2}, \\ G &:= (R_1^{m_1} \otimes_k R_2^{n_2}) \oplus (R_1^{n_1} \otimes_k R_2^{m_2}), \\ F' &:= \mathrm{Hom}_{R_1}(R_1^{m_1}, \omega_1) \otimes_k \mathrm{Hom}_{R_2}(R_2^{m_2}, \omega_2), \text{ and} \\ G' &:= \mathrm{Hom}_{R_1}(R_1^{m_1}, \omega_1) \otimes_k \mathrm{Hom}_{R_2}(R_2^{n_2}, \omega_2) \oplus \\ &\qquad \mathrm{Hom}_{R_1}(R_1^{n_1}, \omega_1) \otimes_k \mathrm{Hom}_{R_2}(R_2^{m_2}, \omega_2). \end{aligned}$$

Then the horizontal maps in the commutative diagram are isomorphisms and the columns are exact (by 2.1)

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R_1 \otimes_k R_2}(M_1 \otimes_k M_2, \omega_1 \otimes_k \omega_2) & & \mathrm{Hom}_{R_1}(M_1, \omega_1) \otimes_k \mathrm{Hom}_{R_2}(M_2, \omega_2) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R_1 \otimes_k R_2}(F, \omega_1 \otimes_k \omega_2) & \longrightarrow & F' \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{R_1 \otimes_k R_2}(G, \omega_1 \otimes_k \omega_2) & \longrightarrow & G', \end{array}$$

and this diagram may be completed to establish the isomorphism whose localization establishes the lemma. □

4. On the conjecture of Tachikawa. We maintain the standard assumptions on $(R_1, \mathfrak{m}_1)$, $(R_2, \mathfrak{m}_2)$ and $R = (R_1 \otimes_k R_2)_{\mathfrak{m}}$. We record the following special case of 3.1:

Theorem 4.1. *Suppose that R_1 and R_2 are Cohen-Macaulay, and the Tachikawa conjecture holds for both R_1 and R_2. Then it also holds for R.*

Proof. By 2.5 we have that R is Cohen-Macaulay. Assume that $\mathrm{Ext}^i_R(\omega, R) = 0$ for all $i > 0$. By 2.7 we have that $\mathrm{Tor}_i^R(\omega, \omega) = 0$ for all $i > 0$, and $\omega \otimes_R \omega$ is maximal Cohen-Macaulay. From 2.3, 2.4, 2.5, 2.8 and 3.2 we obtain that $\mathrm{Tor}_i^{R_j}(\omega_j, \omega_j) = 0$ for all $i > 0$, and $\omega_j \otimes_{R_j} \omega_j$ is a maximal Cohen-Macaulay R_j-module, $j = 1, 2$. By 2.7 again we get that $\mathrm{Ext}^i_{R_j}(\omega_j, R_j) = 0$ for all $i > 0$, $j = 1, 2$. Since the conjecture holds over R_1 and R_2, $\omega_1 \cong R_1$ and $\omega_2 \cong R_2$, and thus $\omega \cong R$. □

Remark. In [**2, 9**] it is proved that if A is an Artinian local ring with radical cube zero, then it satisfies the Tachikawa conjecture; in fact, one only needs to assume that $\mathrm{Ext}^1_A(\omega_A, A) = 0$ to conclude that A is Gorenstein. Thus, if R_1 and R_2 are both of radical cube zero and essentially of finite type over the same field k, and having k as their common residue field, then R satisfies the Tachikawa conjecture even though $\mathfrak{m}^3 \neq 0$; and we too only need to assume that $\mathrm{Ext}^1_R(\omega, R) = 0$. Indeed, since in this case R is Artinian, the canonical module ω is injective, and so by Matlis duality we have $\mathrm{Ext}^1_R(o, R) = 0$ if and only if $\mathrm{Tor}_1^R(\omega, \omega) = 0$. Now 2.3 shows that $\mathrm{Tor}_1^{R_i}(\omega_i, \omega_i) = 0$, $i = 1, 2$, which in turn implies that $\mathrm{Tor}^1_{R_i}(\omega_i, R_i) = 0$, $i = 1, 2$. The result of [**9**] shows that $\omega_i \cong R_i$, $i = 1, 2$, and thus by 2.5, $\omega \cong R$.

Also proved in [**2**] is that the Tachikawa conjecture holds over Cohen-Macaulay rings R in the following circumstances: (1) R is generically Gorenstein; (2) R is the quotient of a generically Gorenstein ring modulo a regular sequence; (3) R is in the linkage class of a complete intersection; (4) the codimension of R is at most 3; (5) R is Golod. Assuming rings essentially of finite type over k and having k as their residue field, each having one of these forms, we may iteratively form local tensors of such to get a larger class of rings satisfying the Tachikawa conjecture.

5. Examples. In this section we give a necessary condition for an R-module to belong to $\mathrm{LT} := \mathrm{LT}(\mathbf{mod}\,(R_1), \mathbf{mod}\,(R_2))$ and two simple examples, one which shows that modules not in LT may have higher syzygies which are in LT, and one which shows that some modules may have no syzygy in LT.

5.1. Write $R_i = Q_i/I_i$, where $(Q_i, \mathfrak{n}_i)$ are regular local rings, $Q := (Q_1 \otimes_k Q_2)_\mathfrak{n}$ with $\mathfrak{n} := \mathfrak{n}_1 \otimes_k Q_2 + Q_1 \otimes_k \mathfrak{n}_2$, and let M be an R-module. For a local ring (A, k), denote by $\mathrm{P}_X^A(t)$ the *Poincaré* series $\sum_{i \geq 0} \dim_k \mathrm{Tor}_i^A(X, k) t^i$ of an A-module X. Then $M = \oplus_{j=1}^r (M_{1,j} \otimes_k$

$M_{2,j})_{\mathfrak{m}}$ has Poincaré series $\mathrm{P}_M^Q(t) = \sum_{j=1}^r \mathrm{P}_{M_{1,j}}^{Q_1}(t)\mathrm{P}_{M_{2,j}}^{Q_2}(t)$. Write $\mathrm{D}_t p(t)$ for the derivative of the polynomial $p(t)$. Since R_i is a Q_i-module of rank zero, $i = 1, 2$, and since rank is additive on exact sequences, -1 is a zero of each of the polynomials $\mathrm{P}_{M_{i,j}}^{Q_i}(t)$, $i = 1, 2$, $j = 1, \dots, r$. Thus we have that M is in LT only if $\mathrm{D}_t\mathrm{P}_M^Q(-1) = 0$.

The property in 5.1 can be used to prove the module M in the following example is not in LT.

Example 5.2. Let $R_1 := k[x]/(x^2)$ and $R_2 := k[y]/(y^2)$. Then $R \cong k[x,y]/(x^2, y^2)$. Let M be the cokernel of the map $R \stackrel{xy}{\to} R$. Then M is not in LT, but its first syzygy is k, which is in LT.

Example 5.3. Let R_1, R_2 and R be as in Example 5.2. Let M be the cokernel of the map $R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)} R^2$. Then no syzygy of M is in LT.

To see that none of the syzygy modules in the last example is in LT, note that a minimal free resolution of M is periodic of period two (meaning that the differentials alternate, the even ones represented by $\left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$ and the odd ones represented by $\left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$), and M is indecomposable. Thus if M had a syzygy in LT it would have the form $(M_1 \otimes_k M_2)_{\mathfrak{m}}$ with M_1 an R_1-module and M_2 an R_2-module. Either M_1 is free, or it has a periodic minimal free resolution over R_1. The same is true for M_2. Now if neither M_i is free, by 2.3 a minimal free resolution of the syzygy M over R is composed of two periodic resolutions, yielding a resolution where the ranks of the free modules grow linearly, but this contradicts that fact that M has a periodic resolution.

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